

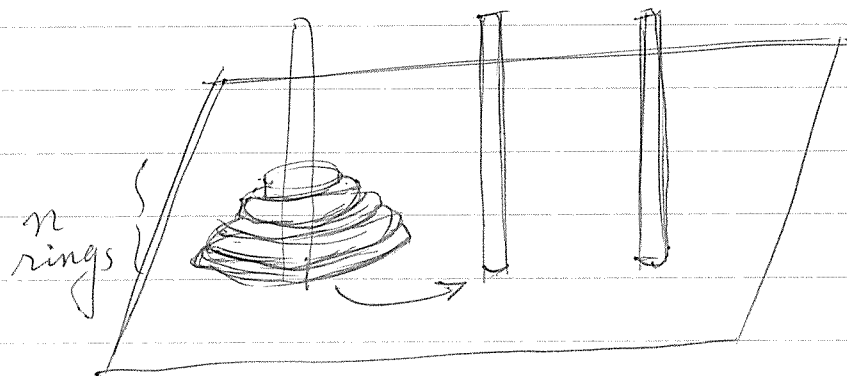
Lecture 6

March 27, 20

Date Examples of Recurrence Relations

No. 6-1

The Tower of Hanoi problem



a_n : the number of moves it takes to transfer n rings from one peg to another

(*) We move $n-1$ rings to the third one, the largest ring into the second one, and then the $n-1$ rings from the third to the second.

$$a_n = 2a_{n-1} + 1$$

↓

Homogeneous solution $a_n = A \cdot 2^n$ ($x-2=0$)

Particular solution: $2^n + C$ ($a_1 = 1 \Rightarrow C = -1$)

$$a_n = 2^n - 1.$$

[What happen if there are "4" pegs?]

Example

A:

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}_{n \times n}$$

1	1	0	0	0	0
1	1	1	0	0	0
0	1	1	1	0	0
0	0	1	1	1	0
0	0	0	1	1	1
0	0	0	0	1	1

$n=6$

The generalization of Tower Hanoi problem

Instead of three pegs, we insert more pegs.

Definition Let $a_{n,k}$ denote the number of moves it takes to transfer n rings from one peg to another provided we have k pegs.

(i) $k=3$ is for Tower Hanoi problem.

(ii) How about more pegs, say $k=4$. (總共 10,000, -NT dollars)
Find $m(n,k)$.

Note This is a type of combinatorial optimization problem

(組合最優化). Since we are aiming at finding the minimum number of moves, we have the following requirements:

(1) Prove that we need at least certain number of moves

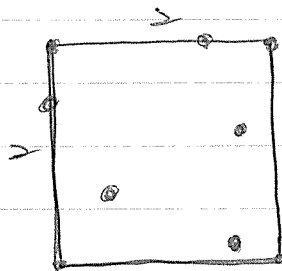
say $m(n,k)$ to transfer the n -rings into another peg

following the basic moving rules.

(2) Then, find a way (algorithm) with $m(n,k)$ moves to transfer the n -rings.

These two steps combined shows that $a_{n,k} = m(n,k)$.

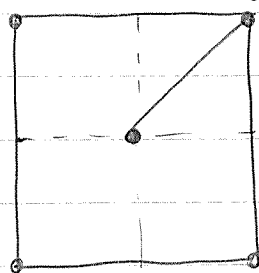
Idea of optimization (Combinatorial)



Problem If we place five points ^{inside} the above square (including its boundary), then what is the maximum "shortest distance" l_5 of these five points?

- (.) There are 10 possible distances between two points.
- (. . .) Clearly, the shortest distance can be found.
- (. . .) We would like to find a layout such that the shortest distance is a maximum.

Idea ① Find an arrangement of points.



Since $\sqrt{2}$ is the shortest, the answer must be at least $\sqrt{2}$, i.e., $l_5 \geq \sqrt{2}$.

- ② Prove that no matter how you place the 5 points, $l_5 \leq \sqrt{2}$. (?)

Problem Find $\det A$. (A in 5-1.)

Sol. $a_n = \det A$ (Expanding w.r.t. 1st column!)

$$a_n = a_{n-1} - a_{n-2}$$

$$\alpha^2 - \alpha + 1 = 0$$

$$\alpha_1 = \frac{1}{2} + \frac{\sqrt{3}}{2}i, \alpha_2 = \frac{1}{2} - \frac{\sqrt{3}}{2}i$$

$$\alpha_1 = \cos \frac{\pi}{3} + i \sin \frac{\pi}{3}$$

$$\alpha_2 = \cos \frac{\pi}{3} - i \sin \frac{\pi}{3}$$

$$\alpha_1 = \rho(\cos \theta + i \sin \theta)$$

$$\text{where } \rho^2 = \left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 = 1$$

$$\text{and } \theta = \tan^{-1}(\sqrt{3}/1/2) = \frac{\pi}{3}.$$

$$a_n = B_1 \alpha_1^n + B_2 \alpha_2^n \quad \text{where } \underline{a_1 = 1, a_2 = 0}$$

$$= B_1' \cos \frac{n\pi}{3} + B_2' \sin \frac{n\pi}{3}$$

$$B_1' = B_1 + B_2 \quad \text{and} \quad B_2' = i(B_1 - B_2)$$

$$B_1' \cos \frac{\pi}{3} + B_2' \sin \frac{\pi}{3} = 1$$

$$B_1' \cos \frac{2\pi}{3} + B_2' \sin \frac{2\pi}{3} = 0$$

$$\frac{1}{2} B_1' + \frac{\sqrt{3}}{2} B_2' = 1$$

$$-\frac{1}{2} B_1' + \frac{\sqrt{3}}{2} B_2' = 0$$

$$\boxed{B_1' = 1 \quad B_2' = \frac{1}{\sqrt{3}}}$$

$$\Rightarrow a_n = \cos \frac{n\pi}{3} + \frac{1}{\sqrt{3}} \sin \frac{n\pi}{3}$$

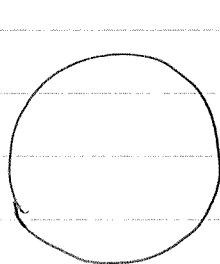
$$\left(\begin{array}{l} B_1 + B_2 = 1 \\ i(B_1 - B_2) = \frac{1}{\sqrt{3}} \end{array} \right)$$

(没用到这两个式子!)

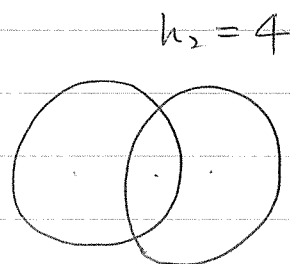
Example

What is the maximum number of regions that n cycles on a plane can achieve?

$$h_0 = 1$$



$$h_1 = 2$$



$$h_2 = 4$$

...

$$h_n = h_{n-1} + 2(n-1)$$

$$h_n - h_{n-1} = 2(n-1)$$

$$h_{n-1} - h_{n-2} = 2(n-2)$$

⋮

$$h_2 - h_1 = 2(2-1)$$

$$\Rightarrow h_n - h_1 = 2(1+2+\dots+(n-1))$$

$$h_n = 2 + n(n-1) = n^2 - n + 2.$$

By G.F.

$$h_0 = 1$$

$$H(x) = \sum_{k=0}^{\infty} h_k x^k$$

$$\sum_{k=1}^{\infty} h_k x^k = \sum_{k=1}^{\infty} h_{k-1} x^k + \sum_{k=1}^{\infty} 2(k-1) x^k$$

$$H(x) - 1 = x \cdot H(x) + 2x^2 \left(\sum_{k=1}^{\infty} (k-1) x^{k-2} \right) = xH(x) + \frac{2x^2}{(1-x)^2}$$

$$H(x)(1-x) = \frac{2x^2}{(1-x)^2} + 1 = \frac{3x^2 - 2x + 1}{(1-x)^2}$$

$$H(x) = \frac{3x^2 - 2x + 1}{(1-x)^3}, \quad h_n = \dots$$

✓ With particular form $b = 4$ Example

$$\begin{cases} h_n = 3h_{n-1} - 4n, & (n \geq 1) \\ h_0 = 2. \end{cases}$$

$$\alpha - 3 = 0, \alpha = 3$$

$$h_n = c_1 3^n,$$

For particular sol. try

compatible to it.

$$\frac{h_n}{an+b}$$

If the answer is correct, then

 $\downarrow h_{n-1}$

$$an+b = 3(a(n-1)+b) - 4n$$

$$an+b = 3a(n-1) + 3b - 4n$$

$$\begin{cases} a = 3a - 4, & \Rightarrow 2a = 4, a = 2 \\ b = 3b - 3a, & \Rightarrow 2b = 6, b = 3. \end{cases}$$

Hence, the solution for h_n is

$$h_n = c_1 3^n + 2n + 3.$$

Since $h_0 = 2$ $2 = c_1 + 3$, $c_1 = -1$.

$$h_n = -3^n + 2n + 3.$$

Example

$$\begin{cases} h_n = 2h_{n-1} + 1 \\ h_0 = 0 \end{cases}$$

particular solution is a constant.

$$h_n = a, h_{n-1} = a,$$

$$a = 2a + 1, a = -1$$

$$h_n = c(2^n) - 1$$

$$h_0 = 0 \Rightarrow c = 1. \quad h_n = 2^n - 1.$$

Another way, $h_n = 2h_{n-1} + 1$

$$= 2(2h_{n-2} + 1) + 1$$

$$= 2^2 h_{n-2} + 2 + 1$$

$$= 2^3 h_{n-3} + 2 + 2 + 1$$

$$= 2^n h_0 + 2^{n-1} + 2^{n-2} + \dots + 2 + 1$$

$$= 2^n - 1.$$

Example $\begin{cases} h_n = 2h_{n-1} + 3^n \\ h_0 = 2 \end{cases}$

$$\underline{h_n = c \cdot 2^n}$$

, particular solution: $a \cdot 3^n$

$$h_n = 2h_{n-1} + 3^n$$

$\uparrow$
 h_n

$$a \cdot 3^n = 2a \cdot 3^{n-1} + 3^n \quad (\div 3^{n-1})$$

$$3a = 2a + 3, \quad a = 3,$$

$$h_n = c \cdot 2^n + 3 \cdot 3^n$$

$$h_0 = c + 3 = 2, \quad c = -1.$$

$$h_n = -2 + 3^{n+1}.$$

$$\checkmark h_0 = 3,$$

$$\Rightarrow h_n = 3^{n+1}$$

$$\checkmark h_0 = 4$$

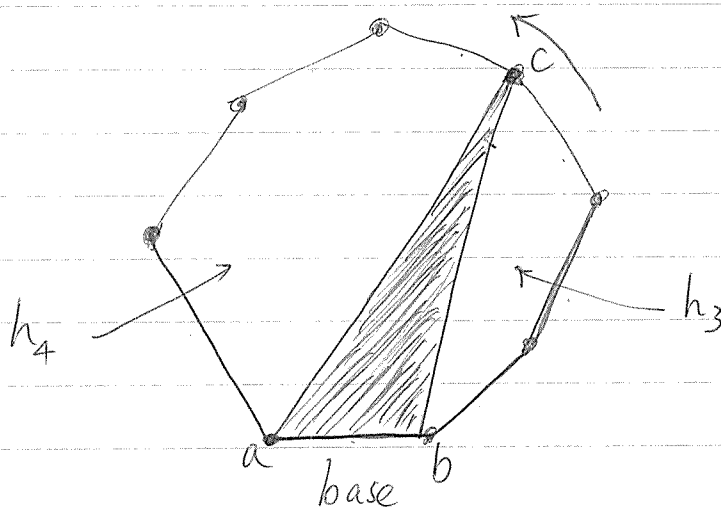
$$\Rightarrow h_n = 2 + 3^{n+1}$$

Initial condition is important!

The most interesting example

Let h_n denote the number of ways of dividing a convex polygonal region with $n+1$ sides into triangular regions (Triangulation). Then, $h_n = \frac{1}{n} \binom{2n-2}{n-1}$, $n \geq 1$ with $h_1 = 1$.

Note that a two sided region is not a triangular region, but for convenience in reduction (recurrence), we let $h_1 = 1$. If we let $h_1 = 0$, then the answer will be changed accordingly. $h_2 = 1$ is also a clear assumption. Consider $n \geq 3$.



$n=8$

$$h_n = \sum_{k=1}^{n-1} h_k h_{n-k}$$

(隨著頂點的改變, 方法數
依序為 $h_1 h_6, h_2 h_5, h_3 h_4, h_4 h_3, h_5 h_2, h_6 h_1$)

Let the generating function for sequence $\langle h_1, h_2, \dots \rangle$ be

$$H(x), \text{ hence } H(x) = \sum_{k=1}^{\infty} h_k x^k.$$

Note that

$$(H(x))^2 = \sum_{j=2}^{\infty} h_j x^j, \quad h_j = \sum_{i=1}^{j-1} h_i \cdot h_{j-i}.$$

$$= H(x) - h_1 x = H(x) - x$$

$$h_2 = h_1 \cdot h_1$$

$$h_3 = h_1 \cdot h_2 + h_2 \cdot h_1$$

$$h_4 = h_1 \cdot h_3 + h_2 \cdot h_2 + h_3 \cdot h_1$$

$\vdots$

$$(H(x))^2 - H(x) + x = 0$$

$$H(x) = \frac{1 \pm \sqrt{1-4x}}{2}. \text{ Since } H(0) = 0, \frac{1 - \sqrt{1-4x}}{2} \text{ is}$$

the answer.

$$\text{Now, } H(x) = \frac{1}{2} - \frac{1}{2} (1-4x)^{\frac{1}{2}}$$

$$= \frac{1}{2} - \frac{1}{2} \sum_{k=0}^{\infty} \binom{\frac{1}{2}}{k} (-4x)^k \quad (\text{Newton's Binomial Theorem})$$

$$= \frac{1}{2} - \frac{1}{2} \sum_{k=0}^{\infty} \frac{(\frac{1}{2})(\frac{1}{2}-1)(\frac{1}{2}-2)\dots(\frac{1}{2}-k+1)}{k!} (-1)^k 4^k x^k$$

$\vdots$ 10 min. later

$$= \frac{1}{2} - \frac{1}{2} \left(1 - 2 \sum_{k=1}^{\infty} \frac{1}{k} \binom{2k-2}{k-1} x^k \right), \quad \left(|x| < \frac{1}{4} \right)$$

$$H(x) = \sum_{k=1}^{\infty} \frac{1}{k} \binom{2k-2}{k-1} x^k.$$

$$\text{Hence, } h_n = \frac{1}{n} \binom{2n-2}{n-1}.$$

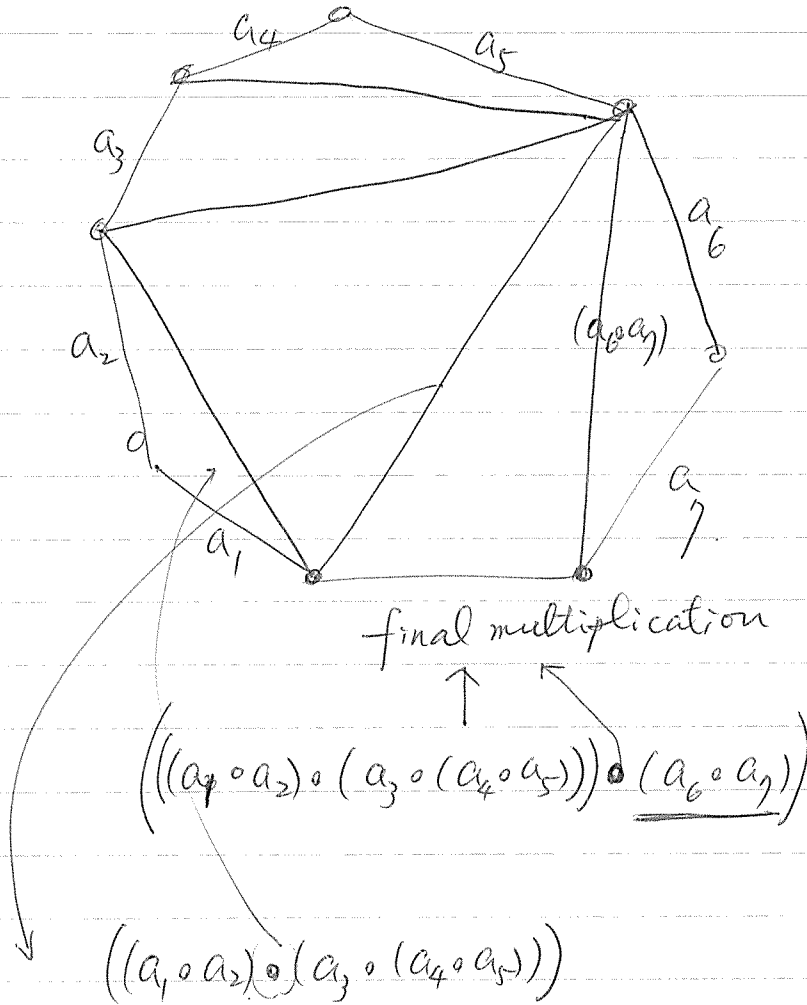
$\Uparrow$ Catalan numbers $\frac{1}{n+1} \binom{2n}{n}$
 $n \rightarrow n+1$

$$\left((a_1 \circ a_2) \circ (a_3 \circ (a_4 \circ a_5)) \right) \circ (a_6 \circ a_7)$$

↖ final multiplication

How many multiplication schemes we shall have?
 for n elements

Let g_n be the number.



Ans. $\underline{g_n = \frac{1}{n} \binom{2n-2}{n-1}}$

(There is a bijection between the triangulation of an $(n+1)$ -gon and schemes for n elements.)

(*) If we have n elements $a_1, a_2, \dots, a_n$, then the possible schemes including the order of elements is more.

$$(G_n = n! \cdot g_n)$$

Theorem The number of sequences $a_1, a_2, \dots, a_{2n}$ with $a_i \in \{-1, +1\}$

satisfying $\sum_{i=1}^k a_i \geq 0$ is equal to $\frac{1}{n+1} \binom{2n}{n}$.
for each $k=1, 2, \dots, 2n$

Proof.

We call $\langle a_1, a_2, \dots, a_{2n} \rangle$ acceptable if $\sum_{i=1}^k a_i \geq 0$ for all $k=1, 2, \dots, 2n$, and unacceptable otherwise.

A_n : # of acceptable sequences

E_n : # of unacceptable sequences

$$A_n + E_n = \binom{2n}{n} = \frac{2n!}{n!n!}.$$

(*) Find E_n first, and then we have A_n .

Let k_0 be the smallest integer such that

$$a_1 + a_2 + \dots + a_{k_0} < 0. \text{ Hence } a_1 + a_2 + \dots + a_{k_0-1} = 0$$

This implies that $a_{k_0} = -1$ and k_0 is odd.

Now, consider the sequence

$$\langle -a_1, -a_2, \dots, -a_{k_0}, a_{k_0+1}, \dots, a_{2n} \rangle.$$

Clearly, this sequence has $n+1$ 1's and $n-1$ (-1)'s.

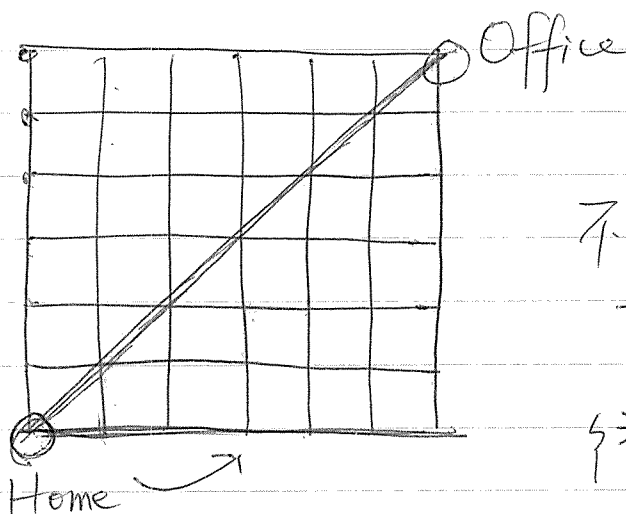
(*) 每一个 unacceptable sequence 都对应一个具有 $n+1$ 1's 及 $(n-1)$ (-1)'s 的排列。

Hence, $E_n = \frac{(2n)!}{(n+1)!(n-1)!}$

$$A_n = \frac{2n!}{n!n!} - \frac{(2n)!}{(n+1)!(n-1)!}$$

$$= \frac{2n!}{n!(n-1)!} \left(\frac{1}{n} - \frac{1}{n+1} \right)$$

$$= \frac{2n!}{n!n!} \cdot \frac{1}{n+1} = \frac{1}{n+1} \binom{2n}{n}.$$



不通过对角线的
下半部。

北(上), $\frac{1}{2} - 1$

(*) $a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}, a_{11}, a_{12}$
 $1, -1, 1, 1, -1, 1, -1, -1, 1, 1, -1, -1$

$$|\text{Acceptable sequences}| = \frac{1}{7} \binom{12}{6} = 132.$$

Problem

Can you find a bijection between the number

of acceptable sequences $\langle a_1, a_2, \dots, a_{2n} \rangle$ and
a triangulation of $(n+2)$ -gon?

Bonus Problem (前面 Lecture)

(想法由吴显良提供)

$$\sum_{k=0}^m \binom{m}{k} (-1)^k (m-k)^n = 0 \text{ 当 } n < m.$$

① 组合意义: 左式[↑]代表从 n -集映成至 m -集的不同函数数量.

由于 $m > n$, 不存在这样的映成函数, 等式成立.

② 组合式的证明

$$\begin{aligned} \text{Consider } f_0(y) &= y^m, \quad f_1(y) = (1+y) \cdot f_0'(y) = (1+y) \cdot m y^{m-1} \\ &= m y^{m-1} + m y^m \\ &= y^{m-1} \cdot (m + m y) \end{aligned}$$

$$\text{For } l > 1, \quad f_l(y) = (1+y) f_{l-1}'(y).$$

$$\begin{aligned} \text{观察 } f_2(y) &= (1+y) \cdot f_1'(y) = (1+y) \cdot [m(m-1) y^{m-2} + m^2 y^{m-1}] \\ &= y^{m-2} [(1+y) \cdot (m(m-1) + m^2 y)] \\ &\vdots \end{aligned}$$

$$f_l(y) = \underbrace{F_l(y)}_{\text{polynomial in } y} \cdot y^{m-l}.$$

$$\text{Now, if } n < m, \quad f_n(y) = F_n(y) \cdot y^{m-n}, \quad f_n(0) = 0.$$

$$\text{On the other hand, } f_0(y) = (-1) + (1+y)^m = \sum_{k=0}^m \binom{m}{k} (-1)^k (1+y)^k.$$

$$\begin{aligned} f_n(y) &= \sum_{k=0}^m \binom{m}{k} (-1)^{m-k} \cdot k^n (1+y)^k \\ &= \sum_{k=0}^m \binom{m}{k} (-1)^k (m-k)^n (1+y)^{m-k}, \quad f_n(0) = \sum_{k=0}^m \binom{m}{k} (-1)^k (m-k)^n \\ &\quad \text{(}(m-k) \text{与 } k \text{ 对换)} \quad \quad \quad \parallel \\ &\quad \quad \quad 0 \end{aligned}$$