

Lecture 2, Feb. 27 and March. 2nd

Date / /

No. 2-1

Selecting subsets from a given set (Multi-sets)

- Set: all elements are distinct
- Multi-set: Some elements may occur more than once

e.g. $\{1, 2, 3\}$ is a set and $\{1, 1, 2, 3, 3, 3\}$ is a multi-set

Problem In how many ways can we choose r -subsets
(r -multi-subsets)
from an n set?

$n = 4, r = 2$. (Subsets). $\binom{n}{r}$: n -choose- r

$S = \{a, b, c, d\}$

$\downarrow$ or C_r^n

ab, ac, ad, bc, bd, cd

$n = 4, r = 2$ (Multi-subsets)

$ab, ac, ad, bc, bd, cd, \underline{aa}, \underline{bb}, \underline{cc}, \underline{dd}$

Combinatorial Meaning

(*) $(1+a)(1+b)(1+c)(1+d)$ (Subsets)

$$= (1 + \underline{a+b+c+d}) + \underline{(ab+ac+ad+bc+bd+cd)} + \underline{(abc+abd+acd+bcd)} + \underline{abcd}.$$

$$\begin{aligned}
 (**) \quad & (1+a+a^2+a^3+a^4)(1+b+b^2+b^3+b^4)(1+c+c^2+c^3+c^4)(1+d+d^2+d^3+d^4) \\
 &= 1 + \underline{(a+b+c+d)} + \underline{(a^2+b^2+c^2+d^2+ab+ac+ad+bc+bd+cd)} + \\
 &\quad \dots + \underline{a^4b^4c^4d^4}
 \end{aligned}$$

(*) Continued ...

If we only consider the number of ways, then we can replace $(1+a)(1+b)(1+c)(1+d)$ by $(1+x)^4 = 1 + 4x + 6x^2 + 4x^3 + x^4$.

Similarly, (**) by $(1+x+x^2+x^3+x^4)^4$.

Clearly, we need a better idea to handle the cases when n and r are getting larger.

=

Combination

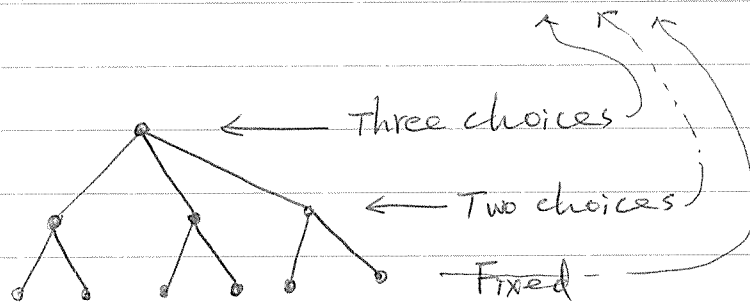
$$\binom{n}{r} = ?$$

First, consider $n, r \in \mathbb{N} \cup \{0\}$ and $r \leq n$.

Review that a permutation of a finite set S with n elements is a bijection from S onto S itself. Furthermore, there are $n!$ permutations in S_n . (Symmetric Group!)

e.g. $S = \{1, 2, 3\}$, Permutations are

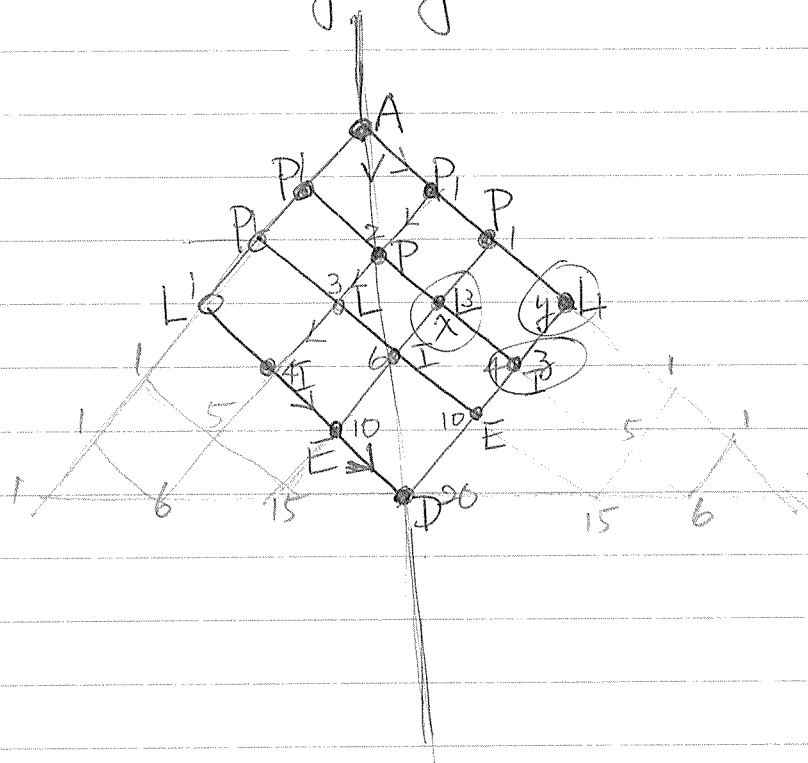
$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}.$$



Fact 1 $\binom{n}{r} = \frac{n(n-1)\dots(n-r+1)}{r!} = \frac{n!}{r!(n-r)!}$

(For completeness, we let $0! = 1$.)

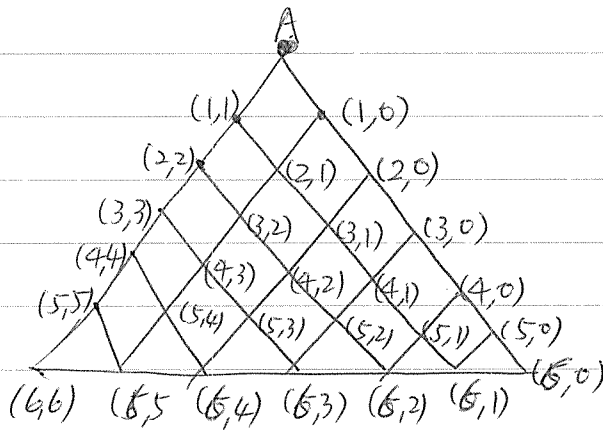
Fact 2 How many ways to read the word "Applied"?



There are 20 ways by using the fact $x+y=z$.

$$x = \binom{3}{1}, y = \binom{3}{0}, z = \binom{4}{1}$$

The number of "Left turn"



Pascal's Triangle

Fact 3 $n, r \in \mathbb{N}, n \geq r$

$$\binom{n}{r} = \binom{n}{n-r}$$

↑Left
 ↑right

Fact 4 $n, r \in \mathbb{N}, n \geq r$

$$\binom{n}{r} + \binom{n}{r-1} = \binom{n+1}{r}$$

(*) Generalized combination

Definition $\binom{x}{k} = \begin{cases} x^{\underline{k}} / k!, & \text{if } k \in \mathbb{N} \cup \{0\}; \text{ and} \\ 0, & \text{if } k < 0. \end{cases}$

For each $x \in \mathbb{R}, k \in \mathbb{Z}$,

Double Counting (Two-way counting)

(.) One of the most important tool in Combinatorics!

Examples

$$1. \sum_{k=0}^n \binom{n}{k} = 2^n.$$

$\binom{n}{k}$: the number of k -subsets

Each subset can be determined by the selection of elements

from ①, ②, ③, ..., ②; (with two possibilities for each element).

(We can also find this answer by using $(1+x)^n = \sum_{k=0}^n \binom{n}{k} x^k$,

$$2. \sum_{k=d}^n \binom{n}{k} \binom{k}{d} = 2^{n-d} \binom{n}{d} \quad (d=0 \Rightarrow \sum_{k=0}^n \binom{n}{k} = 2^n).$$

$$\parallel$$

$$\binom{n}{d} \left[\binom{n-d}{0} + \binom{n-d}{1} + \dots + \binom{n-d}{n-d} \right]$$

$$= \sum_{d'=0}^{n-d} \binom{n}{d} \cdot \binom{n-d}{d'}$$

The number of subsets containing at least

d elements.

$n = n_1 + n_2 + \dots + n_k$ is called a partition of n into k parts.

For example, $5 = 1+1+1+1+1 = 4+1 = 3+1+1 = 2+1+1+1$
 $= 3+2 = 2+2+1 = 5$

5	One part
4+1, 3+2	Two parts
3+1+1, 2+2+1	Three parts
2+1+1+1	Four parts
1+1+1+1+1	Five parts

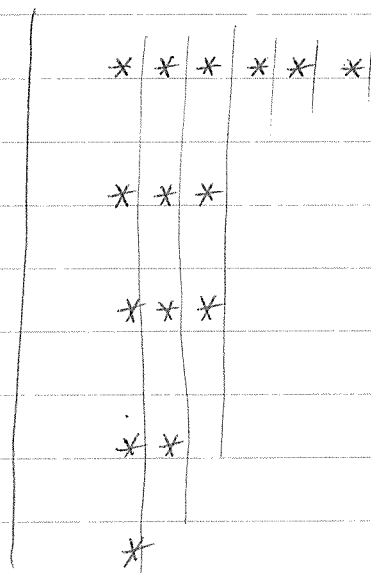
Finding the number of partitions is a very "interesting" and difficult problem.

(..) Prove that, the number of partitions of an integer n into m parts is equal to the number of partitions of an integer n into parts with maximum part m .

Example, $15 = 6+3+3+2+1 \leftrightarrow 5+4+3+1+1+1$

15

5 parts



Ferrers Diagram

$$3. \sum_{i=0}^n \binom{n}{i}^2 = \binom{2n}{n}.$$

Let S be a set of $2n$ balls, n blue and n red.

$$\sum_{i=0}^n \binom{n}{i}^2 = \sum_{i=0}^n \binom{n}{i} \binom{n}{n-i} \quad (i \text{ red and } n-i \text{ blue balls})$$

In total, it is the ways of selecting n balls of from S .

4. n blue and m red balls

$$\binom{n+m}{k} = \sum_{i=0}^k \binom{n}{i} \binom{m}{k-i} \quad (\text{Vandermonde's Convolution})$$

5. Multisets selection

Let S be a set of n distinct elements. Find the number of multi-subsets of size r , denoted by H_r^n .

Solutions

(1) By 1-1, onto (bijection) correspondence ↗ A type of two-way counting

e.g. $(1, 2, -, -, 3, 4, -, -, 5, 6, -, -)$



$\{3, 3, 5, 5, 5, 2, 2\}$ (Multiset of size 7).

In general, $(1, 1, 2, 1, \dots, (n-1), 1)$, there are r inserts. $H_r^n = \binom{n-1+r}{r}$. $H_7^{13} = \binom{13}{7}$

(2) By using power series (generating functions)

H_r^n = the coefficient of x^r in the expansion of

$$(1+x+x^2+\dots) \cdot (1+x+x^2+\dots) \cdot \dots \cdot (1+x+x^2+\dots)$$

~ n-terms ~

$$= (1+x+x^2+\dots)^n$$

$$= \left(\frac{1}{1-x} \right)^n = (1-x)^{-n} \equiv (1+(-x))^{-n}$$

$$= \sum_{k=0}^{\infty} \binom{-n}{k} (-1)^k x^k$$

By the generalization of
Binomial Theorem

$$H_r^n = \binom{-n}{r} (-1)^r = \frac{(-n)(-n-1) \dots (-n-r+1) \cdot (-1)^r}{r!}$$

$$= \frac{n \cdot (n+1) \cdot \dots \cdot (n+r-1)}{r!} = \binom{n+r-1}{r}$$

Another extension of combination (Binomial to)

(Multinomial coefficient) $|S| = n$

$\binom{n}{k}$ can be considered as selecting k elements from S and

put them in the first $\cup$, and then put the other $n-k$

elements in the other $\cup$. Now, if $n = n_1 + n_2 + \dots + n_k$,

we can put a n_1 -subset in $\textcircled{1}$, n_2 -subset in $\textcircled{2}$, $\dots$, n_k -subset

in $\textcircled{k}$, i.e., $\binom{n}{n_1} \cdot \binom{n-n_1}{n_2} \cdot \dots \cdot \binom{n-n_1-n_2-\dots-n_{k-1}}{n_k} = \frac{n!}{n_1! n_2! \dots n_k!}$

$\binom{n}{n_1, n_2, \dots, n_k}$ (Notation)

Example

$$(x+y+z)^3$$

$$= (x^2 + y^2 + z^2 + 2xy + 2yz + 2xz)(x+y+z)$$

$$= (x^3 + \cancel{x^2y} + \cancel{x^2z} + \cancel{2x^2y} + 2xy^2 + \cancel{2x^2z} + \cancel{x^2y} + y^3 + \cancel{yz^2} + 2xy^2 + \cancel{2y^2z} + \cancel{2xy^2} + \cancel{x^2z} + \cancel{yz^2} + z^3 + \cancel{2xyz} + \cancel{2yz^2} + \cancel{2xz^2})$$

$$= x^3 + y^3 + z^3 + 3xy^2 + 3xz^2 + 3x^2y + 3x^2z + 3yz^2 + 3y^2z + 6xyz$$

$$= \binom{3}{3,0,0} x^3 + \binom{3}{0,3,0} y^3 + \binom{3}{0,0,3} z^3 + \binom{3}{1,2,0} xy^2 + \binom{3}{1,0,2} xz^2 + \binom{3}{2,1,0} x^2y + \binom{3}{2,0,1} x^2z + \binom{3}{0,1,2} yz^2 + \binom{3}{0,2,1} y^2z + \binom{3}{1,1,1} xyz.$$

$$\binom{3}{3,0,0} = \binom{3}{0,3,0} = \binom{3}{0,0,3} = \frac{3!}{3!} = 1$$

$$\binom{3}{1,2,0} = \binom{3}{1,0,2} = \binom{3}{2,1,0} = \binom{3}{2,0,1} = \binom{3}{0,1,2} = \binom{3}{0,2,1} = \frac{3!}{2!} = 3$$

$$\binom{3}{1,1,1} = \frac{3!}{1!1!1!} = 6.$$

Conclusion

$$(x_1 + x_2 + \dots + x_k)^n = \sum_{n_i \geq 0, \sum n_i = n} \binom{n}{n_1, n_2, \dots, n_k} x_1^{n_1} x_2^{n_2} \dots x_k^{n_k}.$$

(Multinomial Expansion!)

Find the coefficient of $x^2 y^2 z w^2$ in the expansion of $(x+y+z+w)^7$.

Ans. $\binom{7}{2,2,1,2} = \frac{7!}{2!2!2!} = 7 \times 6 \times 5 \times 3.$

More examples

1. $\sum_{k=0}^n (-1)^k \binom{n}{k} = 0$ if $n > 0$ and 1 otherwise.

2. $\sum_{k=1}^n k (-1)^k \binom{n}{k} = \begin{cases} 0, & n > 1 \\ -1, & n = 1. \end{cases}$

Sol. $(1+x)^n = \sum_{k=0}^n \binom{n}{k} x^k$

两边微分, $n(1+x)^{n-1}$

$$\rightarrow = \binom{n}{1} + 2\binom{n}{2}x + 3\binom{n}{3}x^2 + \dots + n\binom{n}{n}x^{n-1}$$

代入 $x = -1$ ($n > 1$), $\sum_{k=1}^n k \binom{n}{k} x^{k-1} \Rightarrow \sum_{k=1}^n k \binom{n}{k} (-1)^{k-1} = 0$

$$\Rightarrow \sum_{k=1}^n k (-1)^k \binom{n}{k} = 0 \quad (\text{乘上 } (-1))$$

$n = 1$, 只有一项 $k = 1$.

3. $\sum_{k=r}^n (-1)^k \binom{k}{r} \binom{n}{k} = \begin{cases} 0, & n > r; \\ -1, & n = r. \end{cases}$

Beautiful result in Number Theory (Application of $\binom{n}{k}$)

(•) The product of any k consecutive integers is a multiple of $k!$

Example 1. $k=3$

One of them is a multiple of 3. If this number is in the middle, then either one of the other two integers is even or this number (both)

itself is even. In either case; the product is a multiple of $3! = 6$.

(If this number is not in the middle, then one of the other two integers must even, the proof follows.)

Proof of (•)

Let these k consecutive integers be $n, n-1, \dots, n-k+1$.

Case 1. $n > k$,

Since $\frac{n \cdot (n-1) \cdot \dots \cdot (n-k+1)}{k!} = \binom{n}{k}$, the proof follows.

Case 2. $n < 0$. ($-n > 0$)

Now, $n \cdot (n-1) \cdot \dots \cdot (n-k+1) = (-1)^k [(-n) \cdot (-n+1) \cdot \dots \cdot (-n+k-1)]$.

Hence $\frac{n \cdot (n-1) \cdot \dots \cdot (n-k+1)}{k!} = \frac{(-1)^k [(-n+k-1) \cdot (-n+k-2) \cdot \dots \cdot (-n)]}{k!}$
 $= (-1)^k \cdot \binom{-n+k-1}{k}$ an integer, the proof follows.

Case 3. $n \leq k$, but $n \geq 0$. ($0 \leq n \leq k$)

The product is equal to zero. This concludes the proof.