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(*) Constructing the quasigroups with conditions (1)~(6) (not all) is not very difficult. But, for (7) and (8), extra effort is needed.

Of course, if $\langle Q, \circ \rangle$ is itself a group, then there is nothing to do, both (7) and (8) are good. The problem is to find a quasigroup in which "Associative Law" does not hold.

Proposition 1. For each $n \in \mathbb{N} \setminus \{2\}$, there exists an idempotent Latin square of order n .

Proposition 2 For each $n \in \mathbb{N}$, there exists a unipotent Latin square of order n .

Proposition 3. For each $n \in \mathbb{N}$, there exists a commutative Latin square of order n . (Use $\langle \mathbb{Z}_n, + \rangle$.)

Proposition 4. For each odd $n \in \mathbb{N}$, there exists a commutative idempotent Latin square of order n .

Note. Can be obtained from the edge-coloring of K_n which uses n (or total-coloring of K_n) colors.

Proposition 5. For each even $n \in \mathbb{N}$, there exists a commutative unipotent Latin square of order n .

Note. We can use $\chi(K_n) = n-1$.

Definition (Direct product)

Let A and B be two Latin squares based on $\mathbb{Z}_m$ and $\mathbb{Z}_n$ respectively. Then, the direct product of A and B , denoted by $A \otimes B$ is a Latin square of order mn based on $\mathbb{Z}_m \times \mathbb{Z}_n$ such that the entry $A_{ij} = x$ is replaced by (x, B) where (x, B) is a Latin square of order n where the (i', j') entry is filled by $(x, B_{i', j'})$.

e.g.

$$A = \begin{array}{|c|c|} \hline 0 & 1 \\ \hline 1 & 0 \\ \hline \end{array}, \quad B = \begin{array}{|c|c|c|} \hline 0 & 1 & 2 \\ \hline 2 & 0 & 1 \\ \hline 1 & 2 & 0 \\ \hline \end{array}$$

$A \otimes B :$

(0,0)	(0,1)	(0,2)	(1,0)	(1,1)	(1,2)
(0,2)	(0,0)	(0,1)	(1,2)	(1,0)	(1,1)
(0,1)	(0,2)	(0,0)	(1,1)	(1,2)	(1,0)
(1,0)	(1,1)	(1,2)	(0,0)	(0,1)	(0,2)
(1,2)	(1,0)	(1,1)	(0,2)	(0,0)	(0,1)
(1,1)	(1,2)	(1,0)	(0,1)	(0,2)	(0,0)

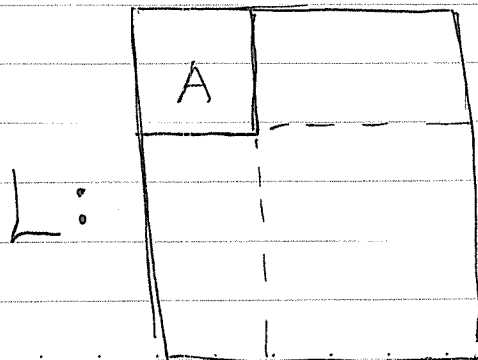
Fact 1 If $L = A \otimes B$, then there are m^2 sub-Latin squares in L each of them is of order n provided A is a Latin square of order m and B is a Latin square of order n .

Definition (sub-Latin square)

If $Q' \subseteq Q$, $\langle Q', \circ \rangle$ and $\langle Q, \circ \rangle$ are quasigroups, then $\langle Q', \circ \rangle$ is called a sub-quasigroup of $\langle Q, \circ \rangle$. Their corresponding Latin squares are "Latin square and Latin subsquare respectively.

Definition (Embedding)

If A is a sub-Latin square (or Latin subsquare) of L , then A is said to be embedded in L . The standard form is the one with A in the upper left hand corner.



Theorem 6 A Latin subsquare of order m can be embedded in a Latin square of order n if and only if $n \geq 2m$.

Fact 11 If L (of order n) has a Latin subsquare A (of order m), then n may not be a multiple of m . (It is true $m|n$ if both L (and A) are corresponding to a group respectively

Proof of Theorem 6 $\leftarrow$ Exercise 1.3. (10 points).

In what follows, we provide some more insight about having subsquare.

Proposition 6 If A is embedded in L and $L(i)$ denotes the element i occurs in L (respectively A, B, C, D), then $A(i) \geq 2m - n$ where A is a Latin square of order m and L is a Latin square of order n .

$\forall i \in \mathbb{Z}_n$,

Proof. $\wedge$ Since $B(i) + D(i) = n - m$, $B(i) \leq n - m$.

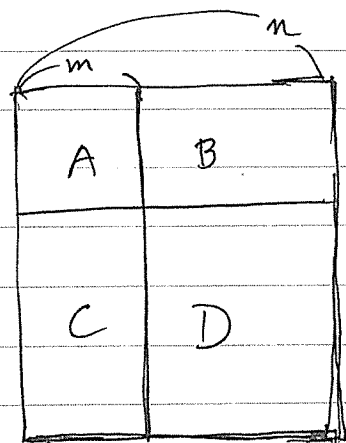
$$A(i) + B(i) = m, \text{ Hence } A(i) = m - B(i)$$

$$\geq m - (n - m) = 2m - n,$$

of Theorem 6

Corollary 7. $(\Rightarrow)$ $\wedge$ is true.

L :

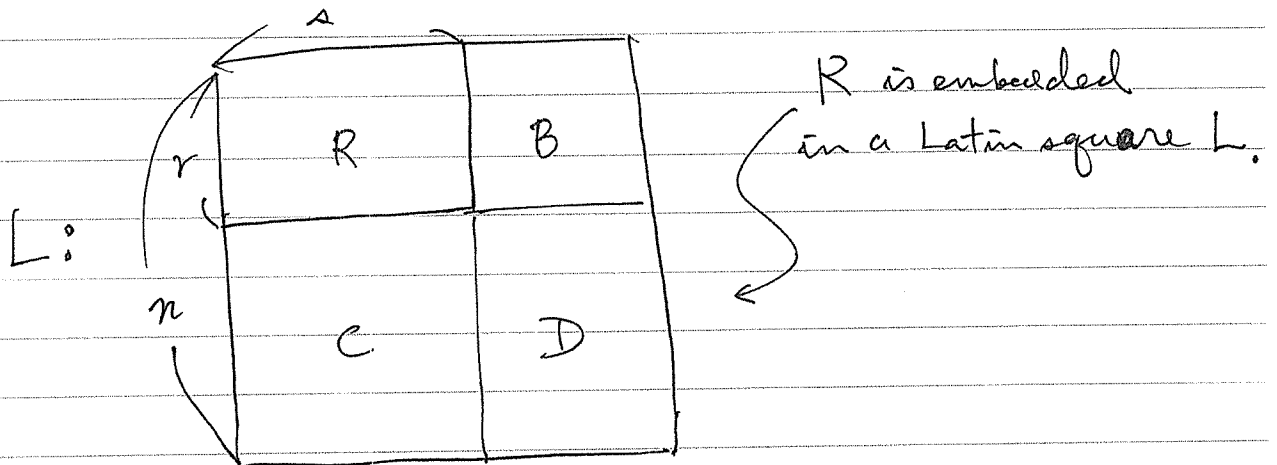


Proof. If $n < 2m$, then every $i \in \mathbb{Z}_n$ has to occur in A which is not possible. ■

In fact, the subsquare A we consider here can be replaced by Latin rectangle or partial Latin rectangle.

Definition (Latin rectangle, partial Latin rectangle)

A Latin $r \times s$ rectangle R based on S is an $r \times s$ array such that each entry of R is filled with an element of S with an extra property: each element of S occurs in each row and each column at most once. If not every cell is filled with an entry and the property holds, then we have a partial Latin rectangle.



✓ If R is embedded in a Latin square L , then (based on S)

Proposition 7. $\forall i \in S, R(i) \geq r + s - n.$

Proof. $R(i) + B(i) = r, B(i) + D(i) = n - s, B(i) \leq n - s \Rightarrow R(i) = r - B(i) \geq r - (n - s).$ SEAS

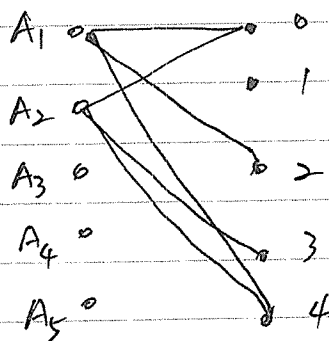
Proposition 8 Let R be an $r \times n$ Latin rectangle based on an n -set S . Then R can be extended to a Latin square of order n .

Proof. Use SDR or König's Theorem. ■

0	1	2	3	4
3	2	4	1	0

A_1, A_2, A_3, A_4, A_5

$$A_1 = \{0, 2, 4\}, A_2 = \{0, 3, 4\}, A_3 = \{0, 1, 3\}, A_4 = \{0, 2, 4\}, A_5 = \{1, 2, 3\}$$



← Regular Bipartite Graph

Fact 12 Let R be an $r \times n$ Latin rectangle. Then R can be extended in a Latin square of order n if and only if $R(i) \geq r + s - n$ for all $i \in S$ (based on S).

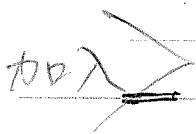
$$R(i) \geq r + s - n, \forall i \in S \quad (|S| = n).$$

(Outline)

Proof. Step 1. Fill all the entries in R , such that the condition $R(i) \geq r + s - n$ holds.

Step 2. Fill in the entries in B. (Obtain an $n \times n$ Latin rectangle.)

Step 3. Complete the Latin square by extending the rectangle.



$S-1, S-2, S-3$

Definition (Partial Latin Square of order n) $PLS(n)$

A $PLS(n)$ is an $n \times n$ array such that each cell is either filled with an entry from an n -set S or empty, moreover, each element in S occurs at most once in each row and resp. once in each column.

Definition (Complete the $PLS(n)$)

Let L' be a $PLS(n)$. L' is said to be completable if we can fill all the empty cells such that the $n \times n$ array is a Latin square.

0	1	
		2

incompletable

0	1	
1		

completable.

Theorem (Hoffman and H.W. Kuhn (1956)).

Let M be a given set. A necessary and sufficient condition for the sets $S_1, S_2, \dots, S_n$ to have an SDR which includes all the elements of M is that, for every $M' \subseteq M$, at least $|M'|$ of the sets $S_1, S_2, \dots, S_n$ have non-empty intersection with M' .

(Note that $\{S_1, S_2, \dots, S_n\}$ has an SDR itself.)

Theorem (P. Hall, 1935)

$\{S_1, S_2, \dots, S_n\}$ has an SDR if and only if the union of any k of them contains at least k elements.

($\Rightarrow$) Trivial.

Proof. ($\Leftarrow$) By induction on n and it's clearly true for $n=1$.

Assume that the assertion is true for all $1 \leq m \leq n$ and consider

$\{S_1, S_2, \dots, S_n\}$.

Case 1 $\forall k \leq n-1, |\bigcup_{j=1}^k S_{i_j}| \geq k+1$

Let $x_n \in S_n$ and consider $\{S_1 \setminus \{x\}, S_2 \setminus \{x\}, \dots, S_{n-1} \setminus \{x\}\}$. By induction

... together with x we have an SDR

Case 2 $\exists h \leq n-1, |\bigcup_{j=1}^h S_{i_j}| = h.$

Let $\bigcup_{j=1}^h S_{i_j} = \tilde{S}$. For convenience, let these h subsets be $S_1, S_2, \dots, S_h$.

Now, consider $S_{h+1} \setminus \tilde{S}, S_{h+2} \setminus \tilde{S}, \dots, S_n \setminus \tilde{S}$. ^{The union of} any k of these sets must

contain at least k elements. For otherwise, the union of $\tilde{S}$ with

these sets will contain less than $h+k$ elements, a contradiction

to the assumption. Hence, $\{S_{h+1} \setminus \tilde{S}, \dots, S_n \setminus \tilde{S}\}$ has an SDR. Also,

$\{S_1, S_2, \dots, S_h\}$ has an SDR. Together, we have an SDR for $\{S_1, S_2, \dots, S_n\}$.

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Bonus problem (Exercise) 20 points

Prove the Theorem by Hoffman and Kuhn and then the details of H.J. Ryser's Theorem.

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We can use this theorem to prove the embedding theorem proved by Ryser in 1951.

Theorem (H. J. Ryser (1951))

Let R be an $r \times s$ latin rectangle based on $S = \{1, 2, \dots, n\}$ (filled partial L.S.). Then, R can be embedded in a L.S. of order n if and only if $R(i) \geq r + s - n$ for all $i \in S$.

Proof. The proof of necessity has been done earlier, we prove the sufficiency in what follows. We claim that R can be enlarged to an $r \times (s+1)$ rectangle R^* such that $R^*(i) \geq r + (s+1) - n$ for all i . Iteration then extends R to an $r \times n$ Latin rectangle Q based on S . Hence, R can be embedded in a Latin square of order n by applying M. Hall's extension theorem.

Let S_j denote the set of elements in S which do not occur in the j th row of R . Now, let M be the set of elements in R occurred exactly $r + s - n$ times. ($|M| \leq r$?) The proof follows by showing $\{S_1, S_2, \dots, S_r\}$ has an SDR such that the set of elements

(*) It is interesting to know whether a $PLS(n)$ can be completed to a Latin square.

Fact 13 A $PLS(n)$ with ^{at most} $n-1$ filled cells can be completed to a Latin square of order n . (Evans' Conjecture)

(In fact, the proof of this fact is quite difficult.)

Proved by B. Smetaniuk (1981). You may refer to "A course in combinatorics" by J.H. van Lint and R.M. Wilson, page 189-193.

Fact 14 It takes about 50 pages to characterize a $PLS(n)$ with at most $n+1$ filled cells which is completable.

(L.D. Anderson and A.J.W. Hilton, 1983, LMS.)