

Lecture 15

NO.

1

DATE

$2-(v, 3, \lambda)$ design

- If $\lambda = 1$, then it is the Steiner triple system of order v .
- How about $\lambda \geq 2$?
- For each $\lambda \geq 2$, and $v \equiv 1$ or $3 \pmod{6}$, a $2-(v, 3, \lambda)$ design exists! If no blocks (triples) are repeated, then the design is called a "simple" design. Clearly, this is not possible, if λ is too large. In fact $\lambda \leq v-2$ if we would like to obtain a $2-(v, 3, \lambda)$ simple design. (?)
- If $\lambda = 2$, then for each $v \equiv 0$ or $1 \pmod{3}$, there exists a $2-(v, 3, 2)$ design.

Proposition $2 \cdot K_v$ is 3-sufficient if and only if $v \equiv 0$ or $1 \pmod{3}$.

Theorem A $2-(v, 3, 2)$ design exists iff $2 \cdot K_v$ is 3-sufficient.
($v \equiv 0$ or $1 \pmod{3}$)

Proof. Direct construction.

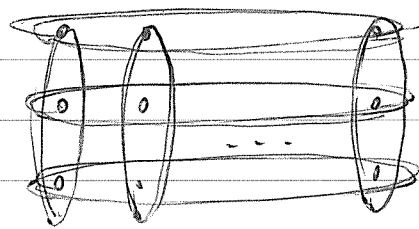
$$v=3, \quad X = \mathbb{Z}_3, \quad B = \{\{0, 1, 2\}, \{0, 1, 2\}\}.$$

$$v=4, \quad X = \mathbb{Z}_4, \quad B = \{\{0, 1, 2\}, \{0, 1, 3\}, \{0, 2, 3\}, \{1, 2, 3\}\}.$$

$$v=6, \quad X = \mathbb{Z}_6, \quad B = \left\{ \begin{array}{cccccc} \underline{013} & \underline{124} & \underline{335} & \underline{340} & \underline{451} & \underline{502} \\ \underline{012} & \underline{234} & \underline{450} & \underline{135} & & \end{array} \right\}$$

$$v=7 \rightarrow 2\text{STS}(7).$$

$$v = 3k, k \geq 3$$

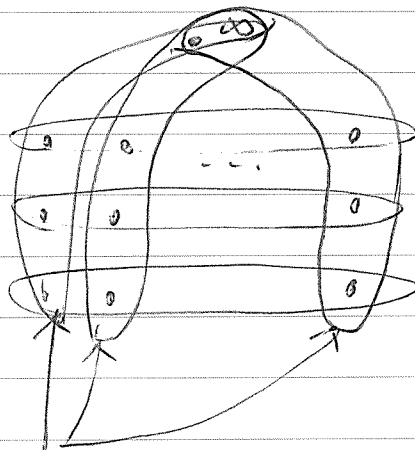


Idempotent $LS(k)$.

$X = \mathbb{Z}_3 \times \mathbb{Z}_k$, $L^{(1)}, L^{(2)}, L^{(3)}$ are idempotent $LS(k)$.
(may be the same)

$$\begin{aligned} & 2 \cdot \{(0, i), (1, i), (2, i)\} + \left\{ \{(0, i), (0, j), (1, L^{(1)}(i, j))\} \mid i \neq j \in \mathbb{Z}_k \right\} \\ & + \left\{ \{(1, i), (1, j), (2, L^{(1)}(i, j))\} \mid i \neq j \in \mathbb{Z}_k \right\} \\ & + \left\{ \{(2, i), (2, j), (0, L^{(2)}(i, j))\} \mid i \neq j \in \mathbb{Z}_k \right\}. \end{aligned}$$

$$v = 3k + 1$$



$2-(4,3,2) \leftarrow \text{Type 1}$

$$X = \{\infty\} \cup (\mathbb{Z}_3 \times \mathbb{Z}_k),$$

Type 2 is similar to that of construction "3k".

Can we use cyclic or recursive constructions?

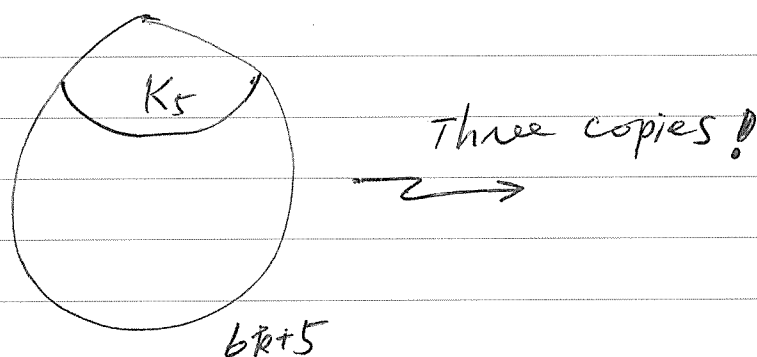
$$\lambda = 3$$

Theorem A $2-(v, 3, 3)$ design exists if and only if $v \equiv 1 \pmod{2}$.

Proof. It suffices to consider the case $v \equiv 5 \pmod{6}$. Moreover,

the proof follows by showing $K_3 \mid 3 \cdot K_5$. (?)

Let $V(K_5) = \mathbb{Z}_5$. Then, the base blocks are $\{0, 1, 2\}$ and $\{0, 2, 4\}$. That is, $\mathcal{B} = \left\{ \begin{array}{l} 012, 123, 234, 340, 401 \\ 024, 130, 241, 302, 413 \end{array} \right\}$. ■



$$\lambda = 4 \sim \lambda = 2 \parallel \lambda = 5 \sim \lambda = 1 \parallel \lambda = 6$$

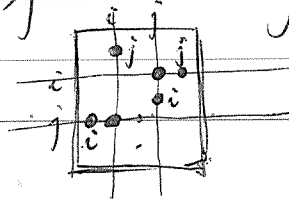
Theorem A $2-(v, 3, 6)$ design exists if and only if $v \neq 2$.

Proof. Let $v = \binom{k}{2}$ and L be an idempotent $LS(k)$. Then, $X = \mathbb{Z}_k$

for each $i \neq j$, let $\mathcal{B} = \{ \{i, j, L(i, j)\} \mid i \neq j, i, j \in \mathbb{Z}_k \}$. Note

that for each pair $i \neq j \in \mathbb{Z}_k$, i, j occur together in exactly

6 blocks (triples), see it?



$v \setminus \lambda$	1	2	3	4	5	0 (mod 6)
1	✓	✓	✓	✓	✓	✓
2	✗	✗	✗	✗	✗	✓ ← except 2
3	✓	✓	✓	✓	✓	✓
4	✗	✓	✗	✓	✗	✓
5	✗	✗	✓	✗	✗	✓
0	✗	✓	✗	✓	✗	✓

(mod 6)

Ex. Give a detail proof of the existence of a $2-(v, 3, \lambda)$ design.
(20 points)

(Bonus) Find a simple $2-(7, 3, 4)$ design. (10 points)

$2-(v, 4, 1)$ design Also known as $(4, 2)$ -design

- A $2-(v, 4, 1)$ design exists if and only if $v \equiv 1 \text{ or } 4 \pmod{12}$

($\Rightarrow$) The two trivial necessary conditions are

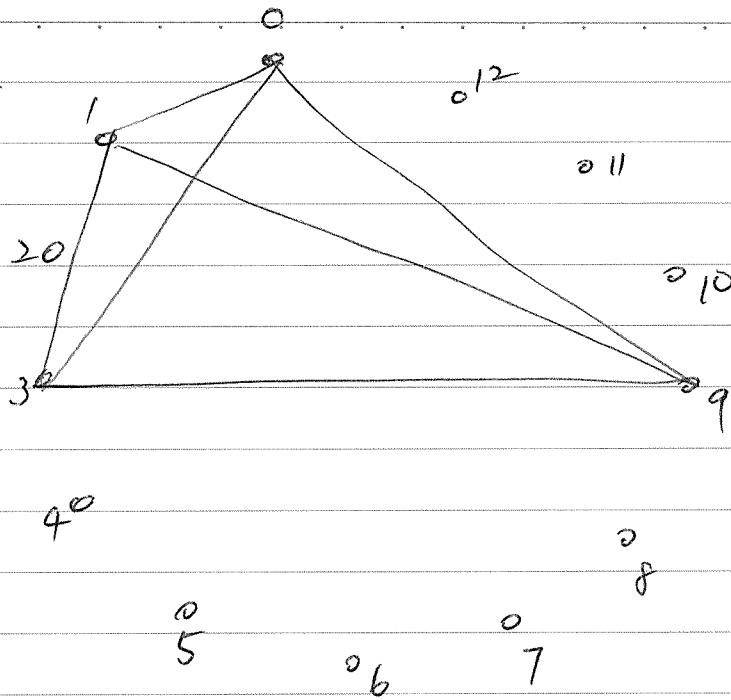
(1) $3 \mid v-1$ and (2) $6 \mid \binom{v}{2}$.

Combining these two conditions, we have $v \equiv 1 \text{ or } 4 \pmod{12}$.

($\Leftarrow$) $v=4$ is the smallest non-trivial case.

$v=13$ can be obtained by projective plane of order 3.

Or
by cyclic method
 $\{0, 1, 3, 9\}$
(base block)

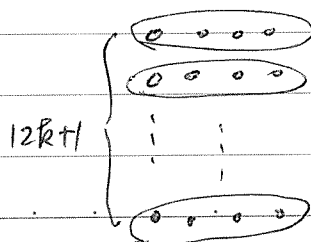


How about general construction?

By the way, if $v=16$, then a $2-(16, 4, 1)$ design exists by the fact of the existence of an Affine plane of order 4.

Proposition If a $2-(v, 4, 1)$ design exists, then a $2-(4v, 4, 1)$ design exists.

Proof. We can use the existence of $MOLS(4)$ and blow up each vertex into 4 vertices.

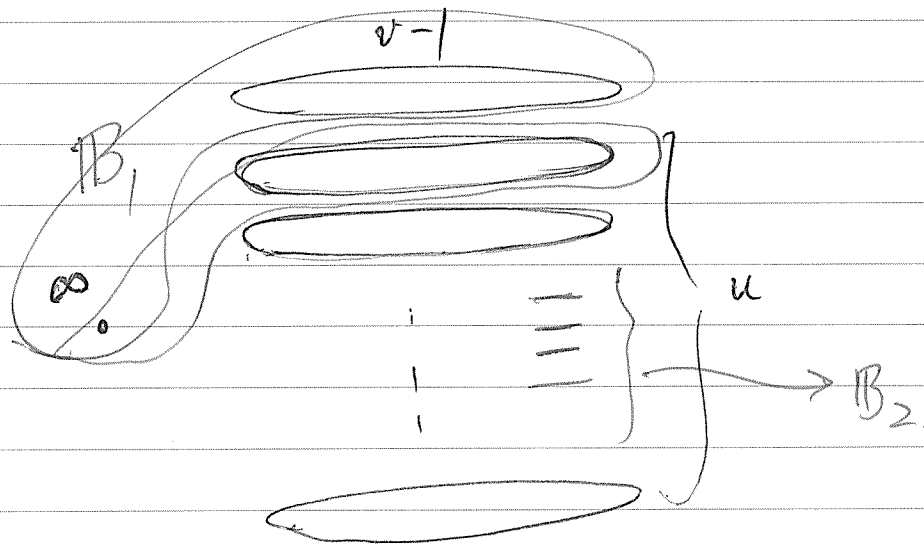


$2-(v, 4, 1)$ design

Corollary If $v \equiv 4$ or $16 \pmod{48}$, then a $2-(v, 4, 1)$ design exists.

Proposition If $2-(u, 4, 1)$ design and a $2-(v, 4, 1)$ design exist, then a $2-(1+u(v-1), 4, 1)$ and a $2-(1+v(u-1), 4, 1)$ design exist.

Proof.



Note that $v-1 \neq 6$.

By recursive construction, we have $v \equiv 1$ or $13 \pmod{48}$.
(Use $u=4$ or $v=4$!)

$v=4 \rightarrow 3u+1 \rightarrow 36k+4, 36k+13.$
 $u=4 \rightarrow 3v+1$

Details : See "Balanced Incomplete Block Designs and

Related Designs by Haim Hanani, Discrete Math., 11 (1975),

255-369.

Packing and Covering K_n with K_3 's

Problem How many edge-disjoint triangles we can find in a graph?

- "Zero" if G is bipartite.
- "Many" if $\delta(G) \geq \frac{3}{4}n$! (Nash-Williams' conjecture).
- Packing K_n with K_3 's

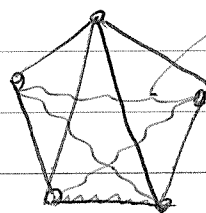
Fact 1. If $n \equiv 1$ or $3 \pmod{6}$, then we have $\frac{\binom{n}{2}}{\binom{3}{2}} K_3$'s.

Definition (Maximum packing)

The maximum packing of a graph G by using H is the maximum number of edge-disjoint subgraphs we can obtain in a decomposition of G (with H).

If $\hat{H}$ is a packing of G , then the subgraph L induced by the edges not in $\hat{H}$ (the union of H 's) is called the

leave of the packing. If $\hat{H}$ is a maximum packing, then L is a minimum leave.
 e.g.



Leave of packing (with K_3 's)

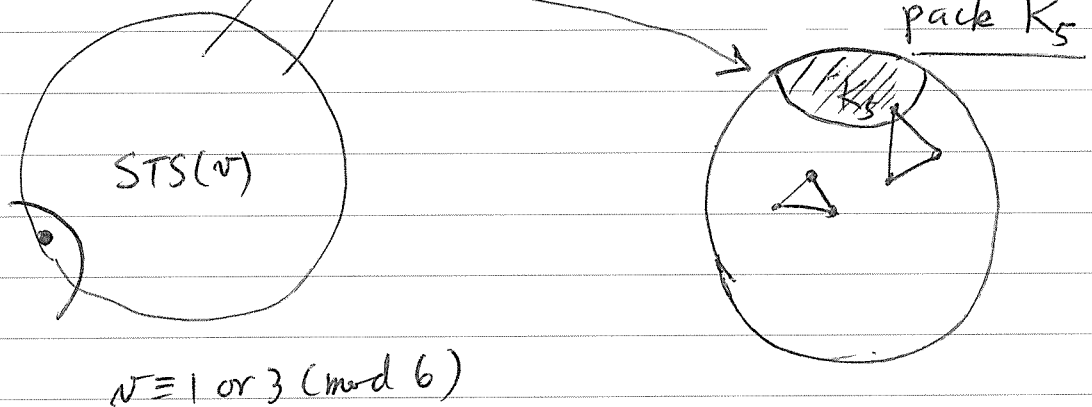
C_4 : Minimum Leave.

Theorem (Packing)

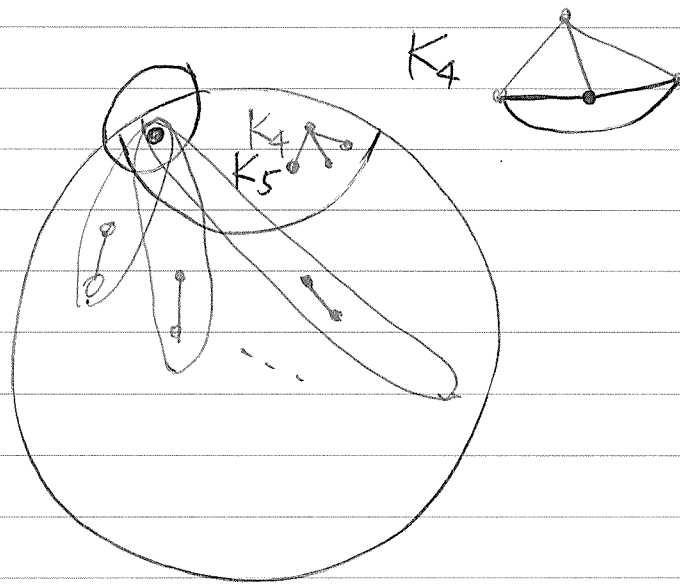
f)	n	1	2	3	4	5	0	(mod 6)
minimum leave	L	$\emptyset$	  ⋮ 	$\emptyset$	  ⋮ 	C_4	  ⋮ 	
			↑		↑		↑	
			1-factor		tripole		1-factor	

- It is not difficult to see, if there is a leave, then the above table shows the minimum leaves for respective cases.

- The proof of $n \equiv 2, 0, 5$ are not difficult to check.



$$n \equiv 4 \pmod{6}$$



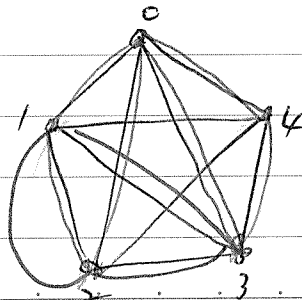
(*) Delete a vertex from K_5 to obtain the desired leave.

Definition (Covering)

A covering of a graph G by a graph H is a collection of subgraphs $H_1, H_2, \dots, H_t$, such that for each i , $H_i \cong H$ and for each $e \in E(G)$, $e \in E(H_j)$ for some j .

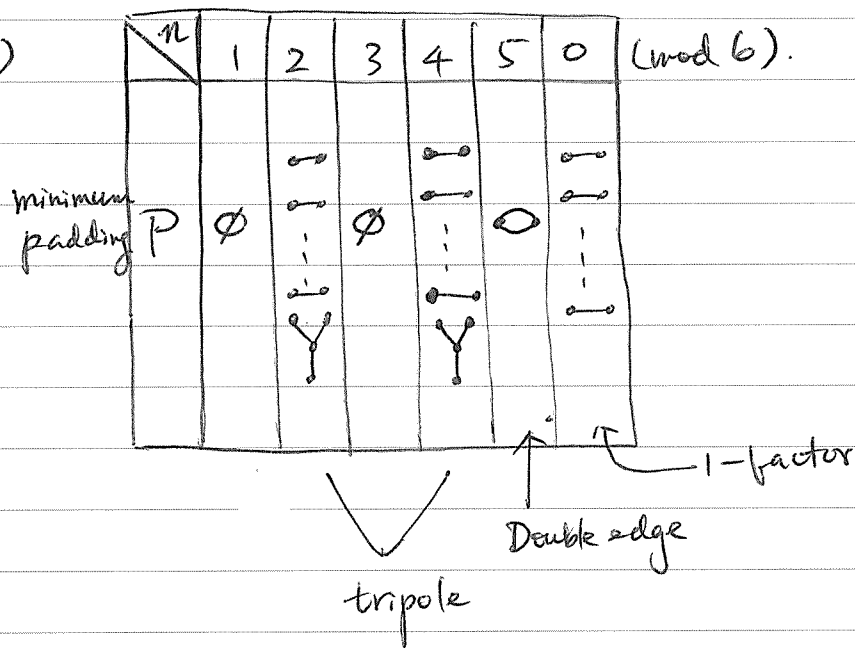
The edges added to G is known as the padding of the covering. If the covering is minimum, then we also have a minimum padding.

e.g.



- 0 1 2
- 0 3 4
- 1 2 4
- 1 3

Theorem
(Covering)



Proof. The proof for minimum covering is slightly more complicated.

The cases (when n is odd) are easy to see. Now, we consider even n and prove the case n is congruent to 2 modulo 6.

Let $n = 6k + 2$. Clearly, each vertex of K_n is of degree $6k + 1$ which is odd. Hence, if P is a padding, then P must be a spanning odd subgraph of K_n . This implies that P is a matching or a tripole if minimum covering is concerned (?). In order that $K_n \cup P$ is of size a multiple of 3, P has to be a tripole. Now, we are ready for the covering.

First, we notice that $K_n = K_{n-1} \cup S_{n-1}$ (a star with $n-1$ edges).

Since $n-1 \equiv 1 \pmod{6}$, $K_3 \mid K_{n-1}$. On the other hand, S_{n-1} UP (as in Figure 1) can be decomposed into K_3 's. This concludes the proof. ▀

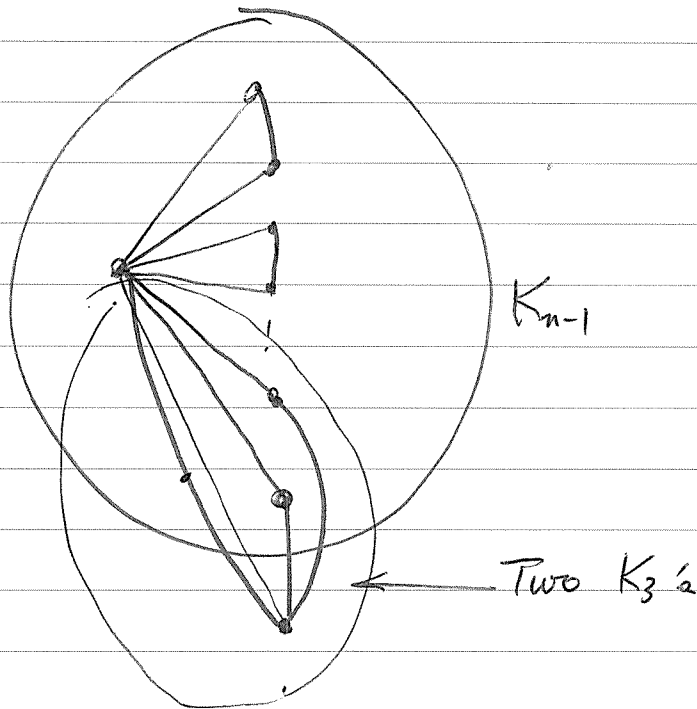


Figure 1. Padding of K_n

(*) Packing is more useful in applications. In experiments, we normally consider "samples" not the whole set of varieties or blocks.