Lecture 2, Feb. 21, 23

Special Latin Squares

Special type of Latin squares can be obtained by direct construction or quasigroups with extra conditions. We start with special quasigroups. Since there is a long list of special quasigroups, we only mention some of them in what follows: let $\langle Q, \circ \rangle$ be a quasigroup.

(1) Idempotent : $\forall\ a \in Q,\ a \circ a = a^2 = a.$

(2) Unipotent : $\forall\ a, b \in Q,\ a^2 = b^2.$

(3) Commutative : $\forall\ a, b \in Q,\ a \circ b = b \circ a.$

(4) Semi-symmetry : $\begin{cases} a \circ (b \circ a) = b & (\text{Right}) \\ (a \circ b) \circ a = b & (\text{Left}) \end{cases}$, $\forall\ a, b \in Q$

(5) Total-symmetry : $\forall\ a, b \in Q$ $\quad a \circ (a \circ b) = b$ & $(a \circ b) \circ b = a.$

$(a \circ (a \circ b)) \circ (a \circ b) = b \circ (a \circ b) = a$

Remark :

Total-symmetry $\rightarrow$ Semi-symmetry

Total-symmetry $\xrightarrow{(?)}$ Commutative.

$b \circ a$
$= b \circ (b \circ (a \circ b))$
$= a \circ b.$

(6) Associative Law : $\forall\ a, b, c \in Q,\ a \circ (b \circ c) = (a \circ b) \circ c.$

(7) Moufang Identity : $(a \circ b) \circ (c \circ a) = [a \circ (b \circ c)] \circ a.$

(8) Neumann's Law : $(a \circ b) \circ (c \circ a) = (a \circ c) \circ (b \circ a).$

($\vdots$ Many others)

Proposition 5. For each even $n \in \mathbb{N}$, there exists a commutative unipotent Latin square of order n.

Note. We can use $\chi'(K_n) = n-1$.

Definition (Direct product)

Let A and B be two Latin squares based on $\mathbb{Z}_m$ and $\mathbb{Z}_n$ respectively. Then, the direct product of A and B, denoted by $A \otimes B$ is a Latin square of order mn based on $\mathbb{Z}_m \times \mathbb{Z}_n$ such that the entry $A_{ij} = x$ is replaced by (x, B) where (x, B) is a Latin square of order n where the (i', j') entry is filled by $(x, B_{i'j'})$.

e.g.

$$A: \begin{array}{|c|c|}\hline 0 & 1 \\ \hline 1 & 0 \\ \hline\end{array}, \quad B: \begin{array}{|c|c|c|}\hline 0 & 1 & 2 \\ \hline 2 & 0 & 1 \\ \hline 1 & 2 & 0 \\ \hline\end{array}, \quad A \otimes B: \begin{array}{|c|c|c|c|c|c|}\hline (0,0) & (0,1) & (0,2) & (1,0) & (1,1) & (1,2) \\ \hline (0,2) & (0,0) & (0,1) & (1,2) & (1,0) & (1,1) \\ \hline (0,1) & (0,2) & (0,0) & (1,1) & (1,2) & (1,0) \\ \hline (1,0) & (1,1) & (1,2) & (0,0) & (0,1) & (0,2) \\ \hline (1,2) & (1,0) & (1,1) & (0,2) & (0,0) & (0,1) \\ \hline (1,1) & (1,2) & (1,0) & (0,1) & (0,2) & (0,0) \\ \hline\end{array}$$

(*) Constructing the quasigroups with conditions (1)~(6) (not all) is not very difficult. But, for (7) and (8), extra effort is needed. Of course, if $\langle Q, \circ \rangle$ is itself a group, then there is nothing to do, both (7) and (8) are good. The problem is to find a quasigroup in which "Associative Law" does not hold.

<u>Proposition 1.</u> For each $n \in \mathbb{N} \setminus \{2\}$, there exists an idempotent Latin square of order n.

<u>Proposition 2</u> For each $n \in \mathbb{N}$, there exists a unipotent Latin square of order n.

<u>Proposition 3.</u> For each $n \in \mathbb{N}$, there exists a commutative Latin square of order n. (Use $\langle \mathbb{Z}_n, + \rangle$.)

<u>Proposition 4.</u> For each odd $n \in \mathbb{N}$, there exists a commutative idempotent Latin square of order n.

<u>Note</u>. Can be obtained from the <u>edge-coloring of K_n</u> (or total-coloring of K_n) which uses n colors.

<u>Fact 1</u> If $L = A \otimes B$, then there are m^2 sub-Latin squares in L each of them is of order n provided A is a Latin square of order m and B is a Latin square of order n.

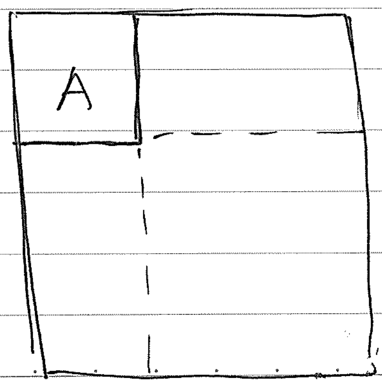
<u>Definition</u> (sub-Latin square)

If $Q' \subseteq Q$, $\langle Q', \circ \rangle$ and $\langle Q, \circ \rangle$ are quasigroups, then $\langle Q', \circ \rangle$ is called a sub-quasigroup of $\langle Q, \circ \rangle$. Their corresponding Latin squares are "Latin square and Latin subsquare respectively.

<u>Definition</u> (Embedding)

If A is a sub-Latin square (or Latin subsquare) of L, then A is said to be embedded in L. The standard form is the one with A in the upper left hand corner.

L:



<u>Theorem 6</u> A Latin subsquare of order m can be embedded in a Latin square of order n if and only if $n \geq 2m$.

<u>Fact 11</u> If L (of order n) has a Latin ~~sub~~subsquare A (of order m), then n may not be a multiple of m. (It is true $m \mid n$ if both L (and A) are corresponding to a <u>group</u> respective

<u>Proof of Theorem 6</u> ← <u>Exercise 1.3. (10 points).</u>

In what follows, we provide some more insight about having subsquare.

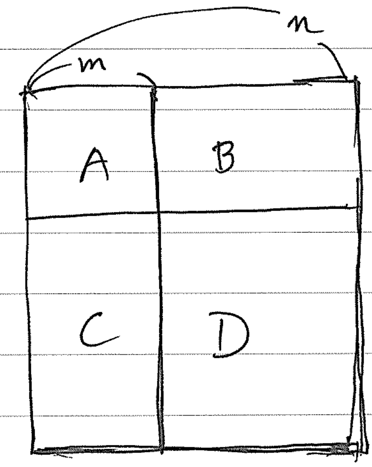
<u>Proposition 6</u> If A is embedded in L and $L(i)$ denotes the element i occurs in L (respectively A, B, C, D), then $A(i) \geq 2m-n$ where A is a Latin square of order m and L is a Latin square of order n.

<u>Proof</u>. $\forall i \in \mathbb{Z}_n$, Since $B(i) + D(i) = n-m$, $B(i) \leq n-m$.

$A(i) + B(i) = m$, Hence $A(i) = m - B(i)$

$\geq m-(n-m) = 2m-n$.

L:



<u>Corollary 7.</u> ($\Rightarrow$) of Theorem 6 is true.

<u>Proof</u>. If $n < 2m$, then every $i \in \mathbb{Z}_n$ has to occur in A which is not possible. ∎

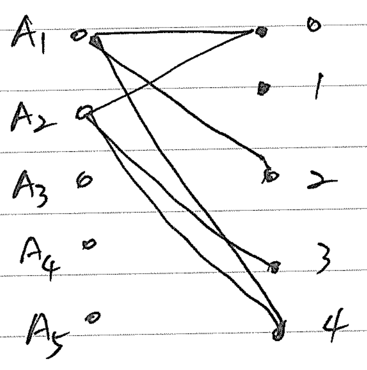
Proposition 8 Let R be an $r \times n$ Latin rectangle based on an n-set S. Then R can be extended (embedded) to a Latin square of order n.

Proof. Use SDR or König's Theorem. ∎

0	1	2	3	4
3	2	4	1	0

A_1 A_2 A_3 A_4 A_5

$A_1 = \{0, 2, 4\}$, $A_2 = \{0, 3, 4\}$, $A_3 = \{0, 1, 3\}$, $A_4 = \{0, 2, 4\}$, $A_5 = \{1, 2, 3\}$



← Regular Bipartite Graph

Fact 12 Let R be an $r \times s$ partial Latin rectangle. Then R can be embedded in a Latin square of order n based on S if and only if $R(i) \geq r + s - n$, $\forall i \in S$ $(|S| = n)$.

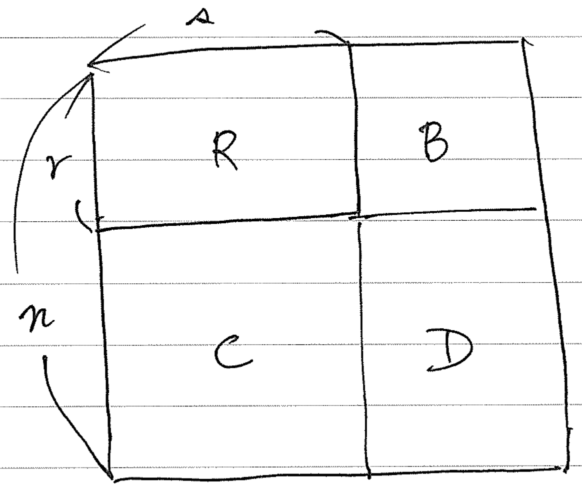
Proof. (Outline) Step 1. Fill all the entries in R, such that the condition $R(i) \geq r+s-n$ holds.

In fact, the subsquare A we consider here can be replaced by Latin rectangle or partial Latin rectangle.

<u>Definition</u> (Latin rectangle, partial Latin rectangle)

A Latin $r \times s$ rectangle R based on S is an $r \times s$ array such that each entry of R is filled with an element of S with an extra property: each element of S occurs in each row and each column at most once. If not every cell is filled with an entry and the property holds, then we have a partial Latin rectangle.

L:



R is embedded in a Latin square L.

If R is embedded in a Latin square L (based on S), then

<u>Proposition 7.</u> $\forall i \in S,\ R(i) \geq r+s-n.$

<u>Proof.</u> $R(i)+B(i)=r,\ B(i)+D(i)=n-s,\ B(i) \leq n-s \Rightarrow R(i)=r-B(i) \geq r-(n-s).$

<u>Step 2.</u> Fill in the entries in B. (Obtain an $r \times n$ Latin rectangle.)

<u>Step 3.</u> Complete the Latin square by extending the rectangle.

<u>Definition</u> (Partial Latin Square of order n) PLS(n)

A PLS(n) is an $n \times n$ array such that each cell is either filled with an entry from an n-set S or empty, moreover, each element in S occurs at most once in each row and resp. once in each column.

<u>Definition</u> (Complete the PLS(n))

Let L' be a PLS(n). L' is said to be completable if we can fill all the empty cells such that the $n \times n$ array is a Latin square.

0	1	
		2

incompletable

0	1	
1		

completable.

(*) It is interesting to know whether a PLS(n) can be completed to a Latin square.

Fact 13 A PLS(n) with at most $n-1$ filled cells can be completed to a Latin square of order n. (Evans' Conjecture)
(In fact, the proof of this fact is quite difficult.)
Proved by B. Smetaniuk (1981). You may refer to "A course in combinatorics" by J.H van Lint and R.M. Wilson, page 189-193.

Fact 14 It takes about 50 pages to characterize a PLS(n) with at most $n+1$ filled cells which is completable.
(L.D. Anderson and A.J.W. Hilton, 1983, LMS.)