# Week 13

Dec. 29

## Random Graphs

**Model A** $G(n,p)$ where $0 \le p \le 1$.

The probability of the existence of an edge (independently) is $p$ and the graph induced by using existent edges is $G_p$.

**Fact 1** Let $H$ be a graph with $n$ vertices and $m$ edges.

Then $P(G_p = H) = p^m \cdot (1-p)^{\binom{n}{2}-m} \approx p^m \cdot q^{N-m}$ where $q = 1-p$ and $N = \binom{n}{2}$.

**Fact 2** The probability of a graph with $m$ edges in $n$ vertices is $\binom{N}{m} p^m \cdot q^{N-m}$.

**Fact 3** We regular use $p = \frac{1}{2}$. (But, it is not necessary.)
(Like or dis-like)

(*) In real world, $p$ is far too small!

~~~

## Expectation

**Definition** (D.P.S., Discrete probabilistic space)
~~~

A D.P.S. is an ordered pair $(S, f)$ where $S$ is a countable set (say finite) and $f: S \to \mathbb{R}$ satisfying (i) $0 \le f(x) \le 1$ and (ii) $\sum_{x \in S} f(x) = 1$.

(*) Let $(S, f)$ be a D.P.S. Then, the probability of $A \subseteq S$ is $\mathbb{P}(A) = \sum_{x \in A} f(x)$.

Definition (Random variable)

Let $(S, f)$ be a D.P.S.. Then, $\mathbb{X}: S \to \mathbb{R}$ is a random variable. We use $\mathbb{X} = k$ to denote the event

$$K = \{ x \in S \mid \mathbb{X}(x) = k \}.$$

For example, let $S = [1, 6]^2$ and $f(x, y) = \frac{1}{36}$ for each $(x, y) \in [1,$

Let $\mathbb{X}((x,y)) = x + y$ (random variable). Then

$$(\mathbb{X} = 7) = \{(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)\}.$$

Definition (Expectation)

Let $\mathbb{X}$ be a random variable in a D.P.S. $(S,f)$. Then the expectation of $\mathbb{X}$, $\mathbb{E}(\mathbb{X}) = \sum_{k \in \mathrm{Range}(\mathbb{X})} k \cdot (\mathbb{P}(\mathbb{X} = k))$.

SEAS@

**Example** One die.

$$\mathbb{E}(X) = 1\cdot\frac{1}{6} + 2\cdot\frac{1}{6} + 3\cdot\frac{1}{6} + 4\cdot\frac{1}{6} + 5\cdot\frac{1}{6} + 6\cdot\frac{1}{6}$$

$$= \frac{21}{6} = \frac{7}{2}.$$

Two dice

$$\mathbb{E}(X) = 2\cdot\frac{1}{36} + 3\cdot\frac{1}{18} + \cdots + 8\cdot\frac{5}{36} + \cdots + 12\cdot\frac{1}{36}$$

$$= 7$$

(*) **Lemma** (Pigeon-hole principle of expectation)

Let $X$ be a random variable of a D.P.S. Then, there exists a $y \in S$ such that $X(y) \geq \mathbb{E}(X)$.

**Lemma** (Linear property of expectation)

Let $X_1, X_2, \cdots, X_m$ be random variables such that $X = \sum_{i=1}^{m} X_i$. Then $\mathbb{E}(X) = \sum_{i=1}^{m} \mathbb{E}(X_i)$.

**Definition** (Indicator Random Variable)

$X : S \to \{0, 1\}$.

SEAS©

✓ $X = \sum_{v \in S} x_v$ where $x_v(y) = \begin{cases} 1, & \text{if } y = v; \text{ and} \\ 0, & \text{otherwise.} \end{cases}$

Theorem (Szele, 1943)

There exists a tournament $T_n$ such that in $T_n$ there are at least $\frac{n!}{2^{n-1}}$ Hamilton paths (distinct).

Proof. In $K_n$, there are $n!$ (Labeled) Hamilton paths. Now, we assign orientation to all the edges in $K_n$. The probabili of obtaining a directed Hamilton path is $\frac{1}{2^{n-1}}$. Hence, the expectation of directed Hamilton paths is $n! \cdot \frac{1}{2^{n-1}}$. By pigeon Hole principle of expectation, there exists a tournament $T_n$ which contains at least $\frac{n!}{2^{n-1}}$ Hamilton paths. ∎

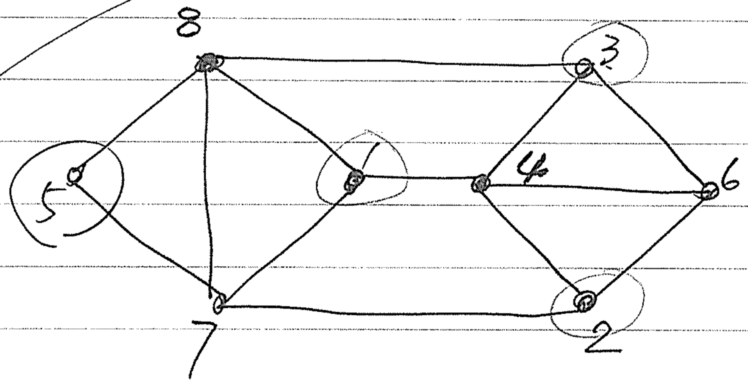
Theorem The independence number of $G$, $\underbrace{\alpha(G) \geq \sum_{v \in V(G)} \frac{1}{1+\deg_G}}_{\text{A famous estimation}}$

Proof. Let $|G| = n$. Use $[1, n]$ to label the vertices of $G$ $(\varphi)$, see example in next page. Now, choose $v_0$ to be included in a set "$S$" if $\varphi(v_0) = \min.\{\varphi(x) \mid x \in N[v_0]\}$.

SEAS©

$v / \varphi(v) =$

So, $S = \{1, 2, 3, 5\}$ is an independent set. (Since we choose "minimum" value, no two of them are adjacent.)




$$\varphi : V(G) \xrightarrow[\text{onto}]{1-1} [1, 8]$$

Now, consider $v_0$. Because the labeling is random, the probability of $v_0$ to have the minimum label is $\frac{1}{1 + \deg_G(v_0)}$. Hence, the expectation of having an independent set is $\sum_{v \in V(G)} \frac{1}{1 + \deg_G(v)}$. That is, there exists a choice of $S$ such that $\alpha(G) \geq |S| = \sum_{v \in V(G)} \frac{1}{1 + \deg_G(v)}$

"Almost all graphs" property

Definition

Given!

We use $G^p$ to denote the distribution of graphs of order $n$ and probability $p$ and $q_n$ the probability of the existence of "Property Q".

Definition If $\lim_{n\to\infty} q_n = 1$, then we say "almost all graphs (of order $n$) have property "Q".

Theorem (Gilbert, 1959)

Let $p$ be a constant such that $0<p\leq 1$. Then, almost all graphs are connected.

Note here that $p$ is the probability of the existence of an edge (independently). So, for example, if a graph $G$ (random) is of order 10 and $p = 0.1$, then $\|G\| \approx (0.1)\cdot\binom{10}{2}$.

(*) If $n$ is small, then the graphs we obtain may not be connected at all! So, keep in mind, we are dealing with the graphs with $n$ "very large".

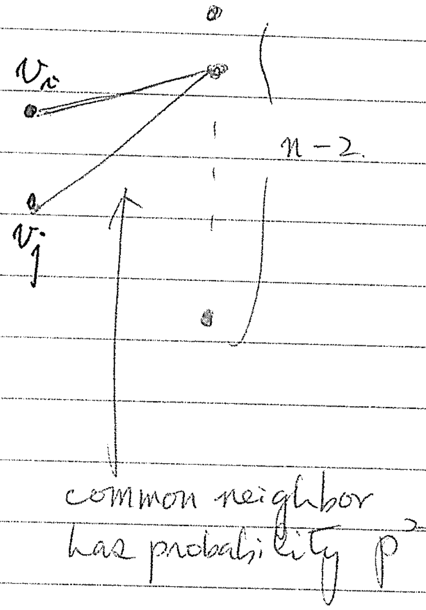
Theorem Let $0<p\leq 1$. Then almost all graphs are of diameter 2.

**Proof.**

Let $\mathbb{X}_{i,j} = \begin{cases} 1, & \text{if } v_i \text{ and } v_j \text{ do not have a common neighbor;} \\ 0, & \text{otherwise.} \end{cases}$

Note that $\mathbb{X}_{i,j}$ is an indicator random variable and

$P(\mathbb{X}_{i,j}=1) = (1-p^2)^{n-2}$.




$$\mathbb{E}(\mathbb{X}) = \sum_{i \neq j} \mathbb{E}(\mathbb{X}_{i,j}) \quad \left(\mathbb{X} = \sum_{i\neq j} \mathbb{X}_{i,j}\right)$$

$$= \binom{n}{2} \cdot (1-p^2)^{n-2}.$$

$$\lim_{n\to\infty} \binom{n}{2} \cdot (1-p^2)^{n-2} = 0$$

$$\Rightarrow \mathbb{E}(\mathbb{X}) \to 0$$

$$\Rightarrow P(\mathbb{X}=0) \to 1$$

Hence, almost every pair of distinct vertices $v_i$ and $v_j$ have a common neighbor. This implies that "$G^p$" is of diameter 2.
(Almost all graphs in). $\blacksquare$

$$\frac{n(n-1)}{2} \cdot (1-p^2)^{n-2} = \frac{n(n-1)}{2} \cdot t^{n-2} \quad \text{where } 0 < t = 1-p^2 < 1.$$

$$a = \frac{1}{t} > 1$$

$$= \frac{n(n-1)}{2 \cdot \left(\frac{1}{t}\right)^{n-2}} = \frac{n(n-1)}{2 \cdot a^{n-2}}$$

$$= \frac{n(n-1)}{2 \cdot e^{(\ln a)(n-2)}} \to 0$$

# Real World (Comments)

Fact 1 Not all edges are of the same probability of existence.

Fact 2 The vertex (vertices) with higher degree has a higher (larger) probability to join a new vertex.

Fact 3 Small world argument shows that the graphs obtained is of diameter not greater than 6.

---

In Graph Theory, if you have published a paper, then the chance of "distance $\leq 6$" to P. Erdös in coauthor's graph is extremely high!

Another fact is about co-actors graph. This graph is also of diameter $\leq 6$.