

Two-colored Complete Graph

(*) We shall use red and blue for the two colors in this note and the edge-induced subgraph of blue edges (red) will be called the "blue subgraph".

Theorem 1 Let K_n be 2-colored. Then either blue graph or red graph is connected.

Proof. If one of the colored subgraph is connected, then (blue or red) we are done. Now, suppose that the red subgraph is disconnected.

We claim that the blue subgraph is connected. Let u and v be two arbitrary distinct vertices in K_n . First, if u and v

are in distinct components of the red graph, then clearly, there is a blue edge connecting u and v . On the other hand,

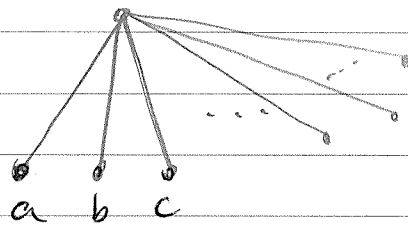
let u and v are in the same component of the red ^{sub-}graph.

Let w be another vertex lies in a distinct component of the

red graph. Then $u-w-v$ is a blue path and we have the claim. Hence, the blue subgraph is connected.

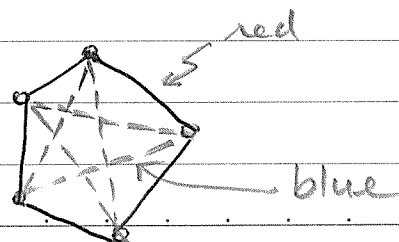
Theorem 2 $\forall n \geq 6$, in a 2-colored K_n , either the blue subgraph or the red subgraph contains a K_3 .

Proof. Let $v \in V(K_n)$. Then $\deg(v) = n-1 \geq 5$ and v is incident to $\lceil \frac{n-1}{2} \rceil$ mono-chromatic edges, say red, see Figure 1. Since $(15) \frac{5}{2}$
 $n \geq 6$, $\lceil \frac{n-1}{2} \rceil \geq 3$. Let a, b, c be three vertices which are incident to v in red subgraph. Now, if any two of them are connected by an red edge, then we have a red K_3 ; otherwise



all of them (a, b, c) are connected by blue edges, then we have a blue triangle.

Notice that if $n=5$, then we have a 2-colored K_5 such that no monochromatic K_3 occurs.



Definition (Ramsey number)

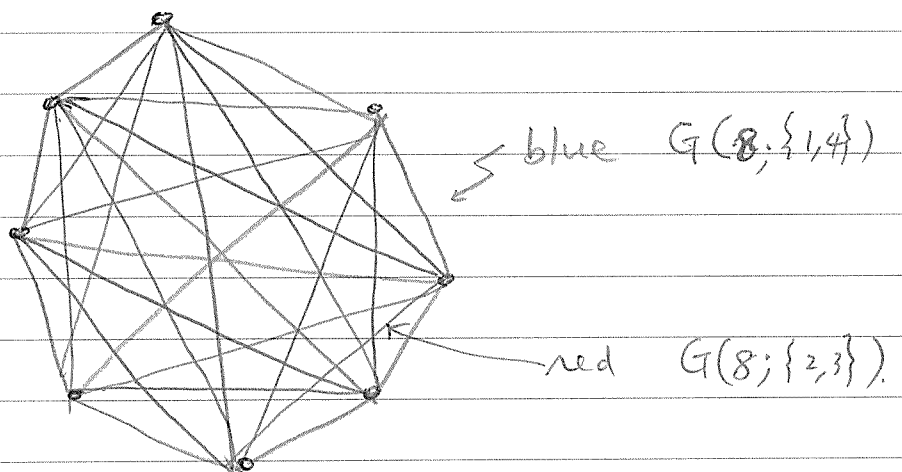
$$R(3,3) = \min. \{n \mid \text{in any 2-colored } K_n, \text{ either we have a blue } K_3 \text{ or a red } K_3\}. \quad (R(3,3) = 6)$$

Remark We can extend K_3 to K_R where $R \geq 3$.

$$R(h,k) = \min. \{n \mid \text{in any 2-colored } K_n, \text{ either we have a blue } K_h \text{ or a red } K_k\}.$$

Theorem 3 $R(3,4) = 9$ (Blue K_3 , Red K_4)

Proof. First, we prove that $R(3,4) \geq 9$. It suffices to construct a 2-coloring of K_8 such that neither blue K_3 nor red K_4 can be found.



Now, we prove that $R(3,4) \leq 9$. Let K_9 be 2-colored and

v be an arbitrary vertex of K_9 . Consider $\deg_b(v)$ the blue degree of v . If $\deg_b(v) \geq 4$, then we obtain either

we have a blue K_3 or a red K_4 , see it? On the other

hand, if $\deg_r(v) \geq 6$, then either we have a red K_4 or

a blue K_3 by using $R(3,3) = 6$. (?) So, it is left to consider

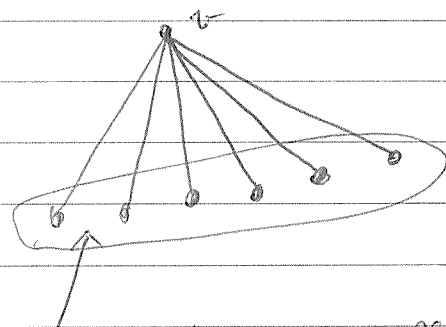
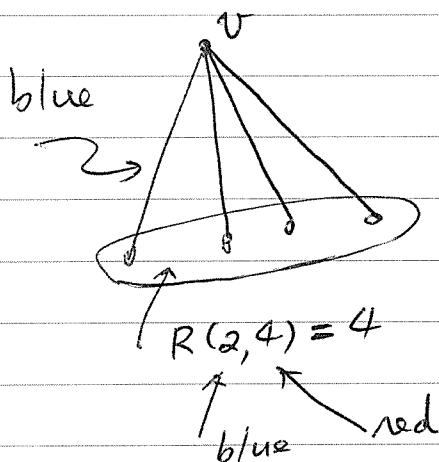
the case that all vertices are of blue degree 3 and red

degree 5. By the fact that we have 9 vertices, no such

subgraphs exist. That is, either we have a vertex of

blue degree at least 4 or a vertex with red degree at least

6. This concludes the proof. ■



A red K_3 gives a red K_4 with the vertex v . Otherwise, we have a blue K_3 .

Find $R(s, t)$.

Fact $R(2, t) = t$ and $R(s, 2) = s$.

Fact $R(s, t) = R(t, s)$. blue red
 ↓ ✓

Theorem 4. For $s \geq 2$ and $t \geq 2$, $R(s, t) \leq R(s, t-1) + R(s-1, t)$.

Proof. Let $n = R(s, t-1) + R(s-1, t)$. Consider a 2-colored

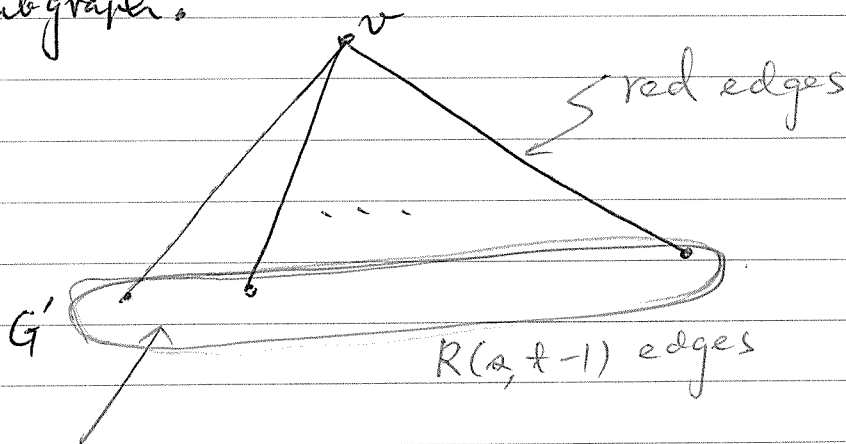
K_n . Then, for each vertex v , $\deg(v) = n-1$. Therefore, either

$\deg_n(v) \geq R(s, t-1)$ or $\deg_b(v) \geq R(s-1, t)$, say the front case,

see the figure below. If G' contains a blue K_2 , then

we are done, Otherwise, we have a red K_{x-1} . Now, with

the vertex v and red edges, we have a K_t which is a red subgraph.



Either a blue K_n or a red K_{t-1} occurs.

Corollary 5 $R(s, t) \leq \binom{s+t-2}{s-1} \text{ or } \binom{s+t-2}{t-1}$

Theorem 6 For each $s \geq 2$, $R(s, s) \leq 2^{2s/s}$ if s is large.
(Thomason, 1988)

$s \backslash t$	2	3	4	5	6	7	8	9	...
2	2	3	4	5	6	7	8	9	
3	3	6	9	14	18	23	28	36	
4	4	9	18	25	...				
5	5	14	25	?	...				
6	6	18	...						
...	...	...	...						

Table for $R(s, t)$

$43 \leq ? \leq 49$
(So far!)

Theorem 7 (Probabilistic Method for lower bound)

$$R(s, s) > \lfloor 2^{\frac{s}{2}} \rfloor \text{ for all } s \geq 3.$$

Proof. First, we claim that if $\binom{n}{s} \cdot 2^{1 - \binom{s}{2}} < 1$, then

$R(s, s) > n$. Consider a random blue-red 2-coloring

of K_n . For a fixed set T of s vertices, let A_T be

the event that $\langle T \rangle$ is monochromatic (either a blue or a red subgraph). Hence $P(A_T) = \left(\frac{1}{2}\right)^{\binom{s}{2}} \cdot 2$ (red or blue).

Since there $\binom{n}{2}$ possible sets for T , the probability

of the existence of a monochromatic subgraph is

$\binom{n}{2} \cdot 2^{1 - \binom{2}{2}} (< 1)$. (By assumption.) This implies that

no monochromatic subgraphs exist is possible. Hence,

$$R(2, 2) < n.$$

Now, if we take $n = \lfloor 2^{\frac{2}{2}} \rfloor$, then

$$\begin{aligned} \binom{n}{2} \cdot 2^{1 - \binom{2}{2}} &< \frac{n^2}{2!} \cdot \frac{2^{1 + \frac{2}{2}}}{2^{\frac{2^2}{2}}} \quad \left(1 - \binom{2}{2} = 1 - \frac{2^2}{2} + \frac{2}{2} \right) \\ &\leq \frac{\left(2^{\frac{2}{2}} \right)^2}{2!} \cdot \frac{2^{1 + \frac{2}{2}}}{2^{\frac{2^2}{2}}} < 1. \text{ Hence, } R(2, 2) > \underbrace{\lfloor 2^{\frac{2}{2}} \rfloor}_n. \end{aligned}$$

Remark This bound is good for larger n .

(*) We can extend "2-coloring" to " k -coloring" ($k \geq 3$).

Theorem 8 $R(3, 3, 3) = 17$.

Proof. By using a similar argument as Theorem 4,

$$R(3, 3, 3) \leq R(2, 3, 3) + R(3, 2, 3) + R(3, 3, 2) - 1 = 17. \text{ Now, the}$$

proof follows by constructing a ^{3-colored complete} graph of order 16 such that no monochromatic K_3 occurs. (?)

Ramsey Theorem (1930)

For any $r \in \mathbb{N}$, there exists an $n \in \mathbb{N}$ ($n(r)$) such that every graph G of order n , either $K_r \leq G$ or $K_r \leq \overline{G}$.

2-colored version

For any $r \in \mathbb{N}$, there exists an $n \in \mathbb{N}$ such that for any 2-colored K_n , either we have a blue K_r or a red K_r .

Proof. $R(r, r) \leq 2^{2r-3}$.

(*) Let $V(K_{16}) = \mathbb{Z}_{16}$. $\forall i, j \in \mathbb{Z}_{16}$, color $\{i, j\}$ by color

k if $i - j \equiv k \pmod{3}$.

Not right!

See (8')

Can you find the 3-coloring?

Observation

In $[1, 15]$, we can obtain three subsets such that in

each subset there exist no "three integers" x, y, z such that $x + y = z$.
(distinct)

$\{1, 2, 4, 8, 13\}, \{3, 6, 10, 12, 14\}, \{5, 7, 9, 11, 15\}$. So, what is the smallest

integer m such that any partition of $[1, m]$ into three

subsets, there exists at least one subset with the above

property, i.e., in S , there exist three integers $x, y, z \in S$,

$x + y = z$.

Bonus problem (2 points, Due 2017, 1, 3, 10:00)

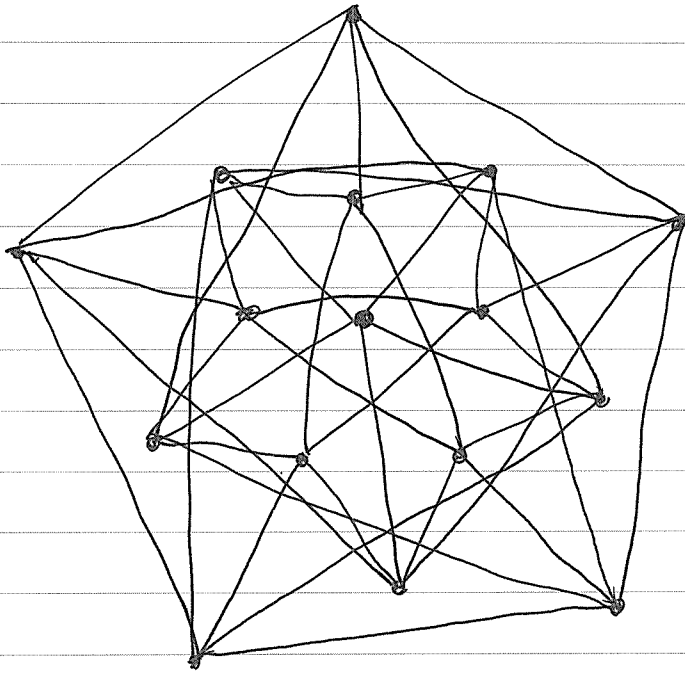
Find the smallest m for the partition of $[1, m]$
(any)

into four subsets such that " $x + y = z$ " has to occur in

at least one set.

used

The unique graph for showing $R(3,3,3) > 16$ is the following graph of order 16: Clebsch graph.



5-regular

Each of the three colors induces a Clebsch graph.
There are two of them: twisted or untwisted drawing.

Theorem (Schur, 1916)

$k=2, m=5$ NO. 9
DATE {1,2,3,4,5}

For every $k \geq 1$ there is an integer m such that every k -coloring of $[1, m]$ contains integers x, y and z of the same color such that $x+y=z$. (Notice that x and y may be the same in this argument.)

Find
Game Three players to find three disjoint subsets of $[1, 16]$? such that in each of their sets, no arithmetic progression of length "3" occurs

→ Proof. Let $m = R(3, 3, \dots, 3) (k\text{-tuples}) - 1$ and $n = R_k(3) = R_k(3)$.

For each k -coloring of C of $[1, m]$, we obtain an induced coloring c' of K_n by letting $V(K_n) = \mathbb{Z}_n$ and $c'(\{i, j\}) = c(|i-j|)$. Since in K_n (after coloring), there is a monochromatic

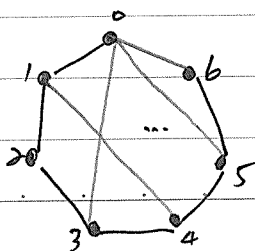
K_3 , say $\{h, i, j\}$, $n \geq j > i > h \geq 1$. $x = i-h, y = j-i$ and

blue red
 $\{1, 2, 4\}, \{3, 5, 6\}$

$z = j-h$ are of the same color.

Moreover $x+y=z$.

(Here, x and y may be the same!)



(*) $R(3,3,3,3) = 62$

(**) If we only consider x, y, z where $x+y=z$ are three distinct integers, then we can not use $R(3,3,3,3)-1$ as its upper bound.

(***) The upper bound and also lower bound for the bonus problem (distinct integers) is exactly "67".