

Extremal GraphsReviewDefinition (Forbidden graphs)

A graph F is said to be forbidden in a class of graphs $\mathcal{G}$ if for each $G \in \mathcal{G}$, G does not contain F as a subgraph, i.e., $G \not\supset F$.

Definition

$$ex(n; F) = \max \{ \|G\| \mid G \not\supset F \text{ and } |G| = n \}.$$

Definition

The graph G_n of order n with $\|G_n\| = ex(n; F)$ is called an extremal graph of order n with forbidden graph F .

Note G is also known as an F -free graph of maximum size.

Remark If we forbid a class of graphs $\mathcal{F}$, then the definitions mentioned above can be revised accordingly, say $ex(n; \mathcal{F})$.

Warming Up (1)

Proposition

If G forbids cycles, then G contains at most $|G|-1$ edges.

Proof. By induction on $|G|=n$.

First, if G is disconnected, then G contains a component G'

and G can be written as the union of G' and G'' . Let

$|G'|=n_1$ and $|G''|=n_2$. Since both G' and G'' are cycle-free

graphs, G' contains at most n_1-1 edges and G'' contains at

most n_2-1 edges. Hence, G contains at most $n-1$ edges.

On the other hand, if G is connected, then $\delta(G)=1$. Let

$\deg_G(v)=1$. Clearly, $G-v$ does not contain cycles, and thus

$G-v$ contains at most $(n-1)-1$ edges. Hence, G contains

at most $n-1$ edges. □

Theorem $ex(n; \mathcal{Z}) = n-1$ where $\mathcal{Z}$ is the collection of cycles.

Proof. By the fact that a tree of order n has exactly

$n-1$ edges, $ex(n; \mathcal{Z}) \geq n-1$. By Proposition above, we have the proof

If we replace $\mathcal{I}$ by a single cycle, then the problem is going to be much more difficult. For example, trying to find an extremal graph for C_4 is not easy at all.

Problem Find $ex(n; C_4)$.

Let $\mathcal{I}_{\text{odd}}$ denote the collection of all odd cycles. Then, the following result is also known.

Theorem $ex(n; \mathcal{I}_{\text{odd}}) = \lfloor \frac{n^2}{4} \rfloor$.

Lemma A graph G is a bipartite graph if and only if G contains no odd cycles.

Proof. ($\Rightarrow$) Easy to see.

($\Leftarrow$) Let $v \in V(G)$, $A = \{u \mid \text{dist}(u, v) \text{ is even}\}$ and $B = \{w \mid \text{dist}(w, v) \text{ is odd}\}$.

It suffices to claim that both A and B are independent sets in G .

Suppose not, say A and $a_i \sim a_j$. Clearly,

one of them can not be v , since the distance is even. (See figure) $\dots$ Then, we have an odd cycle. (?) $\rightarrow$

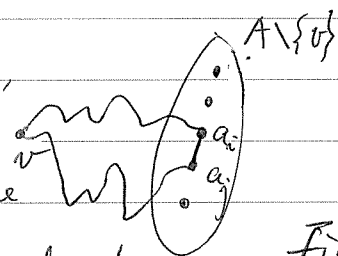


Figure.

Warming Up (2)

Theorem (Pósa)

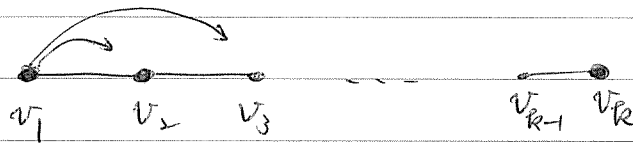
Let G be a ^{connected} graph with $|G| \geq 3$ and $\deg_G(x) + \deg_G(y) \geq k$

for each pair of non-adjacent vertices x and y in $V(G)$.

Then, ^{for $k \leq n$} G contains a path of length k and a cycle of length at least $\frac{k+2}{2}$. (If $k = n$, then G has a Hamilton cycle.)

Proof. We prove the case for paths ^($k \leq n$) and omit the others.

Suppose that we can find a path of length $k-1$ and the condition holds, (Maximal counterexample), see Figure 1.



• v_0

Figure 1. A path of length $k-1$.

Since $k < n$, v_1 is not adjacent to v_k . For otherwise, we

have a path of length k , see Figure 2. (By connectedness) of G

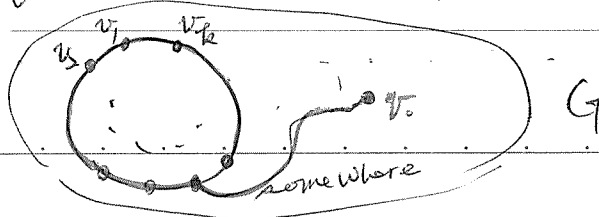
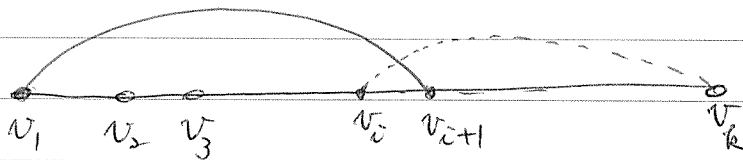


Figure 2;

By the fact that $G \geq C_k$, $\deg(v_1) + \deg(v_k) \leq k-1$, (?)

we have a contradiction.



Remark

$$N(v_1) \subseteq \{v_2, \dots, v_k\} \Rightarrow \deg(v_1) \leq k-1$$

$$N(v_k) \subseteq \{v_1, v_2, \dots, v_{k-1}\} \Rightarrow \deg(v_k) \leq k-1$$

If $v_i v_{i+1} \in E(G)$, then $v_i v_k \notin E(G)$.

$$(\Rightarrow) \deg(v_1) = t \Rightarrow \deg(v_k) \leq k-1-t$$

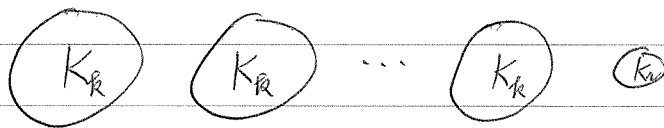
$\Downarrow$

$$\deg(v_1) + \deg(v_k) \leq k-1. \rightarrow \leftarrow$$

Theorem: Let G be a graph of order n such that $G \not\geq P_{k+1}$.

Then, $\|G\| \leq \frac{k-1}{2}n$ and the equality holds iff $k|n$.
all its components are K_k .

Proof. Let $n = qk + r$, $0 \leq r < k$. Let $G = q \cdot K_k \cup K_r$.



Clearly, $G \not\geq P_{k+1}$ and thus $ex(n; P_{k+1}) \geq q \cdot \binom{k}{2} + \binom{r}{2}$

$$\underline{ex(n; P_{k+1}) \leq \frac{k-1}{2} \cdot n}$$

Proof. By induction on n . (Second principle)

First, if G is disconnected, then G contains a component

G_1 , let $G = G_1 + G_2$ and $|G_1| = n_1, |G_2| = n_2$. By induction,

since both G_1 and G_2 do not contain a subgraph P_{k+1} ,

$ex(n_1; P_{k+1}) \leq \binom{k-1}{2} \cdot n_1$ and $ex(n_2; P_{k+1}) \leq \binom{k-1}{2} \cdot n_2$ and thus

G has at most $\binom{k-1}{2} n_1 + \binom{k-1}{2} n_2 = \binom{k-1}{2} n$ edges.

Second, if G is connected, then G contains a vertex x of degree at most $\frac{k-1}{2}$. (For otherwise, $G \geq P_{k+1}$ by Posa's theorem.)

Now, consider $G - x$ and clearly $G - x \not\geq P_{k+1}$. Hence

$G - x$ contains at most $\frac{k-1}{2} \cdot (n-1)$ edges which implies

that G contains at most $\frac{k-1}{2} (n-1) + \frac{k-1}{2} = \frac{k-1}{2} n$ edges.

The equality holds iff $G = \underbrace{K_k + K_k + \dots + K_k}_{q \text{ copies}}$

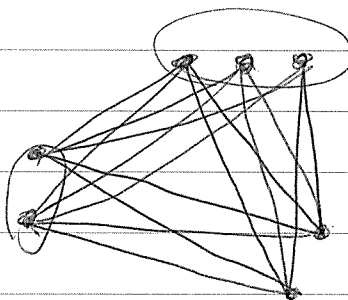
(*) If $n = qk + r$ where $r > 1$, then

$$\binom{k-1}{2} \cdot n = \binom{k-1}{2} \cdot q + \binom{k-1}{2} \cdot r \quad \text{if } r > 0$$

$\underbrace{\binom{k-1}{2} \cdot q}_{\text{VI}}$

Turán's TheoremTurán graph (TG) $\rightarrow T_r(n)$

$T_r(n)$ is the complete r -partite graph such that each partite set is either $\lfloor \frac{n}{r} \rfloor$ or $\lceil \frac{n}{r} \rceil$. It is also known as the balanced, ^{complete} r -partite graph of order n .

Example $T_3(7)$ 

Facts $\|T_2(n)\| = t_2(n) = \lfloor \frac{n^2}{4} \rfloor$.

$$\underline{t_r(n) \geq (1 - \frac{1}{r}) \binom{n}{2}}$$

Example

Consider $r = \frac{n}{q}$. $t_r(n) = \binom{n}{2} - r \cdot \binom{\frac{n}{r}}{2}$

$$= \frac{n(n-1)}{2} - \frac{r \cdot \frac{n}{r} \cdot (\frac{n}{r} - 1)}{2}$$

$$= \frac{n(n-1)}{2} - \frac{r \cdot q(q-1)}{2}$$

$$= \frac{n(n-1)}{2} - \frac{1}{r} \cdot \frac{r q (r q - r)}{2} = \binom{n}{2} - \frac{1}{r} \cdot \frac{n(n-r)}{2}$$

$$\geq \binom{n}{2} - \frac{1}{r} \binom{n}{2}$$

Theorem (Turán, 1940)

For $r, n \geq 2$, $ex(n; K_{r+1}) = t_r(n)$. Moreover, $T_r(n)$ is the unique graph of order n and size $t_r(n)$ that does not contain K_{r+1} .

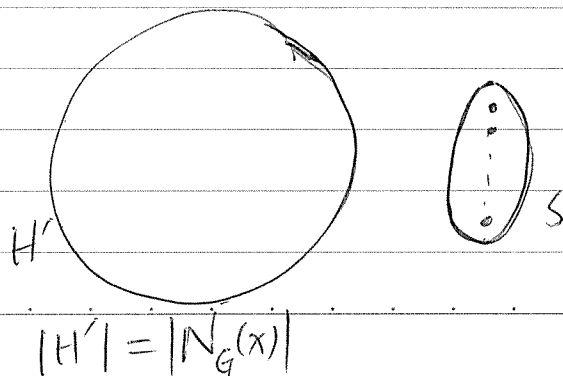
Proof Since $T_r(n)$ does not contain K_{r+1} , $ex(n; K_{r+1}) \geq t_r(n)$.

So, it is left to prove that $ex(n; K_{r+1}) \leq t_r(n)$.

Claim: If G is a graph forbids K_{r+1} , then there exists an r -partite graph H_r such that $\|G\| \leq \|H\|$.
of order n

By induction on r . It is true for $r=1$ and $r=2$. Consider larger r . Let $G \not\supset K_{r+1}$ and $\Delta(G) = \deg_G(x)$. Let G' be the graph induced by $N_G(x)$. Clearly, $G' \not\supset K_r$. Hence, there exists an $(r-1)$ -partite graph H' such that $\|H'\| \leq \|G'\|$.
($\leq G$)

(By induction) Let $S = V(G) \setminus V(H')$. Then $H' \vee S$ (join)



is an r -partite graph

which has

$\|H'\| + (n - \Delta(G)) \cdot \Delta(G)$
edges.

$$\|H'\| + (n - \Delta(G)) \cdot \Delta(G) \geq \|G'\| + \sum_{v \in S} \deg_G(v) \geq \|G\|.$$

This concludes the proof by the fact that $t_r(n) \geq \|H' \vee S\|$.

The "unique" part can also be obtained by using induction.

Let $y \in V(G)$ and $\deg_G(y) = \delta(G)$. Then $\|G - y\| = \|G\| - \deg_G(y) \geq t_r(n-1)$. This implies that $G - y \cong T_r(n-1)$ and the smallest

partite set of $G - y$ has $\lfloor \frac{n-1}{r} \rfloor$ vertices. Moreover, $\deg_G(y) = n - \lfloor \frac{n}{r} \rfloor$ which is equal to $n - 1 - \lfloor \frac{n-1}{r} \rfloor$. So, we conclude that $G \cong T_r(n)$. ■

We have a shorter way to prove the first part.

Proof (2nd proof)

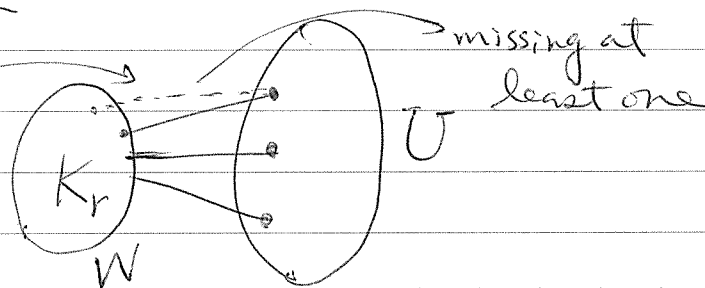
Since $G \not\geq K_{r+1}$, we may add edges to G to contain a subgraph $\tilde{G} \geq K_r$. Let $V(K_r) = W$ and $V(G) \setminus W = U$.

$$\|G\| \leq \binom{r}{2} + (r-1) \cdot (n-r) + \|K[U]\|$$

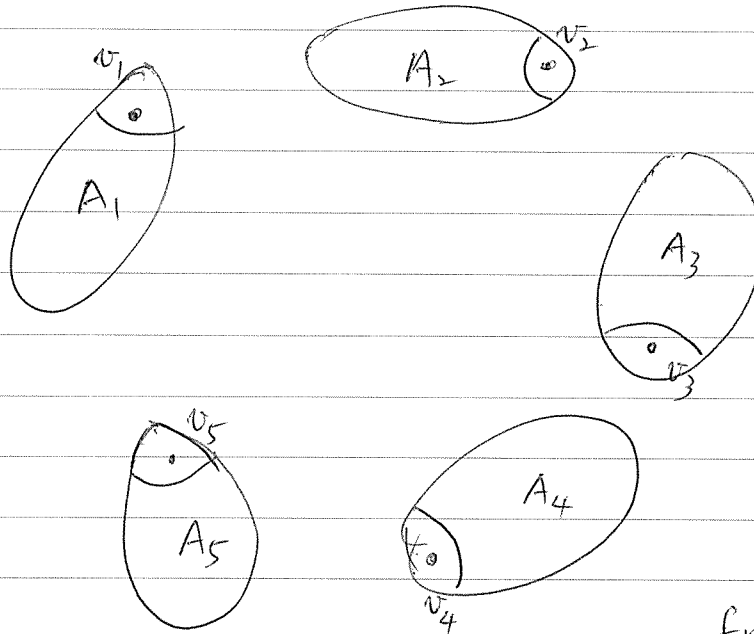
$$\leq \binom{r}{2} + (r-1)(n-r) + t_r(n-r)$$

$$= t_r(n). (?)$$

(Next page.) ■



Example for $r=5$



$$k_5(n) = k_5(n-5) + \binom{5}{2} + 4(n-5)$$

$$\downarrow$$

$$|A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5|$$

Not including v_i 's

$$|\tilde{N}(v_1)| + |\tilde{N}(v_2)| + |\tilde{N}(v_3)| + |\tilde{N}(v_4)| + |\tilde{N}(v_5)|$$

$$\leftarrow$$

$$|A_2| + |A_3| + |A_4| + |A_5|$$

$$\rightarrow$$

$$|A_1| + |A_3| + |A_4| + |A_5|$$

$$= 4(|A_1| + |A_2| + |A_3| + |A_4| + |A_5|)$$

$$= 4(n-5)$$

A combinatorial problem

Let $A_{m \times n}$ be a $(0,1)$ -array such that there exist no sub- $s \times t$ array of all 1's. Find the maximum number of 1's we can have in A .

For example, A is a 4×4 array which contains no 2×2 sub-arrays of 1's.

$$\begin{array}{c}
 \begin{array}{c} a_1 \\ a_2 \\ a_3 \\ a_4 \end{array}
 \begin{array}{c} b_1 \quad b_2 \quad b_3 \quad b_4 \\ \left[\begin{array}{cccc} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array} \right] \end{array}
 \end{array}
 \quad
 \begin{array}{c}
 \begin{array}{c} a_1 \\ a_2 \\ a_3 \\ a_4 \end{array}
 \begin{array}{c} b_1 \quad b_2 \quad b_3 \quad b_4 \\ \left[\begin{array}{cccc} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{array} \right] \end{array}
 \end{array}$$

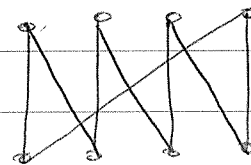
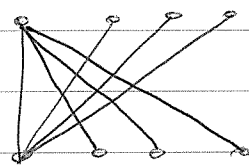
For convenience, $s \times t$ array of all 1's is denoted by $J_{s,t}$.

So, the above problem is dealing with the case $A \neq J_{s,t}$.

(*) An $m \times n$ $(0,1)$ -array can be represented by a bipartite graph

(A_1, A_2) where $|A_1| = m$, $|A_2| = n$ and $a_i \sim b_j$ iff

$$A(i,j) = 1.$$



Bipartite Case

The problem mentioned above can be converted into an extremal graph problem: What is the maximum size of a bipartite graph with partite sets of sizes m and n that does not contain a subgraph $K_{s,t}$. The maximum size is denoted by $z(m, n; s, t)$. The "z" here is to honor Zarankiewicz for mentioning this problem.

There is a good upper bound for $z(m, n; s, t)$. (We omit the proof here.)

Theorem For $m, n, s, t \in \mathbb{N}$, we have

$$z(m, n; s, t) \leq (s-1)^{\frac{1}{t}}(n-t+1) \cdot m^{1-\frac{1}{t}} + (t-1)m.$$

For example, $z(7, 7; 2, 2) \leq 1 \cdot 6 \cdot 7^{\frac{1}{2}} + 7 \approx 22$.

What is the exact value of $z(7, 7; 2, 2)$?

Open problem Determine $z(m, n; 2, 2)$.

Remark We can change $K_{s,t}$ to other graphs, for example

C_4 , C_6 , C_8 , ..., etc.