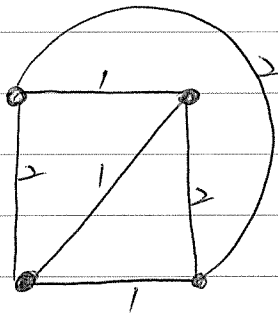
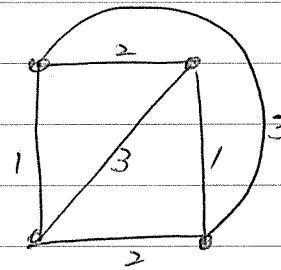


Edge-coloring

Definition A k -edge-coloring π of a graph G is a mapping from $E(G)$ into $\{1, 2, \dots, k\}$. π is a proper k -edge-coloring if incident edges receive distinct image, i.e., $\pi(e) \neq \pi(f)$ provided $|e \cap f| = 1$.

Examples

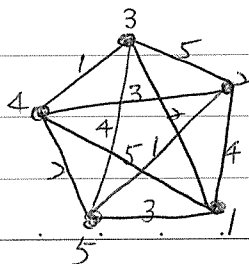
2-edge-coloring



proper 3-edge-coloring

Fact 1 If G has a proper k -edge-coloring, then $k \geq \Delta(G)$.

Fact 2 If K_5 has a proper k -coloring, then $k \geq 5$.



(Each color occurs at most "on" 2 edges!)

Definition (Chromatic index)

The chromatic index of a graph G is the minimum number k such that G has a proper k -edge-coloring.

The chromatic index of G is denoted by $\chi'(G)$.

In what follows, all edge-colorings are "proper".

Fact 3 $\chi'(K_n) = \begin{cases} n-1, & n \text{ is even}; \text{ and} \\ n, & n \text{ is odd.} \end{cases}$

Remark If M_i is a set of edges in G colored " i ", then M_i is a matching, i.e., a set of independent edges.

Proof of Fact 3

Let $V(K_n) = \mathbb{Z}_n = \{0, 1, 2, \dots, n-1\} \pmod{n}$. Let $\pi(\{i, j\}) = t$ provided $i + j \equiv 2t \pmod{n}$. Consider n is odd.

Then π is an n -coloring for

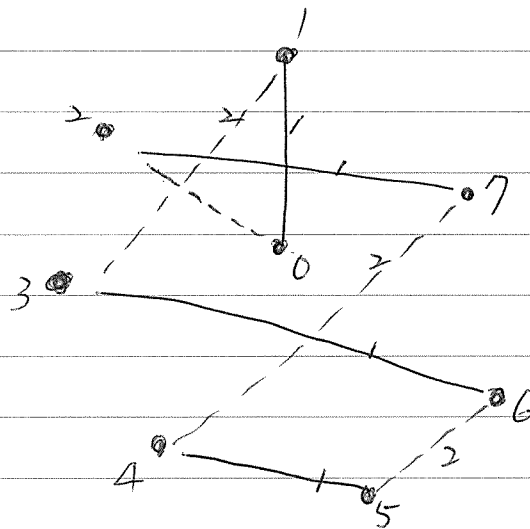
the case n is odd. This implies that $\chi'(K_n) \leq n-1$ when

n is odd. Since each color can occur at most $\frac{n-1}{2}$ times,

$\chi'(K_n) \geq \left\lceil \frac{\binom{n}{2}}{\frac{n-1}{2}} \right\rceil = n$. Hence, $\chi'(K_n) = n$ if n is odd.

n is even

Since $\Delta(K_n) = n-1$, $\chi'(K_n) \geq n-1$ (?). So, it suffices to find an $(n-1)$ -edge-coloring of K_n when n is even. The coloring pattern can be seen as the following case $n=8$.



In fact, the coloring π can be $\pi(\{i,j\}) = t$ if (1) $i=0$ and $j=t$ (2) $i+j \equiv 2t \pmod{n-1}$. Since we have an $(n-1)$ -edge-coloring of K_n (n is even), $\chi'(K_n) \leq n-1$. $\square$

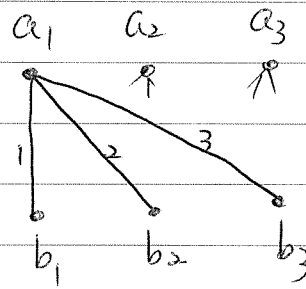
Fact 4 $\chi'(K_{m,n}) = \max\{m,n\}$.

It suffices to prove that $\chi'(K_{n,n}) = n$. Since an n -edge-coloring

of $K_{n,n}$ is corresponding to the existence of a Latin square of order n , the proof follows. (?)

Example

	b_1	b_2	b_3
a_1	1	2	3
a_2	2	3	1
a_3	3	1	2



Latin Square

1	2	3	4	5	6
2	3	4	5	6	1
3	4	5	6	1	2
4	5	6	1	2	3

Latin Rectangle

Definition A Latin square of order n based on $[1, n]$ is an $n \times n$ array such that each element from $[1, n]$ occurs exactly once in each row and each column.

Theorem (Vizing, 1964) (The proof is omitted in this course.)

If G is a simple graph, then $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$.

Definition (Class 1 and Class 2)

A graph G is of Class 1 if $\chi'(G) = \Delta(G)$ and Class 2 otherwise.

(*) Determine whether a graph is of Class 1 or 2 is a very difficult but important problem in Chromatic Theory.

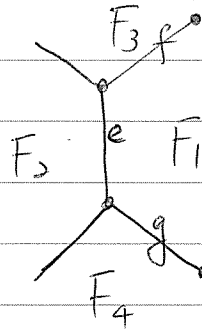
Proposition Petersen graph is of Class 2.

Proof. Note that Petersen graph is a 3-regular graph. Let P denote the graph. Then $\Delta(P) = 3$. Assume that P is of Class 1, i.e., we have $\chi'(P) = 3$. Now, consider the subgraph G of P induced by the edges colored 1 and 2. By observation, G is a 2-regular spanning subgraph of P . The only possibility for such a graph G is a Hamilton cycle. (?) This is not possible since P contains no Hamilton cycles. $\rightarrow \leftarrow$

2-connected

Theorem Let G be a planar, ^{2-connected} cubic graph, _(3-regular) Then $\chi'(G) = 3$.

Proof. Since G is planar, $\chi_f(G) \leq 4$. So, we can use $\{(0,0), (0,1), (1,0), (1,1)\} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ to color the faces such that incident faces receive distinct colors. Now, if e is the boundary of faces ^{of colors on F_1 and F_2} F_1 and F_2 , we color e by using the addition, defined on $\mathbb{Z}_2 \times \mathbb{Z}_2$.



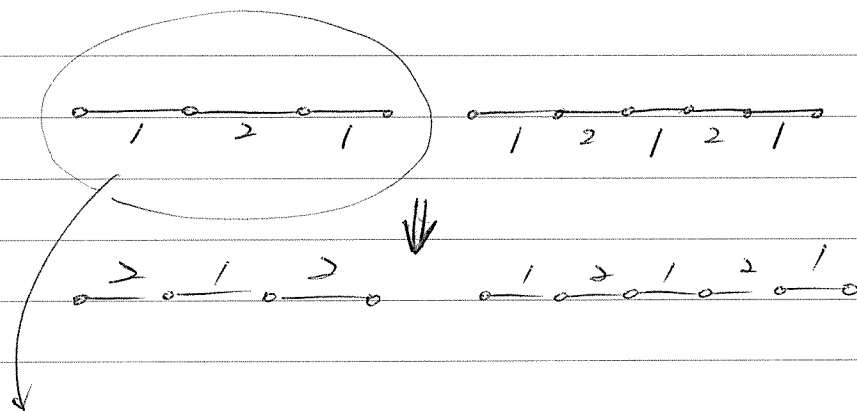
Clearly, we use only three colors, ⁱⁿ $\{(0,1), (1,0), (1,1)\}$ since adjacent faces will receive distinct colors and thus their sum is not $(0,0)$. Moreover, if e and f are incident edges, then the corresponding faces adjacent to F_1 will be different, see figure above. This implies the colors of e and f are distinct. ■

(*) If we don't know $\chi_f(G) \leq 4$, then the proof of this theorem is very difficult. (!)

(**) If G has a k -edge-coloring, then G has an equitable k -edge-coloring such that the difference of colors occurred in the coloring is at most 1.

Proof. By color-switching method.

Remark. If color 1 occurs five times and color 2 occurs three times, then we put these eight edges together to induce a subgraph, see the figure below.

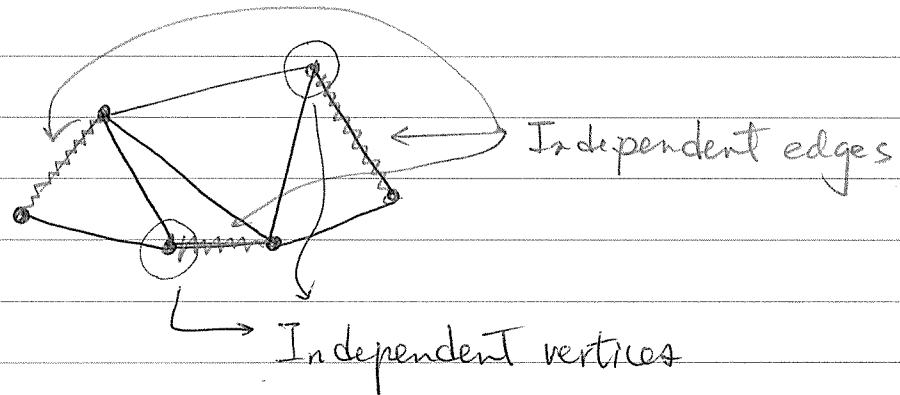


This configuration exists if we have more color "1" than color "2".

Independent Sets and Matchings

Definition

A set of vertices is called independent if no two of its vertices (edges) are adjacent. (incident)



Definition (Matching)

A set of independent edges is also known as a matching.

Remark If $\chi(G) = k$, then G has k independent sets which partition $V(G)$. If $\chi'(G) = k$, then G has k matchings which partition $E(G)$.

(*) We are aiming at finding an independent set of a graph which is of maximum size. The ^{maximum} size of an independent set (of vertices) is called the independence number of G , denoted by $\alpha(G)$.

Easy facts: $\alpha(K_n) = 1$, $\alpha(C_n) = \lfloor \frac{n}{2} \rfloor$ and $\alpha(T) \geq \lfloor \frac{|T|}{2} \rfloor$.
(T is a tree.)

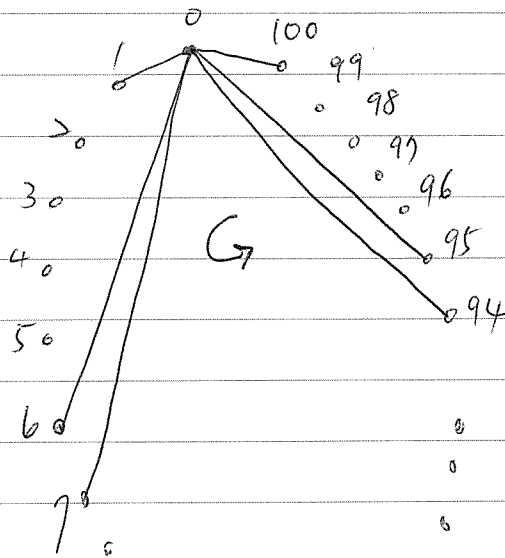
Proposition Let $\chi(G) = k$. Then, $\alpha(G) \geq \frac{|G|}{k}$, or equivalently

$$\chi(G) \cdot \alpha(G) \geq |G|.$$

(*) Finding $\alpha(G)$ for a general graph G is very difficult.

Bonus problem (2 points, Due 12.1.15:00)

Let G be a graph defined on $\mathbb{Z}_{101}$. Two vertices i and j are adjacent if and only if $i - j \in \{1, 6, 7, -1, -6, -7\} \pmod{101}$. Find $\alpha(G)$.



Definition (Matching)

A matching of a graph G is a set of independent edges.

We use $\alpha_1(G)$ to denote the maximum size of a matching.

Clearly, a matching has at most $\frac{|G|}{2}$ edges. If M is a matching of size $\frac{|G|}{2}$ when $|G|$ is even, then M is called a perfect matching of G and the graph induced by M is called a 1-factor.

Definition (k -factor)

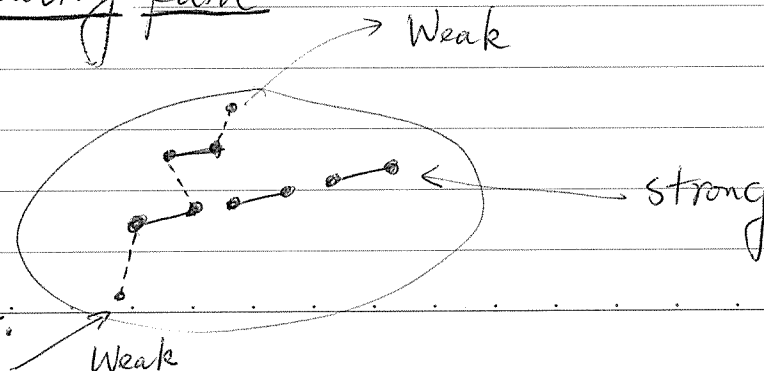
spanning k -regular

A $\wedge$ subgraph of a graph G is called a k -factor.

(*) Finding $\alpha_1(G)$ of a general graph G is comparatively easy. (There exists a polynomial time algorithm to find $\alpha_1(G)$.)

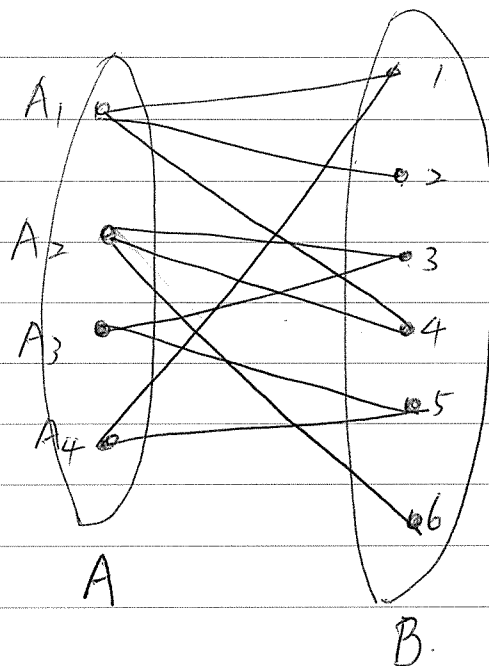
Idea: Alternating path

We can enlarge a matching if we have a path: Weak-Weak:



Marriage Problem

Given a bipartite graph $G = (A, B)$, what is the condition for which G contains a matching saturates A , i.e., a matching incident to every vertex of A .



$$A_1 = \{1, \textcircled{2}, 4\}$$

$$A_2 = \{3, 4, \textcircled{6}\}$$

$$A_3 = \{\textcircled{3}, 5\}$$

$$A_4 = \{\textcircled{1}, 5\}$$

(*) The reason why this is called "marriage problem" follows from the fact that every "one" in A finds a mate in B , but not the same mate.

(**) This problem can be thought as finding representatives from a collection of sets.

Theorem (P. Hall, 1935)

1. Graph Version Let $G = (A, B)$ be a bipartite graph with $A = \{a_1, a_2, \dots, a_m\}$. Then G contains a matching saturates A if and only if for any $1 \leq k \leq m$, $\left| \bigcup_{j=1}^k N_G(a_j) \right| \geq k$.

2. Set Version Let $\mathcal{A} = \{A_1, A_2, \dots, A_m\}$ be a collection of subsets of $[1, n]$. Then, there exists a vector $(a_1, a_2, \dots, a_m)$ where $a_i \in A_i$, $i = 1, 2, \dots, m$ and $a_i \neq a_j$, for $1 \leq i \neq j \leq m$, if and only if for any $1 \leq k \leq m$, $\left| \bigcup_{j=1}^k A_j \right| \geq k$. ----- (1)

Remark In set version, $(a_1, a_2, \dots, a_m)$ is called a system of distinct representatives (SDR in short) and the condition (1) is known as the Hall's condition.

We shall prove the set version in what follows.

Proof. $(\Leftarrow)$ By induction on the number of sets in $\mathcal{A}$, i.e., m .

Clearly, it is true for $m=1$. Assume that the assertion is

true for $1 \leq m \leq h$. Let $\mathcal{A} = \{A_1, A_2, \dots, A_{h+1}\}$ be a collection

of $k+1$ sets satisfying (1). Now, we consider two cases.

Case 1 For any $1 \leq k \leq h$, the union of k sets in $\mathcal{A}$ contains at least $k+1$ elements.

$$\mathcal{A}' =$$

Let $a_{h+1} \in A_{h+1}$ and consider $\bigwedge \{A_1 \setminus \{a_{h+1}\}, A_2 \setminus \{a_{h+1}\}, \dots, A_h \setminus \{a_{h+1}\}\}$.

← By observation, condition (1) holds for $\mathcal{A}'$ and thus $\mathcal{A}'$ has an

SDR $(a_1, a_2, \dots, a_h)$. Combine with a_{h+1} , we have an SDR $(a_1, a_2, \dots, a_{h+1})$

for $\mathcal{A}$.

Case 2 There exists a $1 \leq k' \leq h$, such that the union of some k' sets contains exactly k' elements, say $\mathcal{A}' = \{A_1, A_2, \dots, A_{k'}\}$.

Let $S = \bigcup_{i=1}^{k'} A_i$. Consider the collection of sets $\mathcal{A}'' = \{A_{k'+1} \setminus S, A_{k'+2} \setminus S, \dots, A_h \setminus S\}$.

Now, both $\mathcal{A}'$ and $\mathcal{A}''$ satisfy (1). By induction and the combination

of their SDR's respectively, we conclude the proof.

the union

(Note that any $\bigwedge k$ subsets of $\mathcal{A}''$ contains at least k elements, for

otherwise we have a contradiction. (?))

(For example, if $(A_{k'+1} \setminus S) \cup (A_{k'+2} \setminus S)$ contains only one element,

the union of

then $\bigwedge A_1, A_2, \dots, A_{k'}, A_{k'+1}, A_{k'+2}$ will contain only $k'+1$ elements. $\rightarrow \text{①}$)