

Problem Does Petersen graph contain a Hamilton cycle?

Answer: No.

Proof. Notice that Petersen graph₁^P is 3-regular with girth 5,

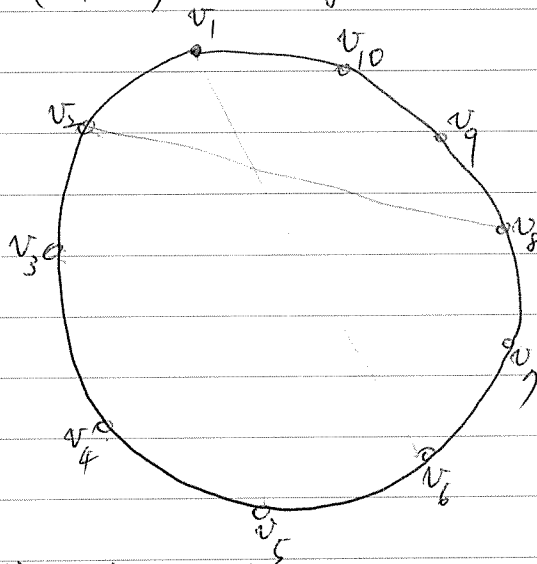
i.e. P contains no C_3 's or C_4 's. Now, if P contains a Hamilton

cycle C , then $P \setminus C$ is a matching with 5 edges. Let

$C = (v_1, v_2, \dots, v_{10})$. Clearly, the edge incident to v_1 must be

incident to one of $\{v_5, v_6, v_7\}$. So, we consider the cases

$v_1 \sim v_5$ and $v_1 \sim v_6$. ($v_1 \sim v_7$ is a symmetric case of $v_1 \sim v_5$.)



$$v_1 \sim v_5 \Rightarrow \begin{cases} v_2 \sim v_7 \Rightarrow v_3 \sim ? \text{ (No where to go!)} \\ v_2 \sim v_8 \Rightarrow v_3 \sim ? \text{ (No where to go!)} \end{cases}$$

$$v_1 \sim v_6 \Rightarrow v_2 \sim v_8 \Rightarrow v_3 \sim ? \text{ (No where to go!)} \quad \blacksquare$$

Bonus Problem (2 points) (Due, Oct. 25, 10:00)

Let G be an n -connected graph where $n \geq 2$. Prove or disprove that any n distinct vertices can be included in a cycle of G .

Hint. By induction on n if the assertion is true.

Remark : If we are dealing with digraph, then a Hamilton cycle is a directed Hamilton cycle.

Tournament (Oct. 20)

Definition (Symmetric digraph)
(asymmetric or oriented)

A digraph D is called symmetric if, whenever (u, v) is an arc of D , then (v, u) is also an arc of D .
(not an arc of D)

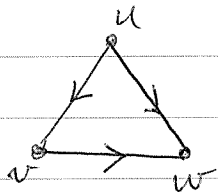
Note that an oriented graph D can be obtained from a graph G by assigning a direction to each edge of G , and thereby transforming G into an asymmetric digraph.

Definition (Tournament)

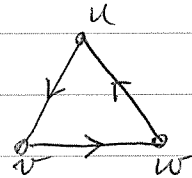
A complete asymmetric digraph is called a tournament.

A tournament with p vertices will be denoted by T_p . A

tournament T_p is transitive if for any u, v, w , (u, w) is an arc of T_p provided (u, v) and (v, w) are arcs of T_p .



(Transitive)



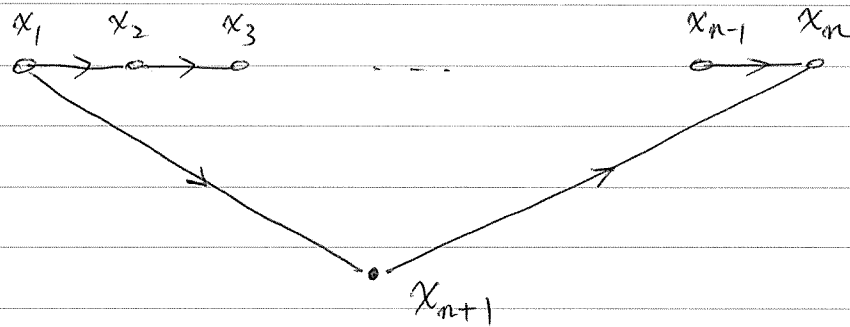
(not transitive)

Theorem

For any tournament T_p , T_p contains a directed Hamilton path.

Proof. (Maximality argument)

Let $\langle x_1, x_2, \dots, x_n \rangle$ be the longest directed path which we can find in T_p . If $n = p$, then we have the proof. Otherwise, let $n < p$ and let x_{n+1} is not on the path. (See Figure below)



Then, (x_1, x_{n+1}) and (x_{n+1}, x_n) are arcs in T_p . If any of these two arcs are in different orientation, we have a longer path. Now, consider the orientations of the arcs between $\{x_2, x_3, \dots, x_{n-1}\}$ and x_{n+1} . (x_2, x_{n+1}) must be an arc ^{in T_p} , otherwise, $\langle x_1, x_{n+1}, x_2, \dots \rangle$ will be longer. Similarly, (x_3, x_{n+1}) is an arc in T_p , so are $(x_4, x_{n+1}), \dots, (x_{n-1}, x_{n+1})$. But, now we have a longer path $\langle x_1, x_2, \dots, x_{n-1}, x_{n+1}, x_n \rangle$, a contradiction. This concludes that $n=p$ and the proof. ▀

Question When does a tournament contain a directed Hamilton cycle?

Theorem

If T_p is a transitive tournament, then all vertices have distinct out-degrees.

Proof. It suffices to claim that no two vertices have the same out-degree. Suppose not. Let $\deg^+(u) = \deg^+(v)$ (in T_p), and W.L.O.G. $(u, v) \in T_p$. First, let $W = N^+(v)$ (out-neighbors). Then, $|W| = \deg^+(v)$. By transitivity of T_p , for each $w \in W$, (u, w) is also an arcs of T_p . ($(u, v) \in T_p$ and $(v, w) \in T_p$) This implies that $\deg^+(u) \geq |W| + 1 = \deg^+(v) + 1$. $\rightarrow \leftarrow$ ▣

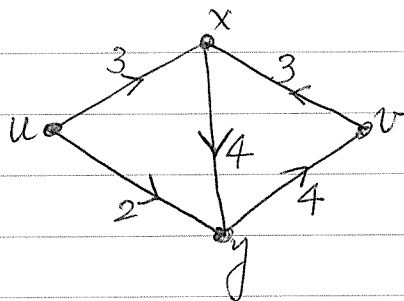
Note

1. This kind of tournaments is good, since we ^{shall} have a unique "king" (maximum out-degree).
2. A transitive tournament does not contain any directed cycles! (Easy to prove.)

Definition

A network N is a digraph D with two distinguished vertices u and v , called the source and sink of N , resp., and a nonnegative integer-valued function c on $A(D)$, i.e. $c: A(D) \rightarrow \mathbb{R}^+ \cup \{0\}$. The digraph is called the underlying digraph of N . The function c is the capacity function of N and its value $c(\vec{a}) = c(x, y)$ on the arc $\vec{a} = (x, y)$ is referred to as the capacity of $\vec{a}$.

e.g.



Definition (Flow)

A flow in a network N (with underlying digraph D , source u , sink v , and capacity function c) is an integer-valued

function f on $A(D)$ such that for every $\vec{a} \in A(D)$,

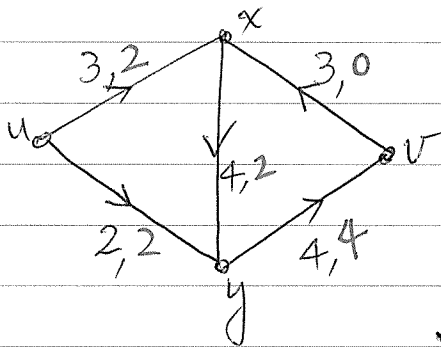
$$0 \leq f(\vec{a}) \leq c(\vec{a}), \text{ and}$$

$$\sum_{y \in N^+(x)} f(x, y) = \sum_{y \in N^-(x)} f(y, x) \quad \forall x \in V(D) \setminus \{u, v\}. \quad \dots (*)$$

(*) is known as the "conservation equation" of the network.

Example of network with flow

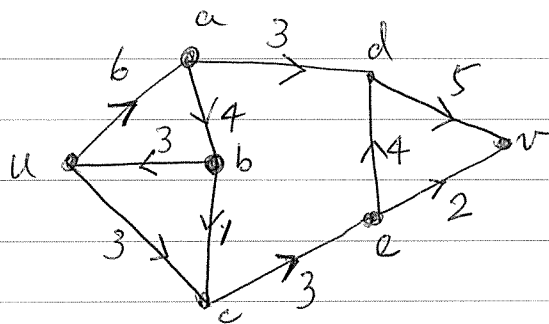
(1)



val f: The value of a flow f in a network, which is defined as the net flow out of the source of N ; equivalently, the net flow into the sink of N .

$\text{val } f = 4$

Another example (2)



What is "maximum flow" from u to v ?

Notation (X, Y)

$$(X, Y) = \{(x, y) \in A(D) \mid x \in X, y \in Y\}.$$

Definition (Cut)

Let N be a network with underlying digraph D , source u , sink v . A cut in N is $(X, \bar{X})$ where $\bar{X} = V(D) \setminus X$, $u \in X$ and $v \in \bar{X}$.

Definition (Capacity of a cut)

Let $K = (X, \bar{X})$ be a cut of network N with capacity c .

Then, the capacity of the cut K , denoted $\text{cap } K = c(X, \bar{X})$

$$= \sum_{(x, y) \in (X, \bar{X})} c(x, y).$$

In example 12).

Let $X = \{u, a, b, c\}$, then $c(X, \bar{X}) = 6$.

$$\bar{X} = \{d, e, v\}.$$

Theorem Let f be a flow in a network N and let

$K = (X, \bar{X})$ be a cut in N . Then

$$\text{val } f = f(X, \bar{X}) - f(\bar{X}, X) \leq \text{cap } K.$$

Minimum cut

A cut K in a network N is called a minimum cut if $\text{cap } K \leq \text{cap } K'$ for every cut K' in N .

(Fact) Let f be a flow and K be a cut in N . If $\text{val } f = \text{cap } K$, then f is a maximum flow and K is a minimum cut.

(A flow f in N is called a maximum flow if $\text{val } f \geq \text{val } f'$ for every flow f' in N .)

Proof. It can be proved (?) that if f^* is a maximum flow and K^* is a minimum cut, then $\text{val } f^* \leq \text{cap } K^*$. Now, f is a flow, hence $\text{val } f \leq \text{val } f^*$; K^* is a minimum cut, hence $\text{cap } K^* \leq \text{cap } K$. Since $\text{val } f = \text{cap } K$ and $\text{val } f \leq \text{val } f^* \leq \text{cap } K^* \leq \text{cap } K$, we conclude that $\text{val } f = \text{val } f^*$ and $\text{cap } K = \text{cap } K^*$. ▀

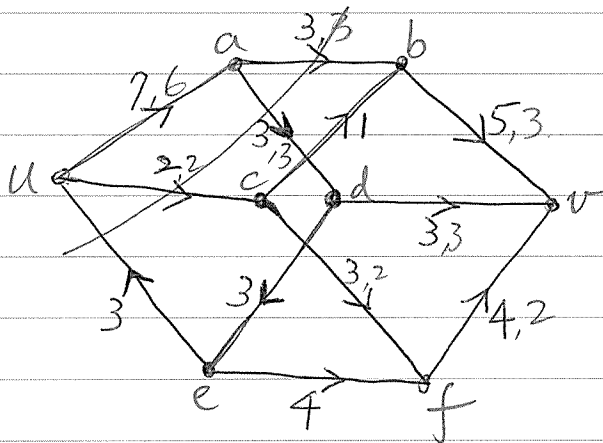
Theorem (Ford and Fulkerson, 1956)

In any network, the value of a maximum flow equals the capacity of a minimum cut.

Proof (Idea)

If the minimum cut has capacity c , then there exists a flow f with $\text{val } f = c$.

Example Find the maximum flow of the following network.



$$X = \{u, a\}, \quad \bar{X} = \{b, c, d, e, f, v\}$$

$$\text{cap}(X, \bar{X}) = 8$$

Can we find a flow f with $\text{val } f = 8$?

Yes. Then $K = (X, \bar{X})$ is a minimum cut and f (above) is a maximum flow.