

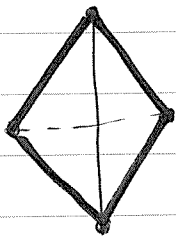
Hamilton Cycles (hamiltonian cycles)

Definition (Hamilton cycles)

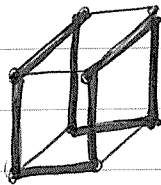
A cycle in G is a hamilton cycle if the cycle contains all vertices of G .

Problem (Thomas P. Kirkman) 1855

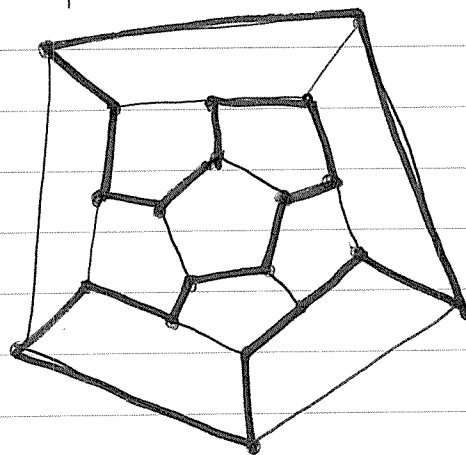
Given the graph of a polyhedron, can one always find a circuit (we call cycle) that passes through each vertex once and only once? (Yes for regular polyhedron!)



(tetrahedron)

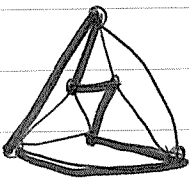


(Cube)

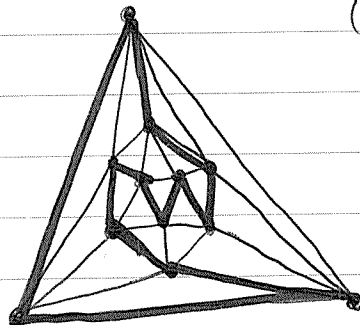


(dodecahedron)

(World Game!)
30 cities



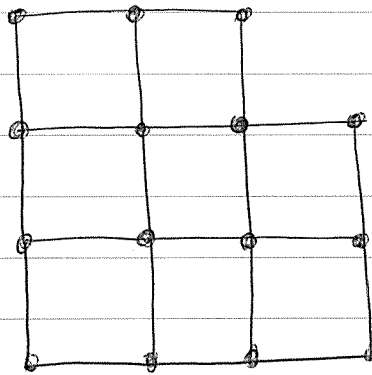
(octahedron)



(icosahedron)

From that time and on, determining whether a graph contains a Hamilton cycle becomes one of the most interested and important problem in graph theory.

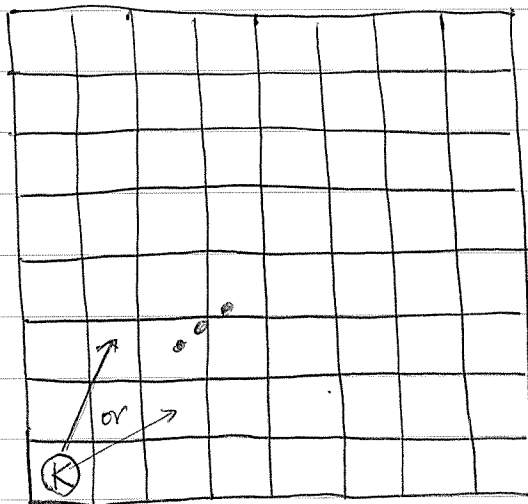
Test Can you find a Hamilton cycle in the following graph?



No! Why?

If a bipartite graph has a Hamilton cycle, then the graph must have even number of vertices.

Test Can you find a "Knight" in a chessboard which jumps all the 64 spots?

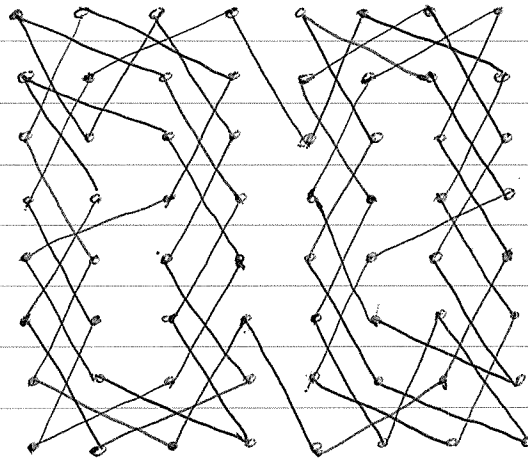


See 2-2'

NO.

2-2'

DATE



One tour of a knight on a chessboard

This implies that if $\deg(v_1) = t$, $\deg(v_n) \leq (n-1) - t$. Hence,

$$\deg(u) + \deg(v) \leq t + (n-1) - t = n-1. \quad \rightarrow \leftarrow \text{We conclude}$$

that G contains a Hamilton cycle.

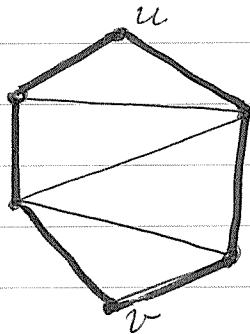
Remark

There are sufficient conditions for the existence of Hamilton cycles
(Quite a few!)

in a graph, but so far, none of them is also necessary.

the condition in

For example, the above theorem is not necessary.



$$\deg(u) + \deg(v) = 4 < 6.$$

(***) Finding a cycle with prescribed length is in general very difficult. (Algorithmic thinking)

Note A graph with Hamilton cycles is called a hamiltonian graph.

Weighted graphs

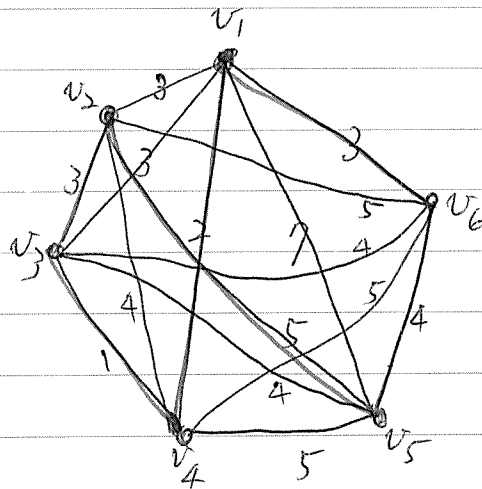
Definition (Weighted graphs)

A graph G is weighted if each edge is assigned a weight by using a weighted function $w: E(G) \rightarrow \mathbb{R}$.

Traveling Salesman Problem (TSP)

In a weighted complete graph G , find a minimum Hamilton cycle, i.e., the sum of all weights in the cycle is minimum comparing the sums of all the weights of the other Hamilton cycles.

e.g.



(Note) If each weight is a finite number, then a greedy algorithm can provide an answer. (May not be minimum.)

Connectivity

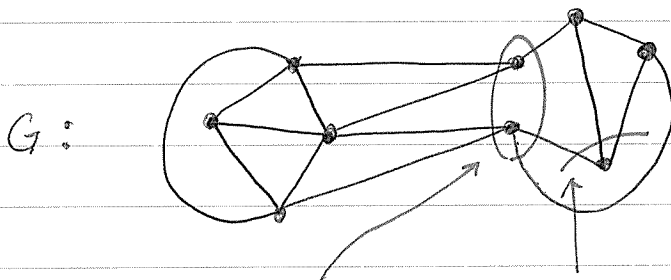
Definition (Connectivity)

 $\kappa(G)$

The connectivity of a graph G , is the minimum number of vertices whose removal from G results in a disconnected graph or a trivial graph (a graph with one vertex).

Definition (Edge connectivity)

The edge connectivity of a graph G , $\kappa_1(G)$, is the minimum number of edges whose removal from G results in a disconnected graph.



$$\kappa(G) = 2, \kappa_1(G) = 3.$$

Theorem For any graph G , $\kappa(G) \leq \kappa_1(G) \leq \delta(G)$.

Proof. Let $v \in V(G)$ and $\deg(v) = \delta(G)$. Then, the deletion of all edges incident to v results in a disconnected graph.

Hence, $\kappa_1(G) \leq \delta(G)$. Now, consider the other inequality. First, if $\kappa_1(G) = 0$, then the G is already disconnected, hence $\kappa(G) = 0$. Assume that $\kappa_1(G) > 0$ and let E' be a set of $\kappa_1(G)$ edges such that $G - E'$ is disconnected. Let S be a set of vertices chosen from the set of vertices incident to edges in E' such that edge in incident to S exactly once. Therefore $|S| \leq |E'|$. Also, $G - S$ is disconnected, since $G - E'$ is disconnected. This implies or a trivial graph that $\kappa(G) \leq |S| \leq |E'| = \kappa_1(G)$. ■

Problem Let $a \leq b \leq c$ be positive integers. Then, there exists a graph G such that $\kappa(G) = a$, $\kappa_1(G) = b$ and $\delta(G) = c$.

Definition (n -connected and n -edge-connected)

A graph G is said to be n -connected (resp. n -edge-connected) if $\kappa(G) \geq n$ (resp. $\kappa_1(G) \geq n$).

(Fact) A graph is n -edge-connected if it is n -connected.

Menger's Theorem

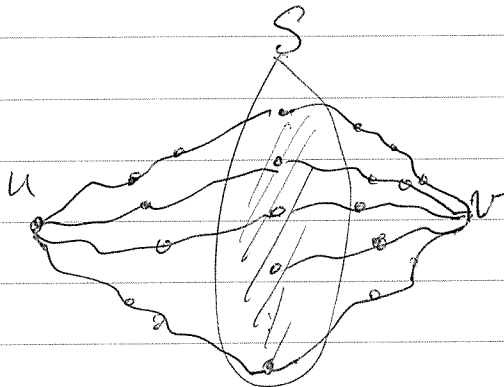
Definition (Separating set)

A set of vertices S of a graph G is said to be a separating set of two vertices u and v of G if $G - S$ is a disconnected graph in which u and v lie in different components. We also say S separates u and v .

(u, v) -separating set

Theorem (Menger, 1927)

Let u and v be nonadjacent vertices in G . Then, the minimum number of vertices that separates u and v is equal to the maximum number of internally disjoint u - v paths in G .



Proof. Many different versions.

Remark

Consider the existence of a Hamilton cycle, it is easy to see that if a graph is of larger connectivity, then the probability of finding a Hamilton cycle is larger. The following theorem is one of this kind.

Theorem (Chvátal and Erdős, 1972)

(Not prove in the class.)

Let G be a graph with at least three vertices. If $\kappa(G) \geq \beta(G)$, where $\beta(G)$ is the independence number G , then G contains a Hamilton cycle.

^{cycle exists}
This implies that $\kappa(G) \geq 2$.

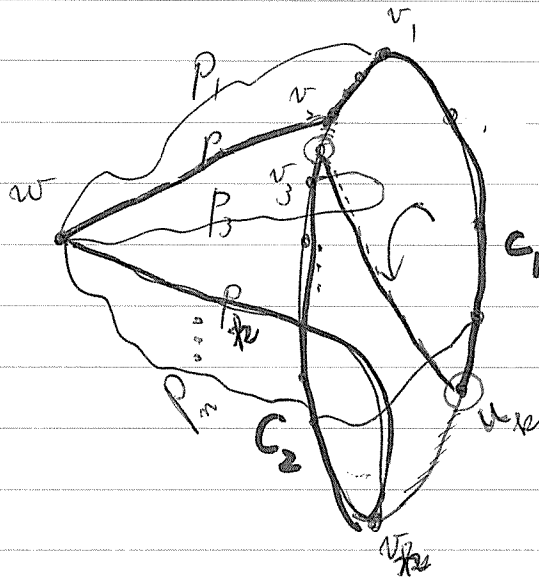
Proof. Clearly, we may assume that $\beta(G) \geq 2$. Let C be the

largest (size) cycle we can find in G . If $|C| = |G|$, then we are done. Otherwise, there is a vertex $w \notin V(C)$. We claim there exists a cycle of length at least $m+1$.

Let $\kappa(G) = n$. Then, by Menger's Theorem, we can find n internal-disjoint paths from w to v_i 's, say $P_1, P_2, \dots, P_n$. ($n \leq m$). ($v_i \in V(C), i = 1, 2, \dots, n$)

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Now, there must be some vertices between v_i and v_{i+1} for $i=1, 2, \dots, n-1$. For otherwise, we already have a cycle of larger size.

Let $u_i \sim v_i$ in counterclockwise direction, $i=1, 2, \dots, n$. Then, $\underline{u_i}$ is not incident to $\underline{w}$ by a similar reason as above. Let $S =$

$\{w, u_1, u_2, \dots, u_n\}$. Then S must be an independent set. Suppose

not, let $u_j u_k$ be an edge in G . Then we have a larger

cycle: $w \xrightarrow{P_1} v_j - C_1 - u_k u_j - C_2 - v_k \xrightarrow{P_k} w$. This concludes the proof.

(*) Finding the independence number of a graph is very difficult in general.