

For reference about Metric Space

Definition (Metric space)

A metric space (defined on M) is an ordered pair (M, d)

where M is a set and d is a metric on M , i.e., a function

$d: M \times M \rightarrow \mathbb{R}$ such that for any $x, y, z \in M$, we have

1. $d(x, y) = 0 \Leftrightarrow x = y$; (identity of indiscernibles)
2. $d(x, y) = d(y, x)$; and (symmetry)
3. $d(x, z) \leq d(x, y) + d(y, z)$. (triangle inequality)

Note $d(x, y) \geq 0$ (non-negativity or separation axiom)

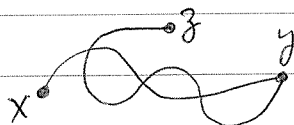
Proof. $2d(x, y) = d(x, y) + d(y, x) \geq d(x, x) = 0$. ■

Fact The "distance" defined on graph G , d , is a metric.

⑥ $d: V(G) \times V(G) \rightarrow \mathbb{R}$

① $d(x, y) = 0$ iff $x = y$, ② $d(x, y) = d(y, x)$, and

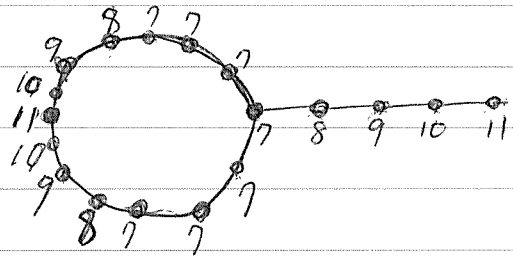
③ $d(x, y) + d(y, z) \geq d(x, z)$. (Why?)



If there exists a walk from x to z , then there exists a path from x to z , and the length of path is not larger than that of the walk. SEASON

Fact For each pair of ^{positive} integers a and b such that $a \leq b \leq 2a$, there exists a graph G with $\text{rad}(G) = a$ and $\text{diam}(G) = b$.

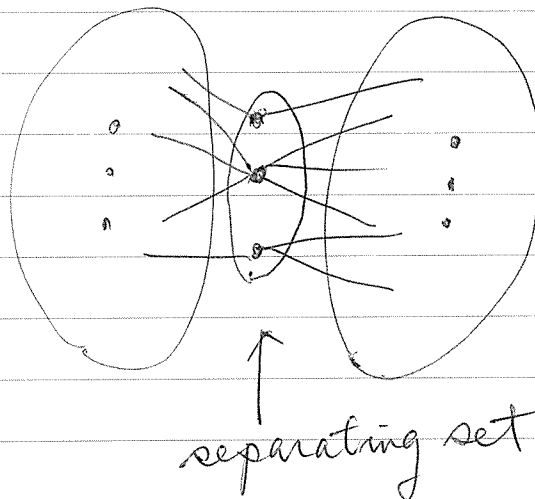
For example, $a=7$ and $b=11$. (A cycle of length $2a+1$ with a "tail" P_{b-a+1}).



(*) $a=7, b=15$ is not possible, why?

Exercise Find a 3-connected graph G such that $\text{rad}(G)=7$ and $\text{diam}(G)=11$.

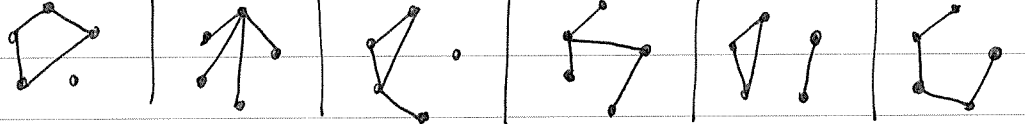
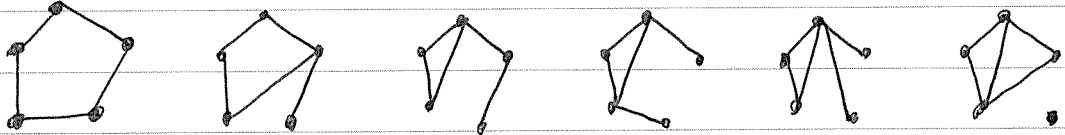
Review A graph G is k -connected if any separating set is of size at least k .



Solution to the 1st Bonus Problem

There are 34 non-isomorphic graphs of order 5.
(simple)

Idea First, we use the number of edges to partition the set of graphs of order 5.

1. $e=0$,		1
2. $e=1$		1
3. $e=2$		2
4. $e=3$		4
5. $e=4$		6
6. $e=5$		6
7. $e=6 \approx e=4$		6
8. $e=7 \approx e=3$		4
9. $e=8 \approx e=2$		2
10. $e=9 \approx e=1$		1
11. $e=10 \approx e=0$		1

Eulerian circuits

NO. 1-22

DATE

Note The first paper of Graph Theory deals with the existence of Eulerian circuits (eulerian circuits).

Walk : A sequence of vertices in G $\langle x_1, x_2, \dots, x_n \rangle$ such that $x_i x_{i+1}$ is an edge of G where $i=1, 2, \dots, n-1$.

Trail : A walk without repeating edges

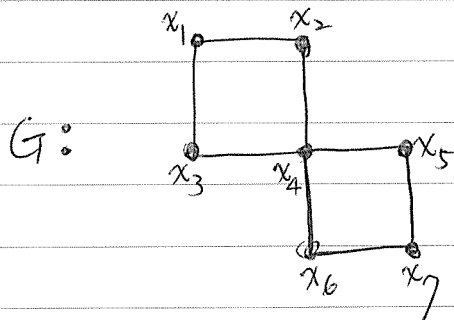
Circuit : A closed trail, i.e., the first vertex is also the last vertex.

Cycle : A circuit without repeating vertices. $(C_n) \Rightarrow \|C_n\| = n$.

Path : A trail (open) without repeating vertices. $(P_n) \Rightarrow \|P_n\| = n-1$.

Definition An eulerian circuit of a graph G (multi-graph)

is a circuit of G which contains all edges of G .



$\langle x_1, x_2, x_4, x_5, x_7, x_6, x_4, x_3, x_1 \rangle$

$= (x_1, x_2, x_4, x_5, x_7, x_6, x_4, x_3)$ is an eulerian circuit

of G .

Theorem (Euler, 1736 (1741))

A graph G contains an eulerian circuit if and only if G is connected and every vertex is of even degree.

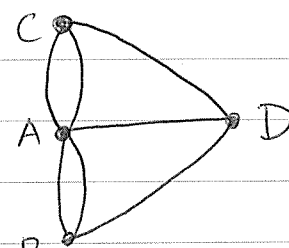
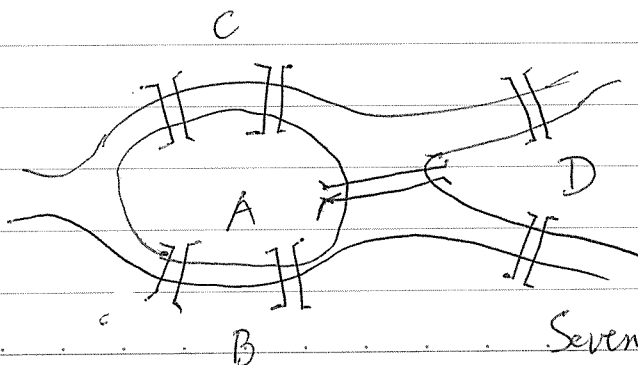
(Remark: The graph considered by Euler is a multi-graph.)

Proof. (Many versions) ($\Rightarrow$) is easy to see, we prove ($\Leftarrow$).

1st proof: Let $T = \langle x_1, x_2, \dots, x_k \rangle$ be the longest trail in G starting from x_1 and ending at x_k . Since each vertex is an even vertex, $x_1 = x_k$. Let E' be the set of edges of this trail. If $\|E'\| = \|G\|$, then the trail is an eulerian circuit. Assume that $E'' = E(G) \setminus E' \neq \emptyset$.

Since G is connected, $V(E'') \cap V(T) \neq \emptyset$. Let x_i be a common vertex and $x_i x_k \in E''$. Now, we have a longer trail starting from x_k ,

$\langle x_k, x_i, x_{i+1}, \dots, x_l, x_2, \dots, x_i \rangle$, a contraction. Hence $E'' = \emptyset$ and T is an eulerian circuit. $\square$



Seven bridges on the Pregel in Königsberg

2nd proof. (The main idea of Euler) By induction on $\|G\|$.

Since each vertex of G is even, $\delta(G) \geq 2$, G contains a cycle. Let C be a circuit in G with the maximal number of edges.

If $\|C\| = \|G\|$, then C is an eulerian circuit of G . On the other hand,

$E(G) \setminus E(C) \neq \emptyset$, i.e., C is not an eulerian circuit of G . As G is connected,

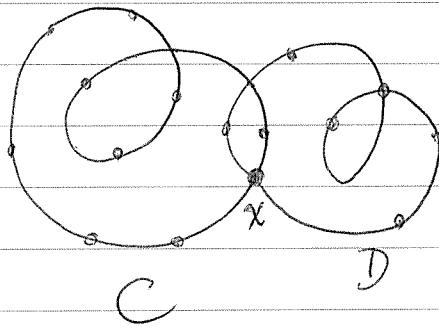
C contains a vertex x in a non-trivial component H of $G - E(C)$.

By induction H has an eulerian circuit D . Since $E(C) \cap E(D) = \emptyset$

and $V(C) \cap V(D) \supseteq \{x\}$, we obtain a larger circuit of G by

concatenating these two circuits. $\rightarrow$ Hence C is an eulerian

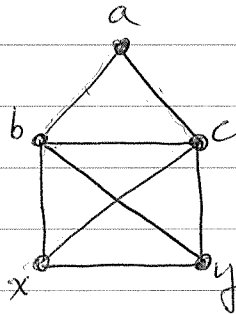
circuit of G . ▣



Remark At the time of "Euler", mathematical induction is not known yet.

Different proofs ...?

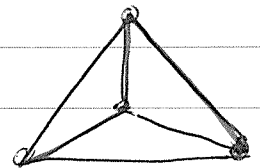
Corollary A connected graph has an eulerian trail from a vertex x to a vertex $y \neq x$ if and only if x and y are the only odd vertices.



Eulerian Trail : $\langle x, b, a, c, x, y, b, c, y \rangle$

Corollary If a connected graph G has $2m$ odd vertices, then the edges of G can be partitioned into m trails such that each trail has an even number of edges except possibly one.

(No easy to prove! Especially the second part.)

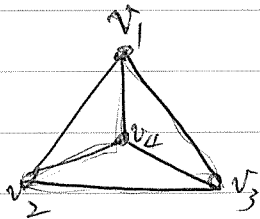


Digraph version (To be continued!)

Proof of the first part

Let the $2m$ odd vertices of G be $\{x_1, x_2, \dots, x_{2m-1}, x_{2m}\}$ and $V(G) = \{x_1, x_2, \dots, x_p\}$. Clearly $p \geq m$. Now, consider the graph $\tilde{G}$ with $V(\tilde{G}) = \{x_0, x_1, x_2, \dots, x_p\}$ and $E(\tilde{G}) = E(G) \cup \{x_0 x_i \mid i = 1, 2, \dots, 2m\}$. Then, $\tilde{G}$ is a connected even graph and thus $\tilde{G}$ has an eulerian circuit. Since the circuit passes x_0 exactly $2m$ times if we start at a vertex which is not x_0 , $\tilde{G} - x_0$ is decomposed into m trails which concludes the proof of the first part.

An example of 2nd part



$\langle v_1, v_4, v_2, v_3 \rangle$ and $\langle v_2, v_1, v_3, v_4 \rangle$
are two trails (Both have length 3)



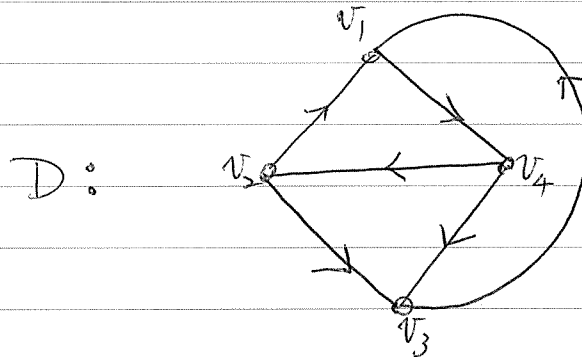
$\langle v_1, v_4, v_2, v_1, v_3 \rangle$ and $\langle v_4, v_3, v_2 \rangle$

are two even trails

Digraph (Directed Graph)

$D =$
Definition A digraph (V, A) is an ordered pair such that V is the set of vertices and A is a set of ordered pairs in (V, V) (arcs) or V^2 . $|A|$ is defined as the size of D (the number of arcs).

e.g.



$$\begin{aligned} \deg_D^+(v_1) &= 1, \deg_D^-(v_1) = 2 \\ \deg_D^+(v_2) &= 2, \\ \deg_D^+(v_3) &= 1, \\ \deg_D^+(v_4) &= 2, \end{aligned}$$

$$V(D) = \{v_1, v_2, v_3, v_4\}$$

$$A(D) = \{(v_1, v_2), (v_1, v_4), (v_2, v_3), (v_3, v_4), (v_4, v_1), (v_2, v_4), (v_4, v_2)\}$$

$$N_D^+(v) = \{u \in V \mid (v, u) \in A\}, \quad N_D^-(v) = \{u \in V \mid (u, v) \in A\}$$

$\uparrow$ out-neighbor $\uparrow$ in-neighbor

$$\deg_D^+(v) = |N_D^+(v)|, \quad \deg_D^-(v) = |N_D^-(v)|$$

$\uparrow$ out-degree of v $\uparrow$ in-degree of v

Theorem Let $D = (V, A)$ be a digraph. Then

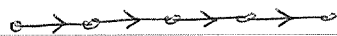
$$\sum_{v \in V(D)} \deg_D^+(v) = \sum_{v \in V(D)} \deg_D^-(v) = |A|.$$

Remark Walks, trails, circuits, cycles and paths in digraph D are defined according the directions, i.e., in a walk $\langle v_1, v_2, v_3, \dots \rangle$ $(v_1, v_2), (v_2, v_3)$ are arcs in D . For clearness, we shall say a directed path in D instead of just a path in D .

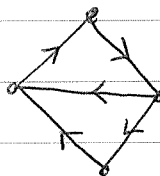
Definition (Connected digraphs)

A digraph D is connected if for any two distinct vertices x and y in $V(D)$, there exists a directed path from x to y (or) from y to x . If we replace "or" by "and", then the digraph is strongly connected.

↓
weakly
connected



weakly connected



strongly connected

Directed eulerian circuits

Theorem A digraph D has an eulerian (directed) circuit if and only if D is strongly connected and for each vertex $v \in V(D)$, $\deg^+(v) = \deg^-(v)$.

Proof. Similarly we can apply the idea used in graphs. ■

In fact, we can count the number of distinct directed eulerian circuits in an eulerian digraph. (A digraph with eulerian circuits is called an eulerian digraph.)

Theorem (BEST Theorem) ($V(D) = \{v_1, v_2, \dots, v_n\}$).

Let D be an eulerian digraph, $s(D)$ be the number of eulerian circuits, and $t_i(D)$ be the number of spanning trees oriented from v_i . Then $s(D) = t_i(D) \cdot \prod_{j=1}^n (\deg^+(v_j) - 1)!$ for every i , $1 \leq i \leq n$. (Note that $t_1(D) = t_2(D) = \dots = t_n(D)$.)

(The theorem was proved by de Bruijn, van Aardenne-Ehrenfest, Smith and Tutte.) We skip the details.

Remark

Counting the number of distinct eulerian circuits in an eulerian graph (not directed) is a very difficult problem.
(Can you do it?)

There are many applications of "eulerian circuits", please refer to lecture notes "Chapter 2". For convenience, I attach them following this page.