

The existence of paths and cyclesProposition 1

Every graph contains a path of length $\delta(G)$ and a cycle of length at least $\delta(G)+1$ (provided $\delta(G) \geq 2$).

Proof. Let $\langle x_1, x_2, \dots, x_k \rangle$ be the longest path we can find

in G . Then $N_G(x_k) \subseteq \{x_1, x_2, \dots, x_{k-1}\}$. This implies that $\delta(G) \leq$

$\deg_G(x_k) \leq k-1$, so, $k \geq \delta(G)+1$. This concludes the first part.

Now, if $\delta(G) \geq 2$, hence $\deg_G(x_k) \geq 2$. Let i be the smallest index

in $\{1, 2, \dots, k-1\}$ such that $x_i x_k \in E(G)$. This implies that $C =$

$(x_i, x_{i+1}, \dots, x_k)$ is a cycle in G . Since $\deg_G(x_k) \geq \delta(G)$,

$i \leq k - \delta$, so, $k - i \geq \delta$. Since the cycle C has $k - i + 1$

vertices, C has at least $\delta+1$ vertices. ■

(Algorithmic aspect) length of the

Remark Determining the longest path(s) in G is also known

as "finding the diameter which is not a difficult problem.

But, finding the longest cycle in a graph is not easy at all.

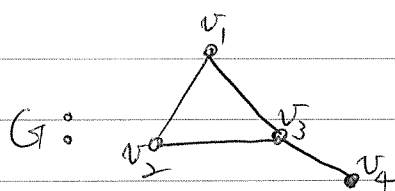
Distance in Graphs

Definition (Adjacency Matrix)

Let G be a graph with $V(G) = \{v_1, v_2, \dots, v_n\}$. Then the adjacency matrix A_G of the graph G is an $n \times n$ matrix such that

$$A_G(i,j) = \begin{cases} 1 & \text{if } v_i \sim_G v_j; \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

Example.



$$A_G = \begin{matrix} & \begin{matrix} v_1 & v_2 & v_3 & v_4 \end{matrix} \\ \begin{matrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{matrix} & \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \end{matrix}$$

Remark

1. A_G is a symmetric matrix with "0"-diagonal.

2. The eigenvalues of A_G can be "found" and they are reals.

(*) The study of the relationship of eigenvalues of G and (for distinct matrices of G) "structure of G " is known as "Algebraic Graph Theory".

3. We are interested in "distance", so, we focus on A_G^k .

$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 3 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix} \quad \left| \quad \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 4 & 1 \\ 3 & 2 & 4 & 1 \\ 4 & 4 & 2 & 3 \\ 1 & 1 & 3 & 0 \end{bmatrix} \right.$$

$v_2 \rightarrow v_3$
3-walk

Theorem (Number of k -walks) (A walk with k edges is a k -walk.)

Let G be the graph with $V(G) = \{v_1, v_2, \dots, v_n\}$ and A_G be its adjacency matrix. Then, the number of different k -walks from v_i to v_j is equal to $A_G^k(i, j)$.

Proof. By induction on k and it is clear for $k=1$.

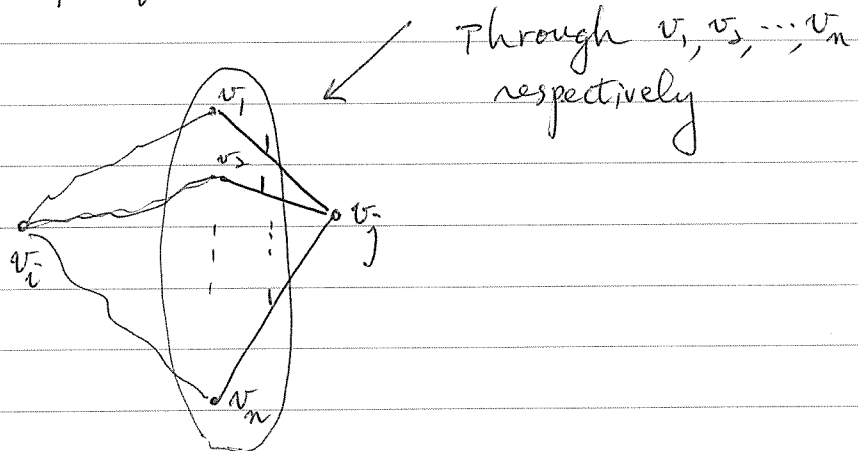
Assume the assertion is true for $1 \leq h < k$. Consider the number of $(h+1)$ -walks in $A_G^{h+1} = (A_G^h) A_G$. By multiplication of two

matrices $A_G^{h+1}(i, j) = (A_G^h(i, 1), A_G^h(i, 2), \dots, A_G^h(i, n)) \cdot \begin{matrix} \downarrow \text{Inner product} \\ A_G(1, j), A_G(2, j), \dots, A_G(n, j) \end{matrix}$.

$$(A_G(1, j), A_G(2, j), \dots, A_G(n, j)) = \sum_{t=1}^n A_G^h(i, t) \cdot A_G(t, j). \quad (1)$$

(transpose of the j th column)

Since the summation of (1) shows the number of $(h+1)$ -walks from v_i to v_j , we have the proof. ■



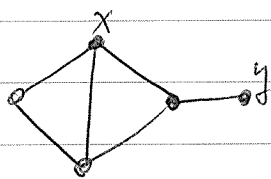
Facts

1. If $\deg_G(v_x) = 1$, then $A_G^{2m+1}(x, x) = 0$ for $m = 0, 1, 2, \dots$.
2. The number of triangles in G is equal to $\frac{c_3(G)}{6} = \frac{1}{6} \sum_{i=1}^n A_G^3(i, i) = \frac{1}{6} \text{tr}(A^3)$.
(trace)
3. For any two vertices v_i and v_j in $V(G)$, if there exists a k such that $A_G^k(i, j) > 0$, then G is a connected graph.
4. The number of 4-cycles in G is equal to $c_4(G) = \frac{1}{8} (\text{tr}(A^4) - 2\|G\| - 2 \sum_{i \neq j} A^2(i, j))$.
5. More ... $c_5, c_6, c_7, c_8, \dots$.

Review

1. The length of a path (or cycles, walk, trail, circuit) is the number of edges on the path (or ...).
2. The distance of two vertices x and y in a graph, denoted $\text{dist}(x, y)$, is the length of a shortest path between x and y .
(min. # of edges)

y.



$$\text{dist}(x, y) = 2.$$

Distance in Graphs (Continued)

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Let $d(x, y)$ denote the distance of x and y in G .
($\text{dist}(x, y)$ or $\text{dist}_G(x, y)$) (Distance is a "metric".)

Notations

1. $e_G(x) = \max \{ d(x, y) \mid y \in V(G) \}$ (Eccentricity of x in G)

2. $\text{rad}(G) = \min \{ e_G(x) \mid x \in V(G) \}$ (Radius of G)

3. $\text{diam}(G) = \max \{ e_G(x) \mid x \in V(G) \}$ (Diameter of G)
 $\parallel$
 $\max \{ d(x, y) \mid x, y \in V(G) \}$

4. x is a central vertex of G if $e_G(x) = \text{rad}(G)$.

5. $Z(G) = \{ x \mid x \text{ is a central vertex} \}$.
(Center of G)

Proposition If G is a connected graph, then $\text{diam}(G) \leq 2 \text{rad}(G)$.

Proof. Let x and y be two vertices in G such that $d(x, y) =$

$\text{diam}(G)$. Let $w \in Z(G)$, i.e., $e_G(w) = \text{rad}(G)$. Since

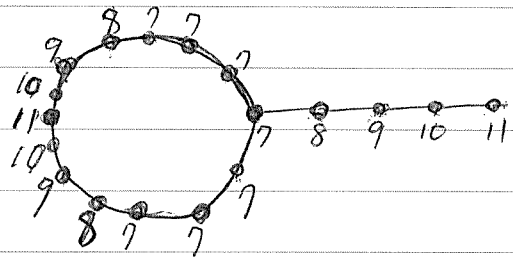
$$d(x, y) \leq d(x, w) + d(w, y) \leq e_G(w) + e_G(w) = 2 \text{rad}(G). \quad \blacksquare$$

(Triangular inequality)

(*) $\text{rad}(G) \leq \text{diam}(G)$ is easy to see.

Fact For each pair of ^{positive} integers a and b such that $a \leq b \leq 2a$, there exists a graph G with $\text{rad}(G) = a$ and $\text{diam}(G) = b$.

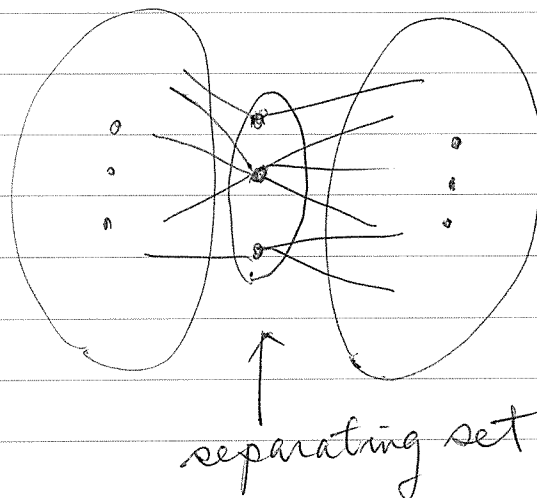
For example, $a=7$ and $b=11$. (A cycle of length $2a+1$ with a "tail" P_{b-a+1}).



(*) $a=7, b=15$ is not possible, why?

Exercise Find a 3-connected graph G such that $\text{rad}(G)=7$ and $\text{diam}(G)=11$.

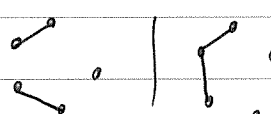
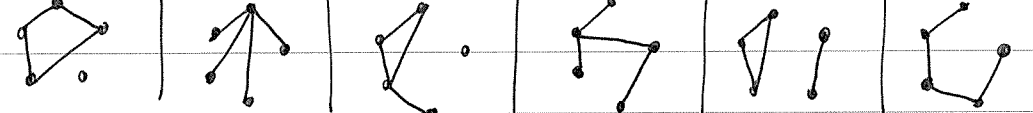
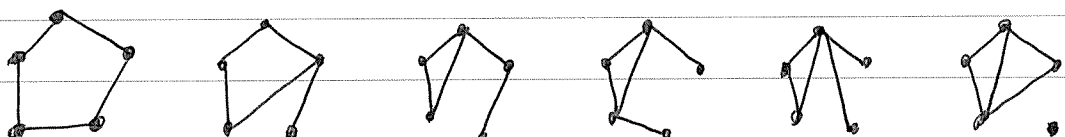
Review A graph G is k -connected if any separating set is of size at least k .



Solution to the 1st Bonus Problem

There are 34 non-isomorphic graphs of order 5.
(simple)

Idea First, we use the number of edges to partition the set of graphs of order 5.

1. $e=0$,		1
2. $e=1$		1
3. $e=2$		2
4. $e=3$		4
5. $e=4$		6
6. $e=5$		6
7. $e=6 \approx e=4$		6
8. $e=7 \approx e=3$		4
9. $e=8 \approx e=2$		2
10. $e=9 \approx e=1$		1
11. $e=10 \approx e=0$		1

Eulerian circuits

Note The first paper of Graph Theory deals with the existence of Eulerian circuits (eulerian circuits).

Walk : A sequence of vertices in G $\langle x_1, x_2, \dots, x_n \rangle$ such that $x_i x_{i+1}$ is an edge of G where $i=1, 2, \dots, n-1$.

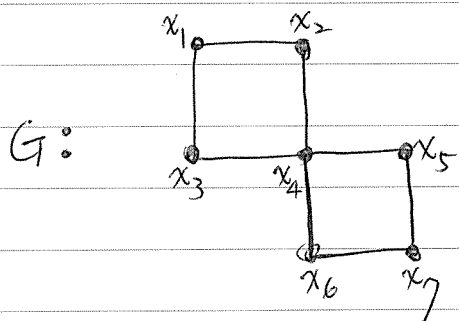
Trail : A walk without repeating edges

Circuit : A closed trail, i.e., the first vertex is also the last vertex.

Cycle : A circuit without repeating vertices. $(C_n) \Rightarrow \|C_n\| = n$.

Path : A trail (open) without repeating vertices. $(P_n) \Rightarrow \|P_n\| = n-1$.

Definition An eulerian circuit of a graph G (multi-graph) is a circuit of G which contains all edges of G .



$\langle x_1, x_2, x_4, x_5, x_7, x_6, x_4, x_3, x_1 \rangle$

$= \langle x_1, x_2, x_4, x_5, x_7, x_6, x_4, x_3 \rangle$ is an eulerian circuit

of G .

Theorem (Euler, 1736 (1741))

A graph G contains an eulerian circuit if and only if G is connected and every vertex is of even degree.

(Remark: The graph considered by Euler is a multi-graph.)

Proof. (Many versions) $(\Rightarrow)$ is easy to see, we prove $(\Leftarrow)$.

1st proof: Let $T = \langle x_1, x_2, \dots, x_k \rangle$ be the longest trail in G starting from x_1 and ending at x_k . Since each vertex is an even vertex,

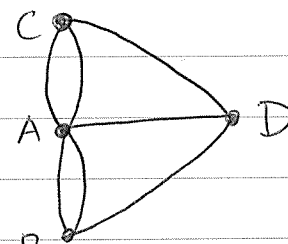
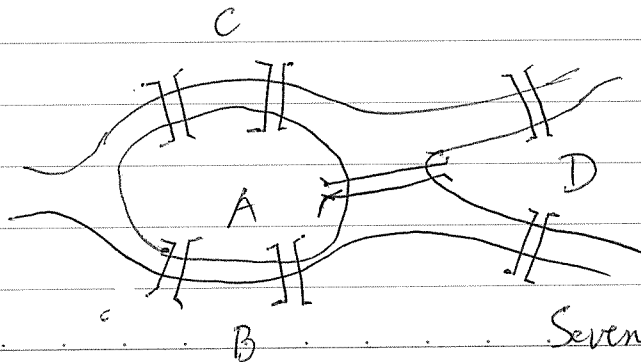
$x_1 = x_k$. Let E' be the set of edges of this trail. If $\|E'\| = \|G\|$,

then the trail is an eulerian circuit. Assume that $E'' = E(G) \setminus E' \neq \emptyset$.

Since G is connected, $V(E'') \cap V(T) \neq \emptyset$. Let x_i be a common vertex

and $x_i x_k \in E''$. Now, we have a longer trail starting from x_k ,

$\langle x_k, x_i, x_{i+1}, \dots, x_i, x_2, \dots, x_k \rangle$, a contradiction. Hence $E'' = \emptyset$ and T is an eulerian circuit. $\square$



Seven bridges on the Pregel in Königsberg

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2nd proof. (The main idea of Euler) By induction on $\|G\|$.

Since each vertex of G is even, $\delta(G) \geq 2$, G contains a cycle. Let C be a circuit in G with the maximal number of edges.

If $\|C\| = \|G\|$, then C is an eulerian circuit of G . On the other hand,

$E(G) \setminus E(C) \neq \emptyset$, i.e., C is not an eulerian circuit of G . As G is connected,

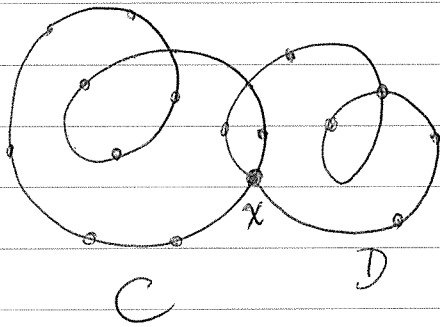
C contains a vertex x in a non-trivial component H of $G - E(C)$.

By induction H has an eulerian circuit D . Since $E(C) \cap E(D) = \emptyset$

and $V(C) \cap V(D) \supseteq \{x\}$, we obtain a larger circuit of G by

concatenating these two circuits. $\rightarrow \leftarrow$ Hence C is an eulerian

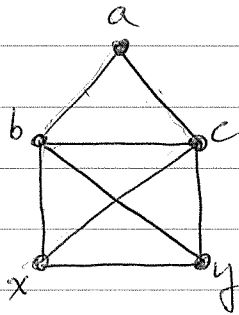
circuit of G . ▣



Remark At the time of "Euler", mathematical induction is not known yet.

Different proofs ...?

Corollary A connected graph has an eulerian trail from a vertex x to a vertex $y \neq x$ if and only if x and y are the only odd vertices.



Eulerian Trail : $\langle x, b, a, c, x, y, b, c, y \rangle$

Corollary If a connected graph G has $2m$ odd vertices, then the edges of G can be partitioned into m trails such that each trail has an even number of edges except possibly one.

(No easy to prove! Especially the second part.)

Digraph version (To be continued!)