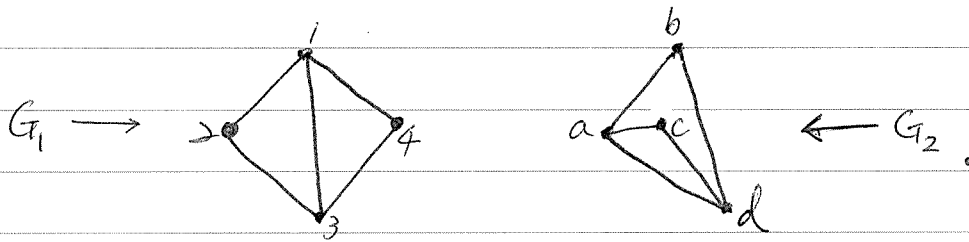


Graph Isomorphism for Simple Graphs

Given two graphs, for example the following:



Are they "the same" or not?

Fact If $V(G_1) \neq V(G_2)$ or $E(G_1) \neq E(G_2)$, then $G_1 \neq G_2$.

Observation

- (1) G_1 and G_2 are of the same order, i.e., $|G_1| = |G_2|$.
- (2) G_1 and G_2 are of the same size, i.e., $\|G_1\| = \|G_2\|$.
- (3) G_1 and G_2 are of the same "incidence" structure:

$$1 \rightarrow a, 3 \rightarrow d, 2 \rightarrow c, 4 \rightarrow b.$$

Review: Two sets A and B are of the same cardinality

if there exists a bijection between A and B , i.e., $\exists$

$$f: A \xrightarrow[\text{onto}]{1-1} B. \quad (A \text{ and } B \text{ can be infinite sets.})$$

Definition (Isomorphic Graphs)

Two graphs G_1 and G_2 are isomorphic if there exists a bijection $f: V(G_1) \rightarrow V(G_2)$ such that

$$\begin{array}{ccc} u \sim_{G_1} v & \iff & f(u) \sim_{G_2} f(v) \\ \parallel & & \parallel \\ uv \in E(G_1) & & f(u)f(v) \in E(G_2) \end{array}$$

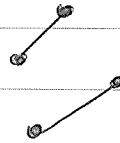
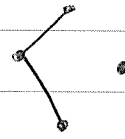
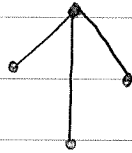
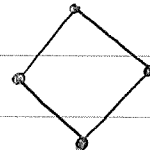
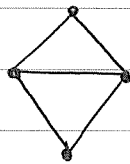
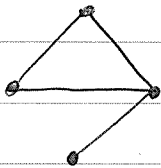
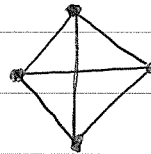
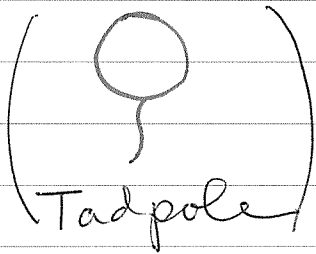
We use $G_1 \cong G_2$ to denote the fact that G_1 is isomorphic to G_2 .

Fact $G_1 \cong G_2 \Rightarrow \left\{ \begin{array}{l} \textcircled{1} |G_1| = |G_2| \\ \textcircled{2} \|G_1\| = \|G_2\| \\ \textcircled{3} \text{ Many others } \dots \\ \text{(properties)} \end{array} \right.$

Fact Two graphs are non-isomorphic if they are not isomorphic.

Fact Isomorphism is an equivalence relation defined on $G \times G$ where G is the set of all simple graphs. (graphs)

Example: Non-isomorphic graphs of order 4
(simple)

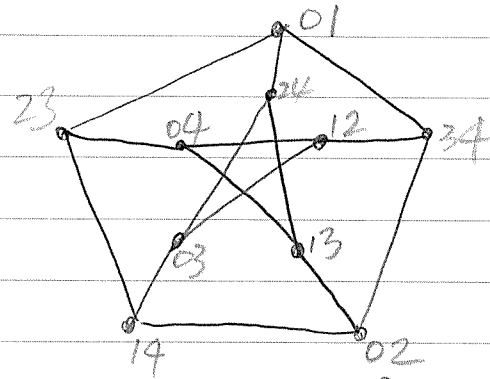
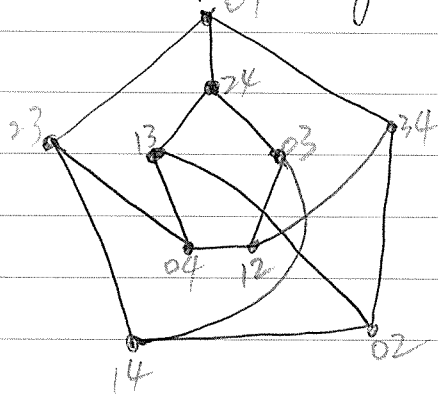
 O_4  O_2UK_2  M_2  P_3UO_1  P_4  S_3  K_3UO_1
or C_3UO_1  C_4  K_4-e  K_4 

(Tadpole)

Bonus problem 1 How many non-isomorphic graphs are
(2 points)
of order 5? (Explain your answer.)
Due Sep. 27
(before 12:00)

(*) All answers should be written and handed in personally to T.A.
(if any)

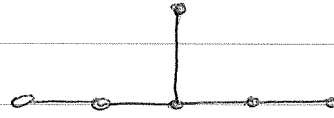
Are the following two graphs isomorphic?



Yes!

Petersen Graph

Are the following two graphs isomorphic?



No!

Fact Let $V = \binom{\mathbb{Z}_5}{2} = \{ \text{2-element subsets of } \mathbb{Z}_5 \}$

$= \{ 01, 02, 03, 04, 12, 13, 14, 23, 24, 34 \}$

and $E = \{ AB \mid A, B \in V \text{ and } A \cap B = \emptyset \}$. The (V, E) is isomorphic

to Petersen graph. (Kneser Graphs!)

Famous Graphs

(!!) So, how to determine whether two graphs are isomorphic or not. (Isomorphism Disease)

Possible approach

Definition (Degree sequence)

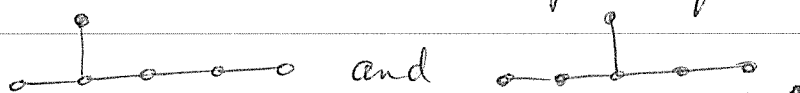
Let G be a graph and $V(G) = \{v_i \mid i = 1, 2, \dots, n\}$. Let $d_i = \deg_G(v_i)$ and $d_1 \geq d_2 \geq \dots \geq d_n$. Then, $\langle d_1, d_2, \dots, d_n \rangle$ is the degree sequence of G .

Review 1. $\forall v \in V(G)$, $N(v) = \{u \in V(G) \mid uv \in E(G)\}$ and $N[v] = N(v) \cup \{v\}$. They are the neighborhood of v and closed neighborhood of v respectively.

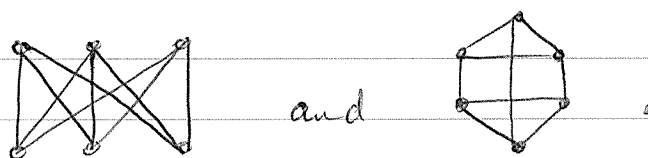
2. $|N(v)|$ is known as the degree of v . If the graph has a loop passing v , then this loop contributes two degrees. denoted by $\deg_G(v)$

3. A vertex is even (or odd) if its degree is even (or odd).

Example 1. $\langle 3, 2, 2, 1, 1, 1 \rangle$ is the degree sequence of both



2. $\langle 3, 3, 3, 3, 3, 3 \rangle$ is the degree sequence of both



Can we say something about isomorphic graphs and degree sequences?

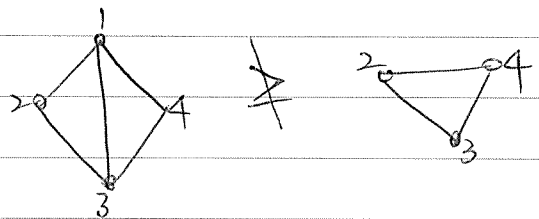
Conclusion (Trivial)

Two graphs may have the same degree sequence, but they are not isomorphic.

(Fact) If two graphs are isomorphic, then they have the same degree sequence.

Subgraphs

Definition (Strong sense)



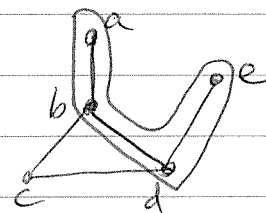
Let G and H be two graphs such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. Then H is a subgraph of G .

Definition (General version)

H is a subgraph of G if H is isomorphic to a subgraph (strong sense) of G , denoted by $H \leq G$.

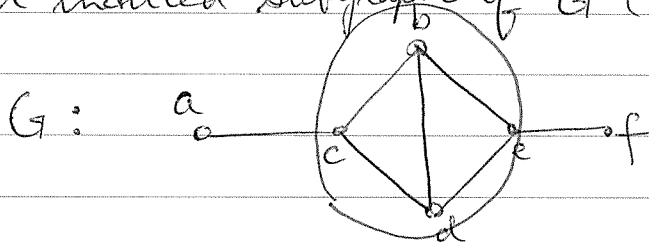


" $\leq$ "
~~~~~

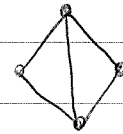


## Definition (Induced subgraphs)

Let  $G$  be a graph and  $S \subseteq V(G)$  be a non-empty (or generated) subset. The graph induced by  $S$ ,  $\langle S \rangle_G$ , is a subgraph of  $G$  such that for any two vertices  $u$  and  $v$  in  $S$ ,  $u \sim_{\langle S \rangle_G} v$  if and only if  $u \sim_G v$ .  $\langle S \rangle_G$  is called an induced subgraph of  $G$  (induced by  $S$ ).



Let  $S = \{b, c, d, e\}$ .  $\langle S \rangle_G$ :

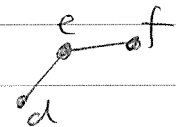


$\langle S \rangle_G \subseteq G$   
↑ induced subgraph

## Definition (Edge-induced subgraphs)

Let  $G$  be a graph and  $\tilde{E} \subseteq E(G)$ . Then the edge-induced subgraph of  $\tilde{E}$ ,  $\langle \tilde{E} \rangle_G$ , is the subgraph of  $G$  whose edge set is  $\tilde{E}$  and vertex-set is the set of vertices in  $G$  which are incident to an edge in  $\tilde{E}$ .

For example, let  $\tilde{E} = \{de, ef\}$ , then  $\langle \tilde{E} \rangle_G$ :



**(Fact)** If two graphs  $G_1$  and  $G_2$  are isomorphic, then  $H$  is a subgraph of  $G_1$  if and only if  $H$  is a subgraph of  $G_2$ .

## Definition ( $H$ -free graphs).

A graph  $G$  is said to be  $H$ -free if  $H$  is not a subgraph of  $G$ .

Example. Petersen graph is  $C_3$ -free and also  $C_4$ -free.

Review. A cycle with  $n \geq 3$  vertices is denoted by  $C_n$ .

## Definition (Extremal graphs)

Consider all graphs on  $n$  vertices. Let

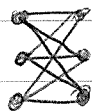
$ex(n; H) = \max\{|E(G)| \mid G \in \mathcal{G}_n \text{ and } G \text{ is } H\text{-free}\}$ . The

graphs  $G$  of order  $n$  and size  $ex(n; H)$  are known as extremal graphs which forbids  $H$ .

Example.  $n=6$ ,  $ex(6; C_3) = 9$ . (why?)

Note Two steps to show.

Step 1. There exists a graph of <sup>order 6 and</sup> size 9 which is  $C_3$ -free.



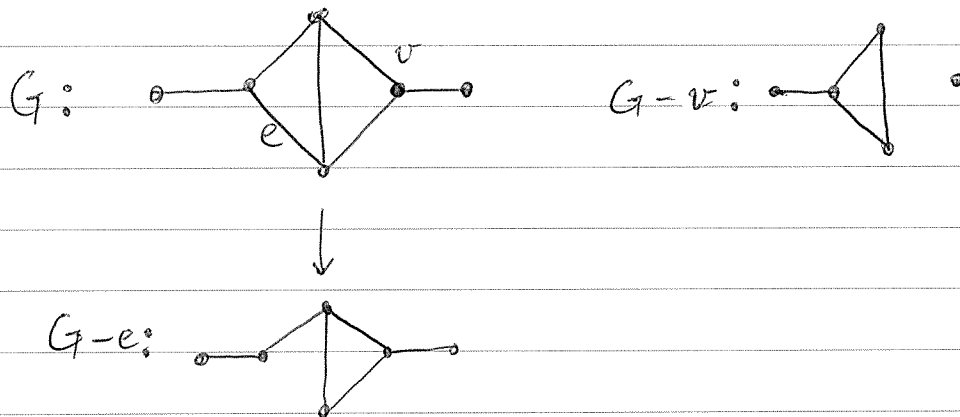
$$(ex(6; C_3) \geq 9)$$

Step 2. All graphs of order 6 and size  $\geq 10$  contain a  $C_3$ .  
(Harder step!).  $(ex(6; C_3) \leq 9)$ .

Remark To determine  $ex(n; H)$  for a given  $H$  is a very difficult problem in general. But, for some special  $H$ , say  $K_k, P_k, \dots$ ,  $ex(n; H)$  can be determined.

Definition (Deletion of vertices (and/or) edges)

Let  $v \in V(G)$ . The graph  $G-v$  is obtained by deleting the vertex  $v$  and all the edges incident to  $v$ . The graph  $G-e$  where  $e \in E(G)$  is a subgraph of  $G$  with the deletion of  $e$  from  $E(G)$ .



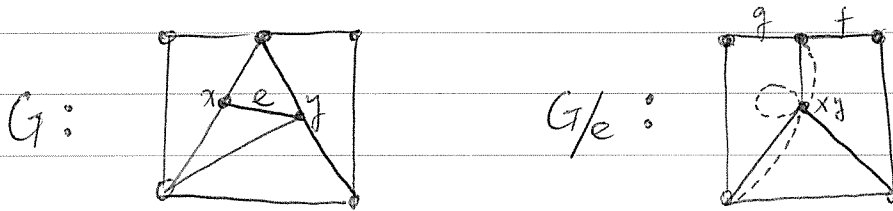
Let  $S \subseteq V(G)$  and  $T \subseteq E(G)$ .  $G-S$  and  $G-T$  can be defined accordingly. (Take away vertices (or edges) one by one.)

## Definition (Graph minors)

A graph  $M$  is called a minor of  $G$  if  $M$  can be obtained from  $G$  by contracting edges, deleting vertices and edges.

## Review (Edge-contraction)

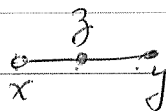
Given an edge  $xy \stackrel{=}{=} e$  of a graph  $G$ , the graph  $G/e$  is obtained from  $G$  by contracting  $e$ ; that is to identify the vertices  $x$  and  $y$  and <sup>deleting</sup> resulting loops and duplicate edges.



Example  $K_4$  is a minor of the above  $G$ .  
(Contracting  $e, f$  and  $g$ .)

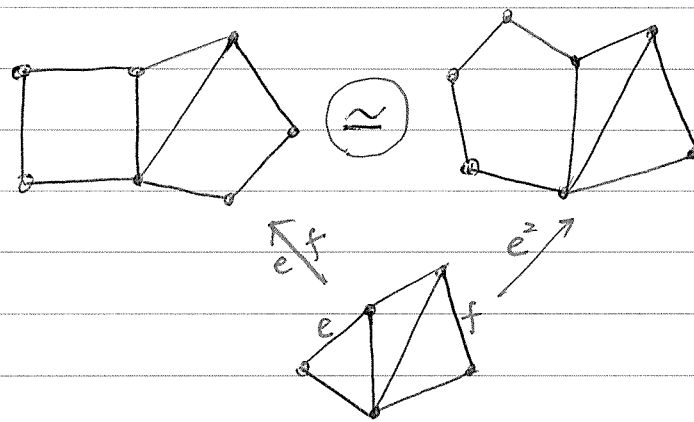
## Definition (Subdivision)

A subdivision of an edge  $xy$  is obtained by adding a new vertex  $z$  such that we have edges  $xz$  and  $zy$ .



## Definition (Homeomorphic Graphs)

Two graphs are homeomorphic if they can be obtained by subdividing edges (consecutively) of a fixed graph.



(These two graphs are "topologically" the same.)

Remark : Two cycles are homeomorphic.

# Basic properties of graphs

NO. 1-12  
DATE

## About "Degrees" in $G$

$\Delta(G)$ : maximum degree

$\delta(G)$ : minimum degree

$d(G)$ : average degree  $(\delta(G) \leq d(G) \leq \Delta(G))$

$\varepsilon(G)$ : average size (of one vertex)  $\varepsilon(G) = \frac{1}{2}d(G)$ .

$$\|G\| = \frac{1}{2} \sum_{v \in V(G)} \deg_G(v) \quad (\text{Total number of edges})$$

$$\varepsilon(G) = \|G\|/|G|$$

(\*)  $\sum_{v \in V(G)} \deg_G(v)$  is even. (The 1<sup>st</sup> theorem of Graph Theory.)

(By shake-hand principle.)

Corollary The number of odd vertices in a graph is "even".

A couple of good examples which use the above corollary.  
(facts)

Example 1 As in Figure 1,  $ABC$  is a triangle,  $A, B$  and  $C$  are labelled 1, 2 and 3 respectively. All the vertices on  $AB, BC$ , and  $CA$  are labelled with  $(1, 2), (2, 3)$  and  $(1, 3)$  respectively. And all the vertices inside  $ABC$  can be labelled with 1, 2 or 3 arbitrarily. Show that there exists a triangle inside  $ABC$  such that its vertices are labelled with 1, 2 and 3 respectively.

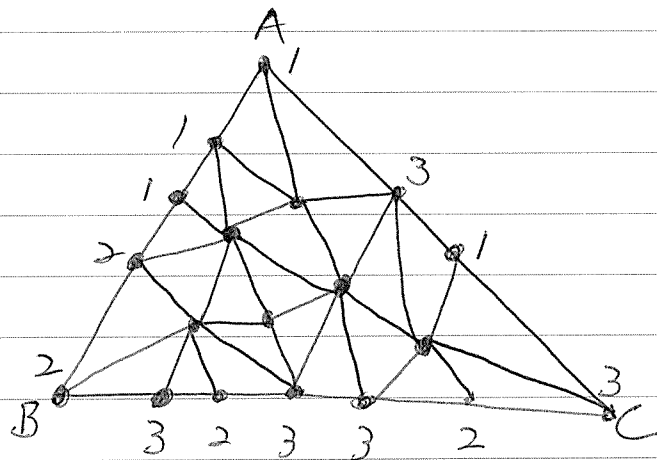


Figure 1, Triangulation of  $ABC$

Hint: Construct a graph and then prove that there exists a vertex of odd degree inside  $ABC$ .

Example 2 As in Figure 2, a large rectangle  $T$  is tiled

with smaller rectangles  $T_1, T_2, \dots, T_k$ . If each  $T_i$  has an integer side then so does  $T$ .

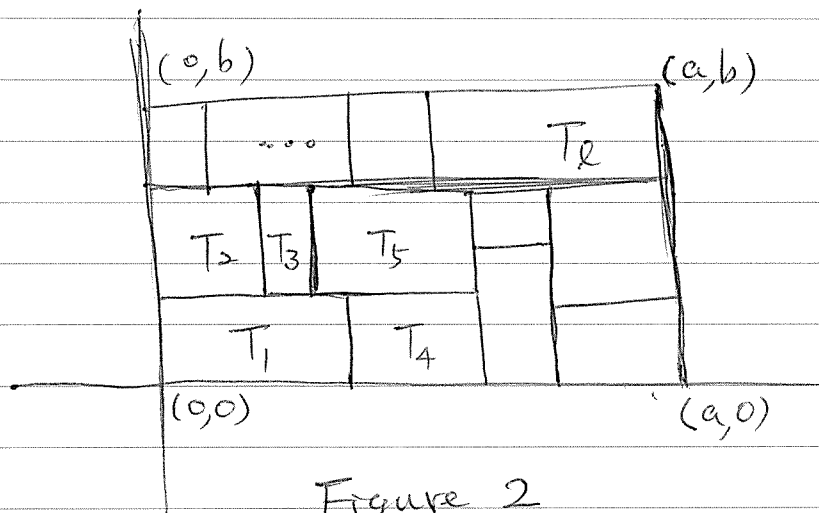
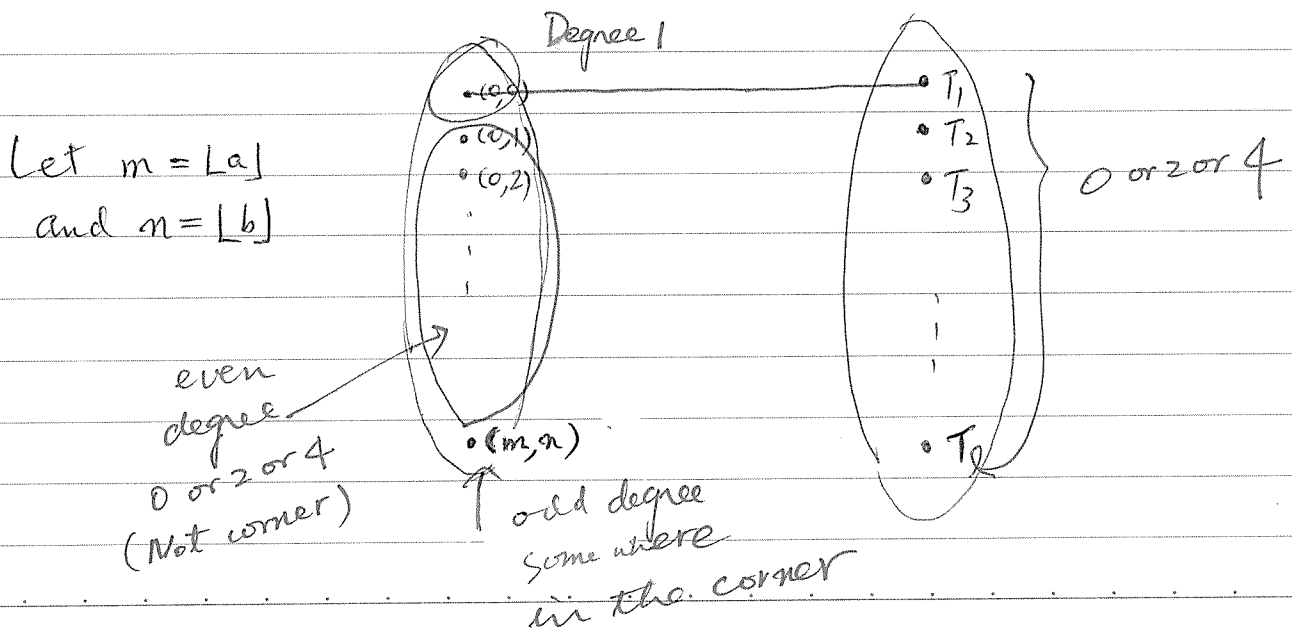


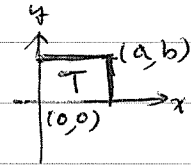
Figure 2

Hint: Construct a bipartite graph and then prove that  $a$  or  $b$  is an integer.



Another proof of Example 2. (Use Calculus.)

Set  $F(x, y) = \sin 2\pi x \cdot \sin 2\pi y$ .



$$\text{Then } \iint_T F(x, y) dx dy = \left(\frac{1}{2\pi}\right)^2 (1 - \cos 2\pi a)(1 - \cos 2\pi b)$$

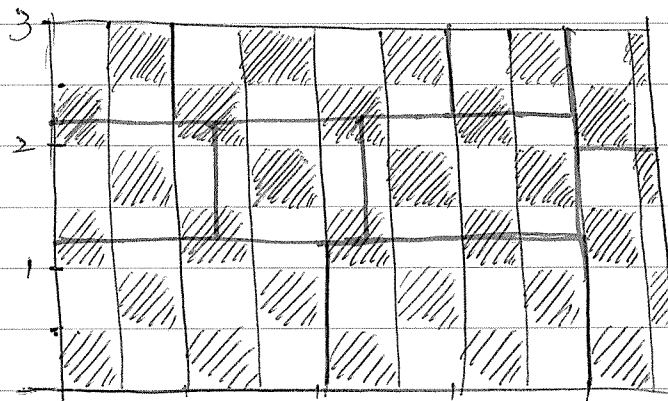
$$= \sum_{i=1}^k \iint_{T_i} F(x, y) dx dy$$

$$= 0. \quad (\text{See next page for details.})$$



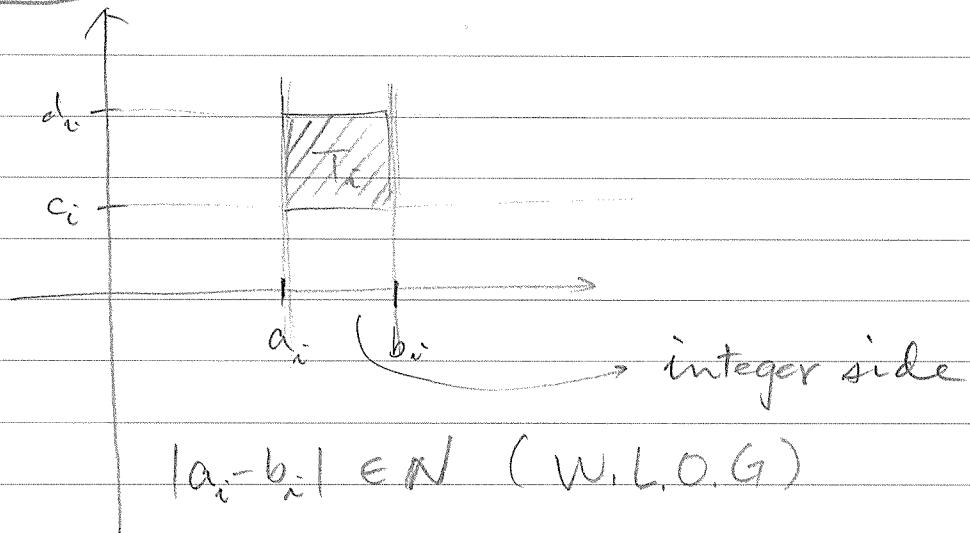
At least one of  $a$  and  $b$  is an integer.

Another proof (High school students)



Use black and white to color  $\frac{1}{2} \times \frac{1}{2}$  grids as above. If a rectangle has an integer side, then the "black area" is equal to "white area". This statement is also true conversely. Hence, we have the proof.

For reference



$$\iint_{T_i} \sin 2\pi x \sin 2\pi y \, dx \, dy$$

$$= \int_{c_i}^{d_i} \int_{a_i}^{b_i} \sin 2\pi x \sin 2\pi y \, dx \, dy$$

$$= \int_{c_i}^{d_i} \left( \int_{a_i}^{b_i} \sin 2\pi x \, dx \right) \sin 2\pi y \, dy$$

$$= \int_{c_i}^{d_i} \sin 2\pi y \cdot \left( \left. \frac{-1}{2\pi} \cos 2\pi x \right|_{a_i}^{b_i} \right) dy$$

$$= \int_{c_i}^{d_i} \sin 2\pi y \cdot \left[ \frac{-1}{2\pi} (\cos 2\pi b_i - \cos 2\pi a_i) \right] dy$$

$$\text{Let } \theta_1 = 2\pi b_i, \theta_2 = 2\pi a_i.$$

$$\cos \theta_1 - \cos \theta_2$$

$$= -2 \sin \frac{\theta_1 + \theta_2}{2} \sin \frac{\theta_1 - \theta_2}{2}$$

$$= -2 \sin \frac{(a_i + b_i)2\pi}{2} \sin \frac{(b_i - a_i)2\pi}{2}$$

$$= -2 \sin \frac{(a_i + b_i)2\pi}{2} \cdot \sin t\pi \text{ where } b_i - a_i = t \in \mathbb{N}$$

//

0

### Remark

1. There are other ways to prove this assertion.
2. This problem can be extended to higher dimensions. For example, in 3-dimension, if the small "boxes" have "k" integer sides,  $k=1$  or  $2$ , (or  $3$ ), then the large one also has (at least)  $k$  integer sides.  
(at least) ✓

Theorem Let a box  $B$  in  $\mathbb{R}^n$  be tiled with boxes  $B_1, B_2, \dots, B_\ell$ .

If each  $B_i$  has at least  $k$  integer sides, then  $B$  has at least  $k$  integer sides.