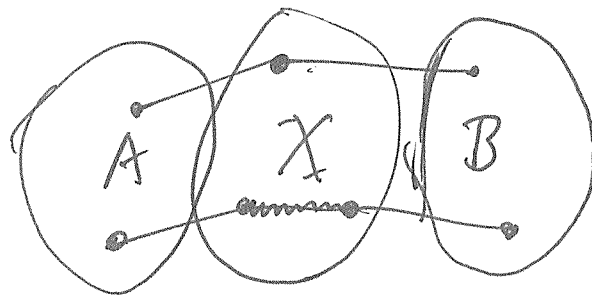


(10,7)

If $A, B \subseteq V$ and $X \subseteq V \cup E$ are such that every A - B path in G contains a vertex or an edge in X , we say that X separates the set A and B in G .

More generally we say that X separates G if $G - X$ is disconnected, that is, if X separates in G some two vertices that are not in X .



Definition (Separator)

A separating set of vertices is a separator.

$X = \{v\} \Rightarrow v$ is a cut vertex.

$X = \{e\} \Rightarrow e$ is a bridge.

G is called k -connected (for $k \in \mathbb{N}$) if $|G| > k$ and $G - X$ is connected for every $X \subseteq V$ with $|X| < k$.

Definition (Connectivity)

The greatest integer k such that G is k -connected is the connectivity $\kappa(G)$ of G .

(*) $\kappa(G) = 0$ iff G is disconnected or $G \cong K_1$.

(*) $\kappa(K_n) = n-1$ for all $n \geq 1$.

Good properties

Fact 1

A vertex v of a connected graph G is a cutvertex of G iff there exist u and w such that v is on every u - w path of G .

(A-B path can be defined accordingly.)

Fact 2

Every nontrivial graph contains at least two vertices that are not cut-vertices.

Fact 3

The vertices of a connected graph G can always be enumerated, say as $v_1, v_2, \dots, v_n$, so that $\langle \{v_1, v_2, \dots, v_i\} \rangle$ is connected for each $i \in \{2, 3, \dots, n\}$.
with $V = \{v_1, v_2, \dots, v_n\}$

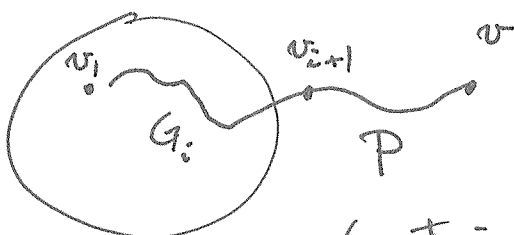
Proof. By induction. Pick any vertex as v_1 .

$i = 2$. True.

Let $v_1, v_2, \dots, v_i$ be chosen so that $\langle \{v_1, v_2, \dots, v_i\} \rangle = G_i$

is connected. Consider $G - G_i$ and let $v \in V(G - G_i)$.

Since G is connected, there is a $v_1 - v$ path P .



Choose v_{i+1} as the first vertex (not in G_i) which is on P .

Then v_{i+1} has a neighbor in G_i and thus

$\langle G_i \cup \{v_{i+1}\} \rangle_G$ is connected. This concludes the proof by induction hypothesis.

Fact 4

The vertices of a tree T (connected graph of minimum size) can always be enumerated, say as, $v_1, v_2, \dots, v_n$ so that every v_i with $i \geq 2$ has a unique neighbor in $\langle \{v_1, v_2, \dots, v_{i-1}\} \rangle_T$.

Note T is a tree if and only if for any two distinct vertices in $V(T)$ there exists a unique path.

Block

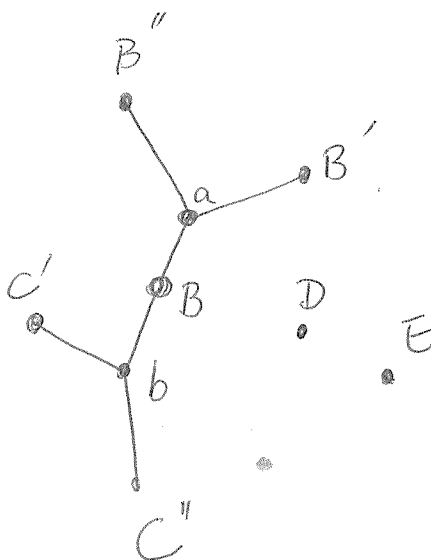
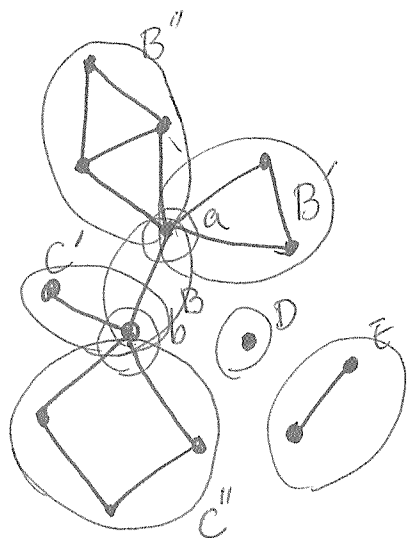
10,14

4

A maximal connected subgraph without a cutvertex is called a block.

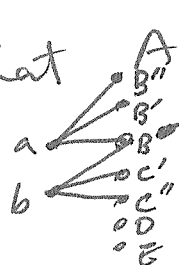
Fact

Every block of a graph G is either a maximal 2-connected subgraph, or a bridge (with its ends), or an isolated vertex.



Let A denote the set of cutvertices of G , and B the set of its blocks. Then, we have a bipartite graph (A, B) where $a \in A$ is incident to $B \in B$ iff a is a vertex of $V(B)$.

The above example shows that $A = \{a, b\}$ and $B = \{B', B'', B, C', C'', D, E\}$



Fact 5

The block graph of a connected graph G is a tree. Moreover, if G is 2-connected, then the block graph contains exactly one vertex.

Review

What is a tree? A forest?

Definition A tree is an connected acyclic graph.

or

Definition A tree is a ^(of order n) connected graph with minimum size _($n-1$).

Equivalent statements:

- ①
- ②
- ③
- ④
- ⑤
- ⋮

Important Idea

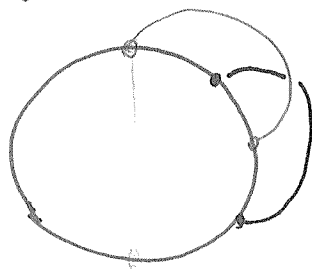
The first type of graphs you "check" are trees!

For examples:

- ① Find the independence number, $\alpha(G)$, of a graph G .
- ② Find the maximum matching of a graph G , $\alpha'(G)$.
- ③ Find the prime or graceful labeling of a graph G .
- ④ ...

The structure of 2-connected graphs

A graph G is 2-connected iff it can be constructed from a cycle by successively adding H-paths to graphs H already constructed.



H-H paths
called H-paths

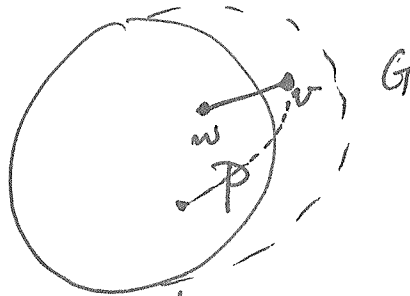
Proof.

Clearly, every graph constructed as described is 2-connected.

Conversely, let G be a 2-connected graph which is given. Then G contains a cycle C , and hence we may start from this cycle C to construct a maximal subgraph H constructible as above.

Since $\forall xy \in E(G) \setminus E(H)$ with $x, y \in H$ would define (b.e.) an H -path, H is an induced subgraph of G .

Thus, if $H \neq G$, i.e. $V(G) \setminus V(H) \neq \emptyset$, then by the connectedness of G there is an edge $vw \in E(G)$ such that $v \in V(G) \setminus V(H)$ and $w \in V(H)$.



Since G is 2-connected $G - w$ contains a v - H path P .

Then wvP is an H -path in G , and $H \cup wvP$ is a constructible subgraph of G larger than H . $\rightarrow \square$

The structure of 3-connected graph

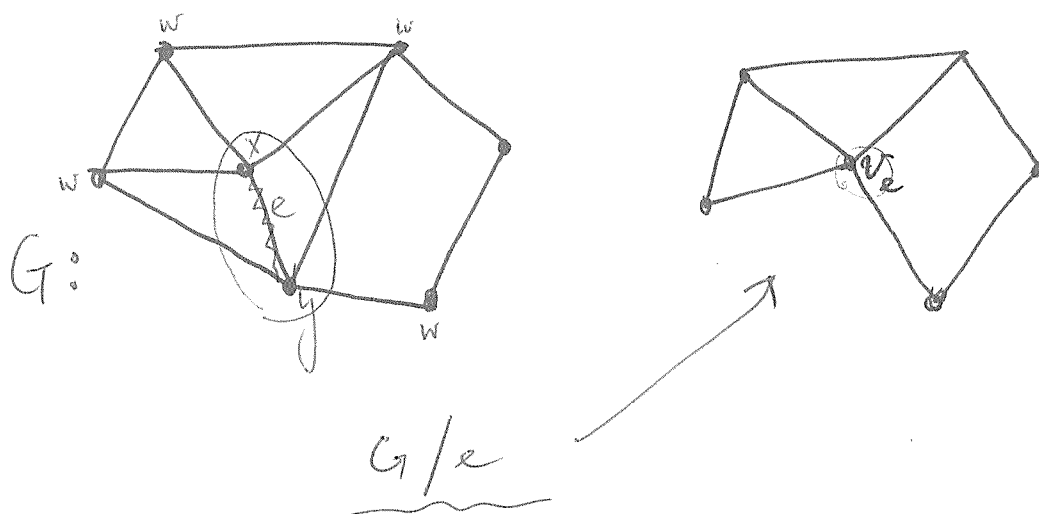
8

Definition (Contraction of an edge)

Let $e \stackrel{\text{def}}{=} xy$ be an edge of G , the graph obtained by contracting e , denoted by G/e , is the graph $G/e =$

(V', E') with $V' \stackrel{\text{def}}{=} (V \setminus \{x, y\}) \cup \{v_e\}$ and

$E' = \{vw \in E \mid \{v, w\} \cap \{x, y\} = \emptyset\} \cup \{v_e w \mid xw \in E \setminus \{e\} \text{ or } yw \in E \setminus \{e\}\}$

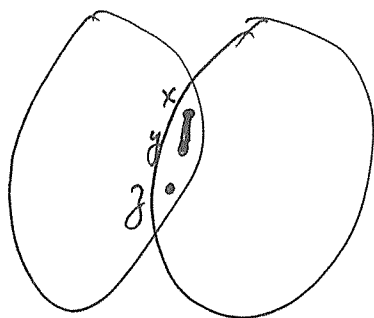


Lemma If G is 3-connected and $|G| > 4$, then G has an edge e such that G/e is also 3-connected.

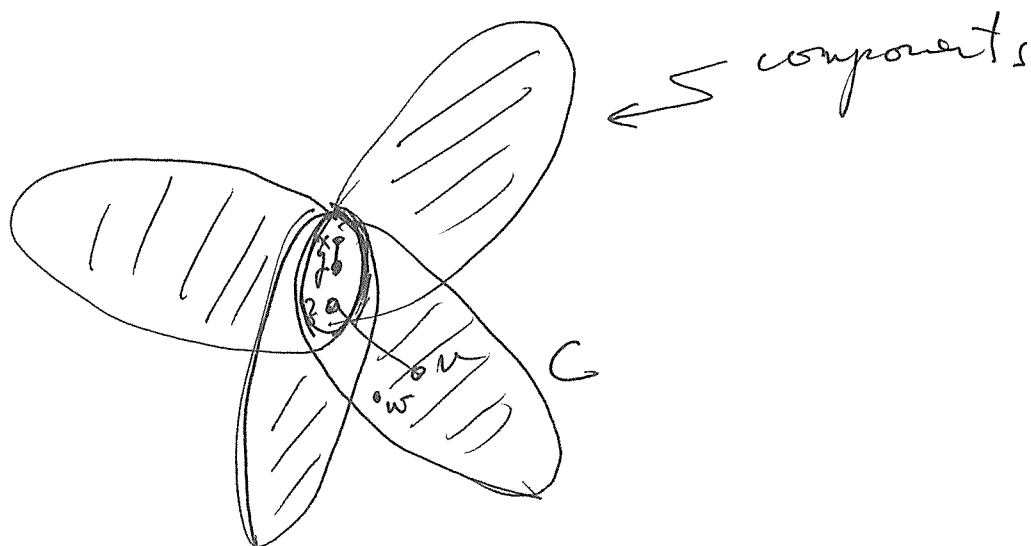
Proof. Suppose not. Then, $\forall xy \in E(G)$, G/xy has a separator S of size $\wedge 2$ at most. Since $\kappa(G) \geq 3$, $v_{xy} \in S$ and $|S| = 2$.

Let z be the other vertex in S , i.e., $\{v_{xy}, z\}$ is a separator of G/xy . This implies $T = \{x, y, z\}$ is a separator of G .

9.



Since no proper subset of T separates G , every vertex of T has a neighbor in every component of $G - T$.



We choose the component " C " which contains xy and z ; and $|C|$ is minimum (among all components).

Let v be a neighbor of z in C . Again $G_{/zv}$ is not 3 -connected.

By similar argument, there exists a ^{proper} component D in C which does not contain $\{x, y\}$, i.e., D is also a component of $G - T$, but

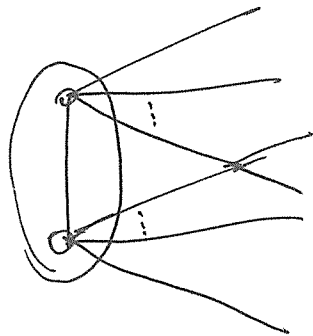
$D \subseteq C$.

Theorem (Tutte, 1961)

A graph G is 3-connected if and only if there exists a sequence $G_0, G_1, \dots, G_n$ of graphs with the following properties:

- (i) $G_0 = K_4$ and $G_n = G$;
- (ii') G_{i+1} has an edge xy with $\deg(x), \deg(y) \geq 3$ and

$$G_i = G_{i+1} / xy, \text{ for every } i < n.$$



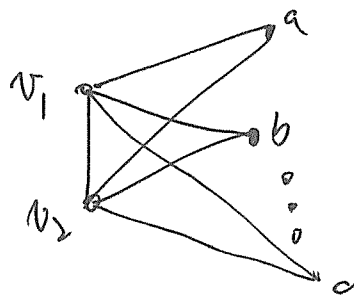
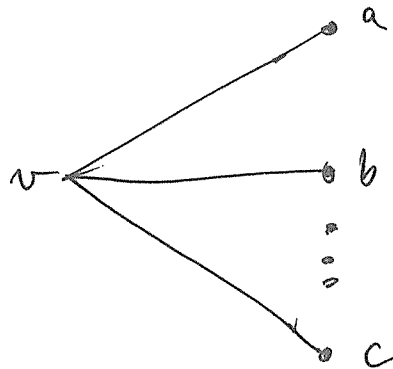
($\overset{v}{\circ} \rightarrow \overset{v}{\circ} \circ$), Splitting a vertex of degree 3 and join them!)

Theorem (Tutte, Later)

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G is a 3-connected graph iff G itself is a wheel graph W_k , $k \geq 3$, or G can be obtained by the following two operations:

- (1) Add a new edge; or
- (2) split a vertex of degree ≥ 3 (at least) into v_1 and v_2 and join v_1 and v_2



Using degree sequence ($|G|=p$)

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(10, 16)

$$d_1 \leq d_2 \leq \dots \leq d_p, \quad 1 \leq n \leq p-1.$$

$$\forall \quad 1 \leq k \leq \left\lfloor \frac{p-n+1}{2} \right\rfloor, \quad d_k \leq k+n-2 \Rightarrow d_{p-n+1} \geq p-k$$

$\Rightarrow G$ is n -connected.

Proof. Suppose not. $\exists S, |S| \leq n-1$ such that $G-S$ is not connected. Let H be a component of $G-S$ with minimum order, say $|H|=k$. Clearly, $k \leq \left\lfloor \frac{p-(n-1)}{2} \right\rfloor$.

Now, consider the degrees of the vertices in H .

$\deg_H(u) \leq (k-1) + (n-1)$ for each $u \in V(H)$. Since H has k vertices, this implies that $d_k \leq (k-1) + (n-1)$ i.e. $d_k \leq k+n-2$. By assumption, $d_{p-n+1} \geq p-k$.

Since, the vertices in $V(G) \setminus (V(H) \cup S)$ can have degree at most $(p-1)-k = p-k+1$, only vertices in S can have such large degree $p-k$. Now, $d_{p-n+1} \geq p-k$ implies that $d_{p-n+2}, d_{p-n+3}, \dots, d_p \geq p-k$. In total, there are n vertices which are of degree at least $p-k$. But, $|S|=n-1$, a contradiction. ▀

Two vertices version of (Menger's Theorem) 13

$$A = \{a\}$$



$$B = \{b\}$$

Theorem (Menger, 1927)

Let $\kappa(u, v)$ denote the minimum number of vertices in G which separates u and v in G . Then, G contains exactly $\kappa(u, v)$ vertex-disjoint $u-v$ paths.

Proof. By induction on $\kappa(u, v)$ and it is clearly true when $\kappa(u, v) = 1$. Clearly, if there are k vertex-disjoint paths from u to v , then we need at least k vertices to separate u and v .

Hence, it suffices to prove that if $\kappa(u, v) = h$, then there exist at least (exactly) h vertex-disjoint $u-v$ paths.

Suppose not. Let G be the minimal counterexample, i.e., for each $e \in E(G)$, such that $\kappa_{G-e}(u, v) = h-1$ and we can find at most $h-1$ vertex-disjoint $u-v$ paths whenever $\kappa_G(u, v) = h$.

Now, we characterize the structure of such a graph G .

(1) $N_G(u) \cap N_G(v) = \emptyset$

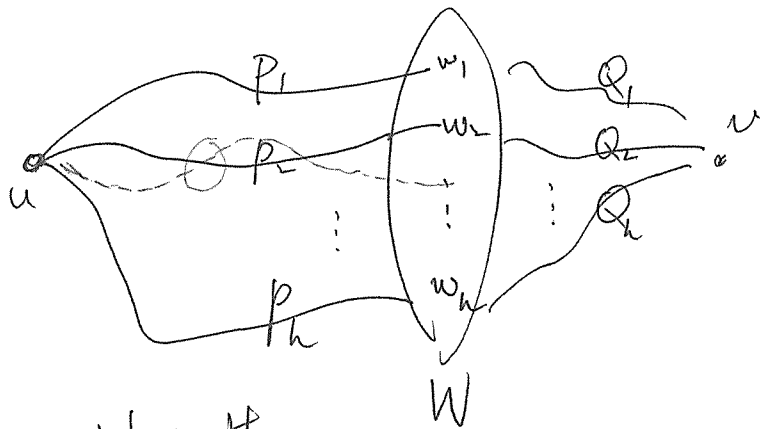
Suppose not. Let $w \in N_G(u) \cap N_G(v)$ and let $e = uw$.

Let $S(e)$ be a separating set of $G-e$. By assumption $|S(e)| = h-1$.

Observe that w is not incident to any vertex of $S(e)$. For 14
 otherwise $S(e)$ is not a separating set of $G-e$. This is because
 of $G-e$ has $h-1$ vertex disjoint $u-v$ paths (by induction)
 which pass each vertex of $S(e)$ respectively. But, now we
 have h vertex disjoint paths in G by adding $u-w-v$ path
 in G .
 $\wedge$
 $u-v$ in G (special one).

(2) If W is a separating set of size h , then either
 $W \subseteq N_G(u)$ or $W \subseteq N_G(v)$.

By observation, there are h vertex disjoint $u-W$ paths.
 For otherwise, we can obtain a smaller separating set.



Also, there are h $v-W$ paths, $Q_1, Q_2, \dots, Q_h$.
 vertex-disjoint

Now, if there exist paths P_i and Q_j which are of
 length ≥ 2 , then we can define two graphs G_u and G_v
 by replacing $\{Q_1, Q_2, \dots, Q_h\}$ with $\{vw_1, vw_2, \dots, vw_h\}$ and
 $\{P_1, P_2, \dots, P_h\}$ with $\{uw_1, \dots, uw_h\}$ respectively from G .

Then both G_u and G_v are of order less than that of G . By induction, since W is still a separating set of size h , we have h vertex-disjoint (u,v) -paths in both G_u and G_v . Now, combining the above two graphs suitably, we have h vertex-disjoint (u,v) -paths in G , a contradiction.

Finally, w.l.o.g. let p be a shortest $u-v$ path in G . By (1), p is of length at least 3. Let $p = \langle u, u_1, u_2, \dots, v \rangle$. By considering the edge $e = u_1u_2$, we notice that there exists a set of size $h-1$, $S(e)$, such that both $S(e) \cup \{u\}$ and $S(e) \cup \{u_2\}$ are separating set of h (of G). From p , we see that $u \sim_G u_1$ and $u_1 \not\sim_G v$. This implies that u is incident to every vertex of $S(e) \cup \{u_1\}$. By (1), no vertex of $S(e)$ is incident to v . So, for separating set $\underline{S(e) \cup \{u_2\}}$, each vertex is incident to u and thus $u \sim_G u_2$. $\rightarrow \leftarrow$. This concludes the proof.

Menger's Theorem ¹⁶ (Cornerstone of graph theory)

(Menger, 1927)

Let $G = (V, E)$ be a graph and $A, B \subseteq V$. Then the minimum number k of vertices separating A from B in G is equal to the maximum number k of disjoint A - B paths in G .

Proof. There are many different proofs in literatures, we provide one here.

Clearly, if there are k disjoint A - B paths in G , then an A - B separating set contains at least k vertices.

So, it suffices to prove that if an A - B separating set is of $\wedge$ ^{minimum} size k , then we can find k A - B vertex-disjoint

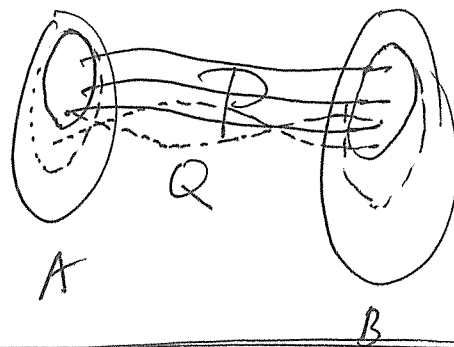
A - B paths. In fact, in what follows, we prove a stronger statement:

(*) If P is any set of fewer than k disjoint A - B paths in G , then G contains a set of $|P|+1$ disjoint A - B paths exceeding P .

(Note)

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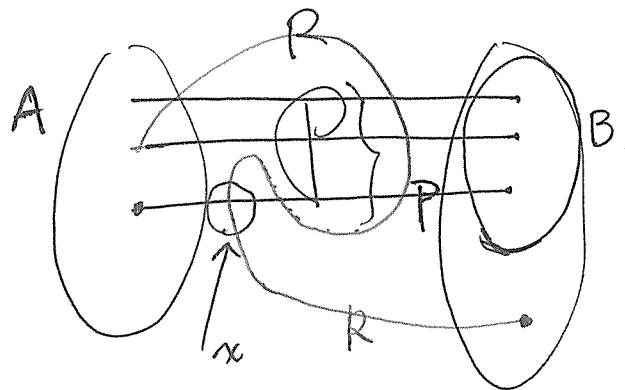
Let $\mathcal{P}$ be a set of disjoint A - B paths, and let $\mathcal{Q}$ be another such set. We say $\mathcal{Q}$ exceeds $\mathcal{P}$ if the set of vertices in A that lie on a path in $\mathcal{P}$ is a proper subset of vertices in A that lie on a path in $\mathcal{Q}$, and likewise for B . Then, in particular $|\mathcal{Q}| \geq |\mathcal{P}| + 1$.



$$\mathcal{P} = \{P_1, P_2, \dots, P_k\}$$

$$UP = \bigcup_{i=1}^k V(P_i)$$

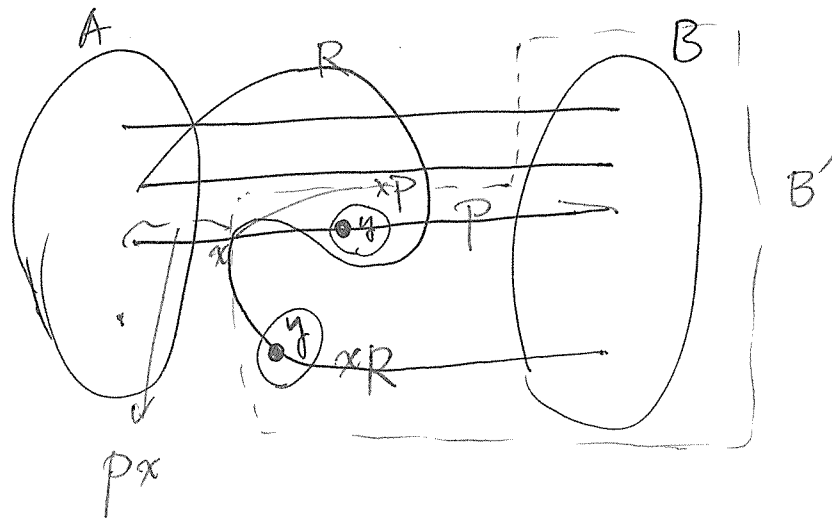
Keeping G and A fixed (as given), we let B vary and use induction on $|UP|$, i.e., the set of vertices which are on paths in $\mathcal{P}$.



Let R be an A - B path which uses the vertex in $B \setminus UP$ as its end vertex. If $V(R) \cap UP = \emptyset$, then $\mathcal{P} \cup \{R\}$ exceeds $\mathcal{P}$, as desired.

On the other hand, see above Figure, let x be the last vertex of R that lies on some $P \in \mathcal{P}$.

Now, let $B' =_{\text{def.}} B \cup V(xP \cup xR)$ and $\mathcal{P}' = (\mathcal{P} \setminus \{P\}) \cup \{P_x\}$
 $|\mathcal{P}'| = |\mathcal{P}|.$



Clearly, $|\mathcal{P}'| < |\mathcal{P}|$ and the size of (A, B) separating set is not greater than (A, B') separating set.
 that of an

By induction, there is a $\mathcal{Q}$ of $|\mathcal{P}|+1$ disjoint $A-B'$ paths exceeding $\mathcal{P}'$. This implies that $\mathcal{Q}$ contains a path Q ending in x , and a unique path Q' whose last vertex is not among the last vertices of the paths in $\mathcal{P}'$.

Case 1. $y \notin xP$.

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Let Q be obtained from Q' by adding QxP and $Q'yR$ and deleting Q and Q' . So,

$$|Q| = |P| + 1.$$

Case 2. $y \in xP$.

Use $Q'yP$ and QxR for Q and Q' .

In both cases Q contains $|P| + 1$ vertex-disjoint $A-B$ paths which exceeds P . ▣