

The Decycling Number of Graphs and Digraphs

Hung-Lin Fu

Department of Applied Mathematics,
National Chiao Tung University, Taiwan

This is a joint work with Huilan Chang, Min-Yun Lien and
Chie-Huai Shih.

Outline

- Decycling Problem and Applications
- Decycling Number and Cycle Packing Number
- Decycling Number
 - Path Product
 - Generalized Kautz Digraph
- Total decycling number
- Objective Work

Definition and Applications

Definition (Decycling Problem)

Given a directed/undirected graph $G = (V, E)$, find a minimum set $D \subset V$ such that $G \setminus D$ is acyclic.

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Process A is waiting for the resource on Process B so it can't release its own resource.

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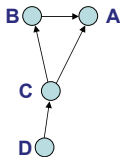
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Process A is waiting for no resource so it can release its resource.

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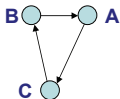
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Deadlock:

Competing actions are each waiting for the other to finish, and thus neither ever does.

Solution:

Remove some processes to **break such cycles** and put them in a waiting queue.

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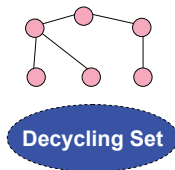
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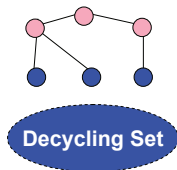
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Example: A 4-regular graph (such as toroidal mesh network).



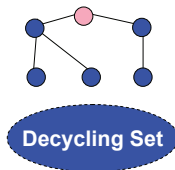
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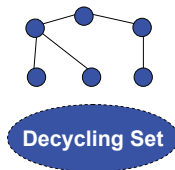
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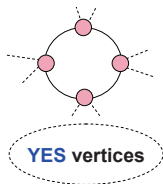
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NO vertices on the cycle remain NO.
Failed!

Applications

- Constraint satisfaction problem (Dechter 1990).
- Bayesian inference in artificial intelligence (Bar-Yehuda et al. 1998).
- Converters' placement problem in optical networks (Kleinberg and Kumar 1999).
- VLSI chip design (Festa et al. 2000).

Complexity

- ▶ Has been extensively studied.
 - NP-hard
Reduction from VERTEX COVER (R. Karp 1972).
Even for planar graphs.

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- ▶ Is related or equivalent to
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- Feedback vertex set problem (Wang et al. 1985).
- Hitting cycle problem
- Maximum induced forest problem (Erdős, Saks and Sós 1986 Maximum induced trees in graphs).

At least $\frac{71n + 72}{128}$ for triangle-free n -vertex planar graphs (Kowalik et al. 2010).

Variations

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 - Graph Bipartization Problem
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Variations

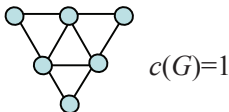
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Seek a minimum-weight decycling set D where each vertex has a weight.
 - **Loop Cut Set Problem**
Given $A(C) \subseteq V(C)$ for each cycle C of G , find a minimum set D such that $D \cap A(C) \neq \emptyset$.

Relations with Cycle Packing Number

- Is often compared with the following graph parameter.
- Cycle Packing Number $c(G)$: the maximum number of vertex-disjoint cycles of G .



- **Definition:** $\nabla(G) :=$ the decycling number (minimum size of decycling set) of G .

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- **Definition:** $\nabla(k) := \max\{\nabla(G) : c(G) = k\}$.
- ▷ $\nabla(1) = 3$; $c(K_5) = 1$ and $\nabla(K_5) = 3$ (Bollobás 1964).
- ▷ $\nabla(2) = 6$ and $9 \leq \nabla(3) \leq 12$ (Voss 1968).

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- ▷ $\nabla(2) = 6$ and $9 \leq \nabla(3) \leq 12$ (Voss 1968).
- ▷ $c_1 k \log k \leq \nabla(k) \leq c_2 k \log k$ for some constants c_1 and c_2 (Erdős and Pósa 1964).

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Theorem (Chen, Fu and Shih 2010)

For every planar graph G , $\nabla(G) \leq 3c(G)$.

Decycling number of outerplanar graphs

Consider outerplanar graphs:

Theorem (Kloks, Lee and Liu 2002)

For every outerplanar graph G , $\nabla(G) \leq 2c(G)$.

- An outerplanar graph G is called *lower-extremal* if $\nabla(G) = c(G)$ and *upper-extremal* if $\nabla(G) = 2c(G)$.

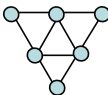
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- ▶ An outerplanar graph G is called *lower-extremal* if $\nabla(G) = c(G)$ and *upper-extremal* if $\nabla(G) = 2c(G)$.
- ▶ Upper-Extremal Results:
 - We define a *sun graph* S_3 as follows.



- $\nabla(S_3) = 2 = 2c(S_3)$.

Decycling number of outerplanar graphs

Theorem (Chang, Fu, Lien, 2011)

An outerplanar graph G is upper-extremal if and only if G is an S_3 -tree.

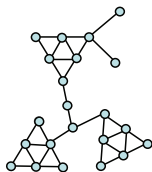
- A graph is an S_3 -tree of order t if it has exactly t vertex-disjoint S_3 -subdivisions and every edge not on these S_3 -subdivisions belongs to no cycle.

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- Example:



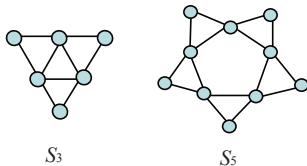
An S_3 -tree G of order 3,
where $\tau(G) = 6 = 2c(G)$.

Decycling number of outerplanar graphs

► Lower-Extremal Results:

- The following graphs are NOT lower-extremal ($\nabla(G) \neq c(G)$):

Sun graphs S_k with odd number k :



- $\nabla(S_k) = \lceil \frac{k}{2} \rceil$ and $c(S_k) = \lfloor \frac{k}{2} \rfloor$.

Decycling number of outerplanar graphs

Theorem (Chang, Fu, Lien, 2011)

For an outerplanar graph G , if G has no S_k -subdivision for all odd number k , then G is lower-extremal.

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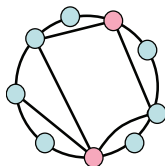
For an outerplanar graph G , if G has no S_k -subdivision for all odd number k , then G is lower-extremal.

Lemma

If G is a 2-connected outerplanar graph with no S_k -subdivision for all odd number k , then G is lower-extremal.

(Proved by induction on $|E(G)|$.)

• Example:



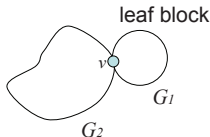
Decycling number of outerplanar graphs

- A graph property is called *monotone* if it is closed under removal of vertices.
- ▷ "Without S_k -subdivision for all odd number k " is a monotone graph property.

Lemma

Suppose that a 2-connected graph is lower-extremal provided that it satisfies a monotone property $\mathcal{P}$. Then a graph is lower-extremal if it satisfies $\mathcal{P}$.

Proof. By induction on $|G|$. Suppose G satisfies $\mathcal{P}$. Consider that G has a cut vertex v .



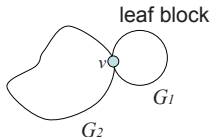
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$$\begin{aligned}c(G) &= c(G_1) + c(G_2) \text{ or} \\c(G_1) + c(G_2) - 1. \\ \nabla(G) &\leq \nabla(G_1) + \nabla(G_2).\end{aligned}$$

Decycling number of outerplanar graphs

Suppose to the contrary that $\nabla(G) > c(G)$. Then

(1) $c(G) = c(G_1) + c(G_2) - 1$ and

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By the monotonicity of $\mathcal{P}$ and the induction hypothesis,
 $c(G^* - v) = \nabla(G^* - v) = \nabla(G^*) = c(G^*)$, a contradiction. ■

Lower Bound of Undirected Graphs

Lemma (Beineke 1997)

If G is a connected graph with p vertices ($p > 2$), q edges, and maximum degree Δ , then $\nabla(G) \geq \frac{q-p+1}{\Delta-1}$.

Cartesian Product

Definition ($G \square H$)

$$V(G \square H) = \{(u, v) | u \in V(G) \text{ and } v \in V(H)\}$$

$$E(G \square H) = \{(u, v)(u', v') | u = u' \text{ and } (v, v') \in E(H) \text{ or } (u, u') \in E(G) \text{ and } v = v'\}$$

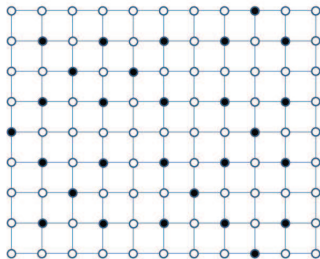
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Figure : $P_9 \square P_{11}$



Decycling number of $C_m \square C_n$

Theorem (Pike, Zou 2005)

$$\nabla(C_m \square C_n) = \begin{cases} \lceil \frac{3n}{2} \rceil & \text{if } m = 4, \\ \lceil \frac{3m}{2} \rceil & \text{if } n = 4, \\ \lceil \frac{mn+2}{3} \rceil & \text{otherwise.} \end{cases}$$

Decycling number of Path Product $P_m \square P_n$

Lower bound

Theorem (Luccio 1998)

If $m, n \geq 2$, then $\nabla(P_m \square P_n) \geq \left\lceil \frac{(m-1)(n-1)+1}{3} \right\rceil$.

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- $|E| - 4k + 1 \leq |V| - k - 1$.

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- $k \geq \left\lceil \frac{(m-1)(n-1)+1}{3} \right\rceil$.

Decycling number of Path Product $P_m \square P_n$

Theorem (Madelaine and Stewart 2008)

Table :

$m \backslash n$	0	1	2	3	4	5
0	B	A	B	B	A	B
1	A	A	A	A	A	A
2	B	A	B	B	A	B
3	B	A	B	B	A	C
4	A	A	A	A	A	A
5	B	A	B	C	A	C

 : increasing the lower bound in this paper
 : decreasing the upper bound in this paper

In Table, A: $\nabla(P_m \square P_n) = F_{m,n}$, B: $\nabla(P_m \square P_n) \leq F_{m,n} + 1$, C: $\nabla(P_m \square P_n) \leq F_{m,n} + 2$, where $F_{m,n} = \left\lceil \frac{(m-1)(n-1)+1}{3} \right\rceil$.

Decycling number of Path Product $P_m \square P_n$

New Lower bound

Proposition (Observation)

If $m \geq 5$ and $f_{m,n} = \frac{(m-1)(n-1)+1}{3}$ is an integer, then each decycling set S of size $f_{m,n}$ satisfies the following two properties:

- (1) S contains exactly one vertex of degree 3 and contains no vertex of degree 2; and*
- (2) S induces a subgraph of $P_m \square P_n$ with no edges.*

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Theorem

If $m \geq 5$, mn is even and $f_{m,n}$ is an integer, then

$$\nabla(P_m \square P_n) \geq f_{m,n} + 1 = F_{m,n} + 1, \text{ where } f_{m,n} = \frac{(m-1)(n-1)+1}{3}.$$

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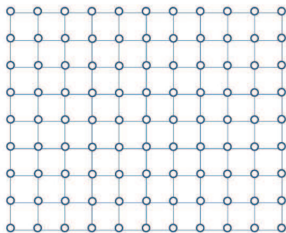
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By above Proposition, we may let $v_{i,1}$ be the vertex of S with degree 3 where $2 \leq i \leq \lfloor \frac{m}{2} \rfloor$.



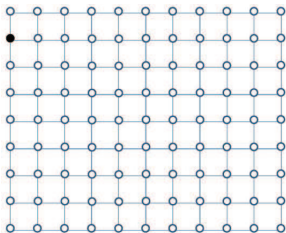
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By above Proposition, we may let $v_{i,1}$ be the vertex of S with degree 3 where $2 \leq i \leq \lfloor \frac{m}{2} \rfloor$.



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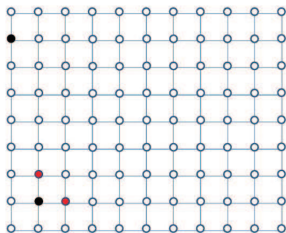
Decycling number of Path Product $P_m \square P_n$

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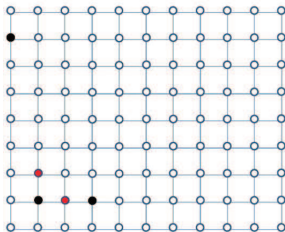
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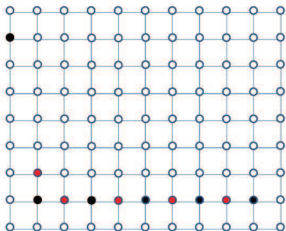
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$v_{m-1,2}, v_{m-1,4}, \dots, v_{m-1,n-1} \in S$.

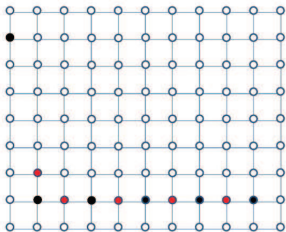
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Hence, $n-1$ is even and n is odd.

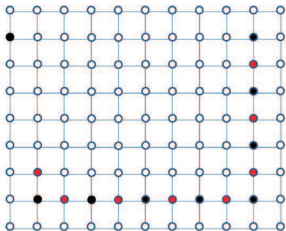
Decycling number of Path Product $P_m \square P_n$

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By above Proposition, we may let $v_{i,1}$ be the vertex of S with degree 3 where $2 \leq i \leq \lfloor \frac{m}{2} \rfloor$.



Similarly,

$v_{m-3,n-1}, v_{m-5,n-1}, \dots, v_{2,n-1} \in S$.

Therefore, m is odd, a contradiction.

Decycling number of Path Product $P_m \square P_n$

New Lower bound

Corollary

For $m \geq 5$, if $m \equiv 0 \pmod{6}$ and $n \equiv 2 \pmod{3}$ or $(m, n) \equiv (3, 2) \pmod{6}$, $\nabla(P_m \square P_n) \geq F_{m,n} + 1$.

Decycling number of Path Product $P_m \square P_n$

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Theorem

For $m \geq 5$, if $(m, n) \equiv (0, 2), (0, 5), (3, 2), (2, 0), (5, 0), (2, 3) \pmod{6}$, then $\nabla(P_m \square P_n) = F_{m,n} + 1$.

Decycling number of Path Product $P_m \square P_n$

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Theorem

For $m, n \geq 6$, $\nabla(P_m \square P_n) \leq F_{m,n} + 1$.

Decycling number of Path Product $P_m \square P_n$

Theorem (Madelaine and Stewart 2008)

Table :

$m \backslash n$	0	1	2	3	4	5
0	B	A	B	B	A	B
1	A	A	A	A	A	A
2	B	A	B	B	A	B
3	B	A	B	B	A	C
4	A	A	A	A	A	A
5	B	A	B	C	A	C

 : increasing the lower bound in this paper
 : decreasing the upper bound in this paper

In Table, A: $\nabla(P_m \square P_n) = F_{m,n}$, B: $\nabla(P_m \square P_n) \leq F_{m,n} + 1$, C: $\nabla(P_m \square P_n) \leq F_{m,n} + 2$, where $F_{m,n} = \left\lceil \frac{(m-1)(n-1)+1}{3} \right\rceil$.

Decycling number of Generalized Kautz Digraphs

- de Bruijn digraph $B(d, n)$

$$V(B(d, n)) = \{x_1x_2 \cdots x_n : x_i \in \{0, 1, \dots, d-1\}, 1 \leq i \leq n\}.$$

$$\text{Edge} : X = x_1x_2 \cdots x_n \longrightarrow Y = x_2x_3 \cdots x_n\alpha \text{ where } \alpha \in \{0, 1, \dots, d-1\}.$$

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- Kautz digraph $K(d, n)$

$$V(K(d, n)) = \{x_1x_2 \cdots x_n : x_i \in \{0, 1, \dots, d\}, 1 \leq i \leq n \text{ and } x_i \neq x_{i+1}, 1 \leq i \leq n-1\}.$$

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$$K(d, n) \subseteq B(d+1, n).$$

Decycling number of Generalized Kautz Digraphs

- generalized de Bruijn digraph $GB(d, n)$

$$V(GB(d, n)) = \{0, 1, \dots, n-1\}.$$

$$E(GB(d, n)) = \{(x, y) | y \equiv dx + i \pmod{n}, 0 \leq i \leq d-1\}.$$

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- generalized Kautz digraph $G_K(d, n)$

$$V(K(d, n)) = \{0, 1, \dots, n-1\}.$$

$$E(K(d, n)) = \{(x, y) | y \equiv -dx - i \pmod{n}, 1 \leq i \leq d\}.$$

Example of de Bruijn Graph and Generalized de Bruijn Graph

Figure : $B(2,3)$

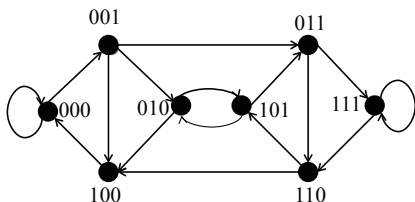
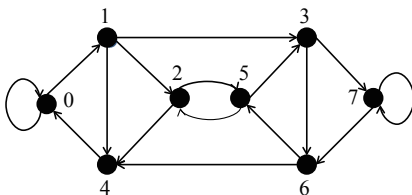


Figure : $GB(2,8)$



$$B(d, k) = GB(d, d^k).$$

The de Bruijn digraph is a special case of $GB(d, n)$.

Decycling number of Generalized Kautz Digraphs

- de Bruijn digraphs

Decycling number of Generalized Kautz Digraphs

- de Bruijn digraphs

Theorem (Xu et al. 2010)

For any integers $d \geq 2$ and $n \geq 2$

$$\nabla(B(d, n)) = \begin{cases} \frac{1}{n} \sum_{i|n} d^i \varphi\left(\frac{n}{i}\right) & \text{for } 2 \leq n \leq 4; \\ \frac{d^n}{n} + O(nd^{n-4}) & \text{for } n \geq 5, \end{cases}$$

where $i|n$ means i divides n , and $\varphi(i)$ is the Euler totient function.

$$\varphi(n) = n \prod_{p|n} \left(1 - \frac{1}{p}\right).$$

Decycling number of Generalized Kautz Digraphs

- Generalized de Bruijn digraphs

Decycling number of Generalized Kautz Digraphs

- Generalized de Bruijn digraphs

Theorem (Xu et al. 2009)

$$\nabla(GB(d, n)) \leq \begin{cases} 4 + 7m + \lfloor \frac{5t+3}{8} \rfloor + \lfloor \frac{3t}{4} \rfloor - \lfloor \frac{5t}{8} \rfloor - \lfloor \frac{t+1}{2} \rfloor, \\ \quad \text{where } m = \lfloor \frac{n}{32} \rfloor, n \equiv t \pmod{32}, \text{ for } d = 2, \\ 4 + 6m + \lfloor \frac{7t}{9} \rfloor - \lfloor \frac{t+1}{3} \rfloor, \\ \quad \text{where } m = \lfloor \frac{n}{18} \rfloor, n \equiv t \pmod{18}, \text{ for } d = 3, \\ 2 + t + d(2d - 3)m + \lceil \frac{(d+1)t+d-1}{d^2} \rceil - \lfloor \frac{t}{d} \rfloor, \\ \quad \text{where } m = \lfloor \frac{n}{d(2d+3)} \rfloor, n \equiv t \pmod{d(2d+3)}, \text{ for } d \geq 4. \end{cases}$$

Decycling number of Generalized Kautz Digraphs

- Generalized Kautz digraphs:

Decycling number of Generalized Kautz Digraphs

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Theorem

$$\nabla(G_k(d, n)) \leq \begin{cases} 8m + 3t + 1, \\ \quad \text{where } m = \lfloor \frac{n}{36} \rfloor, n \equiv t \pmod{36}, \text{ for } d = 2, \\ 12m + t + \lceil \frac{t}{2} \rceil + \lfloor \frac{3t}{4} \rfloor + 5, \\ \quad \text{where } m = \lfloor \frac{n}{36} \rfloor, n \equiv t \pmod{36}, \text{ for } d = 3, \\ (\frac{1}{2} - \frac{d-1}{2d^2})n + \frac{d}{2}(d - t + 5) - 2, \\ \quad \text{where } m = \lfloor \frac{n}{d+1} \rfloor, n \equiv t \pmod{(d+1)}, \text{ for } d \geq 4. \end{cases}$$

Decycling number of Generalized Kautz Digraphs

Notation:

For $\alpha \in V(G_K(d, n))$, $R(\alpha) = \{-d\alpha - i \pmod{n}, 1 \leq i \leq d\}$.

For a set $S \subset V(G_K(d, n))$, $R(S) = \bigcup_{v \in S} R(v)$.

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For convenience, we use the notation as follows:

$$[a, b] = \begin{cases} \{a, a+1, \dots, b\} & \text{if } a \leq b, \\ \{a, a+1, \dots, n+b-1, n+b\} & \text{otherwise.} \end{cases}$$

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$$R([a, b]) = [-db - d, -da - 1] \pmod{n}.$$

Decycling number of Generalized Kautz Digraphs

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Decycling number of Generalized Kautz Digraphs

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Decycling number of Generalized Kautz Digraphs

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- $V_{n,c} = V(G_K(d, n)) - V_C$.

Wants:

- V_d : decycling set.
- ▷ $V_C = \emptyset$ or $V_{nc} = V(G_K(d, n))$.

Decycling number of Generalized Kautz Digraphs

$$V_{nc} = V(G_K(d, n)).$$

Decycling number of Generalized Kautz Digraphs

$$V_{nc} = V(G_K(d, n)).$$

Lemma (Main Idea)

For any arbitrary subset $S \subset V_r$, if $R(S) \subset V_{nc}$, then $S \subset V_{nc}$.

Lemma

$$V_d \subseteq V_{nc}.$$

Decycling number of Generalized Kautz Digraphs

$$G_K(d, n), d \geq 4.$$

Decycling number of Generalized Kautz Digraphs

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Let $m = \lfloor \frac{n}{d+1} \rfloor$ and $n \equiv t \pmod{d+1}$, then $n = (d+1)m + t$, where $0 \leq t \leq d$.

Decycling number of Generalized Kautz Digraphs

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Let $m = \lfloor \frac{n}{d+1} \rfloor$ and $n \equiv t \pmod{d+1}$, then $n = (d+1)m + t$, where $0 \leq t \leq d$.

$$A_1 = [0, m],$$

$$A_2 = [\lfloor \frac{n}{d} \rfloor + \lfloor \frac{n}{d^3} \rfloor, 2m + 1],$$

$$A_i = [\lfloor \frac{(i-1)n}{d} \rfloor + \lfloor \frac{(i-1)n}{d^3} \rfloor, im + (i-1)], \text{ for } i = 3, 4, \dots, d.$$

We will show that

$$\bigcup_{i=1}^d A_i \text{ is a decycling set of } G_K(d, n).$$

Decycling number of Generalized Kautz Digraphs

Example.

Let $n = 378, d = 4, m = \lfloor \frac{n}{d+1} \rfloor = 75, t = 3$, where $n = (d + 1)m + t$.

Decycling number of Generalized Kautz Digraphs

Example.

Let $n = 378$, $d = 4$, $m = \lfloor \frac{n}{d+1} \rfloor = 75$, $t = 3$, where $n = (d + 1)m + t$.

Step 1.

$$A_1 = [0, 75] \subseteq V_{nc}.$$

Decycling number of Generalized Kautz Digraphs

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Let $n = 378, d = 4, m = \lfloor \frac{n}{d+1} \rfloor = 75, t = 3$, where $n = (d + 1)m + t$.

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$$A_1 = [0, 75] \subseteq V_{nc}.$$

$$S_1 = [76, 93].$$

$$R(S_1) = [-4 \cdot 93 - 4, -4 \cdot 76 - 1] = [-376, -305] =$$

$$[2, 73] \subseteq V_{nc}.$$

$$[0, 93] \subseteq V_{nc}.$$

Decycling number of Generalized Kautz Digraphs

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$$R(S_1) = [-4 \cdot 93 - 4, -4 \cdot 76 - 1] = [-376, -305] = [2, 73] \subseteq V_{nc}.$$

$$[0, 93] \subseteq V_{nc}.$$

$$X_1 = [355, 377].$$

$$R(X_1) = [0, 91] \subseteq V_{nc}.$$

$$[0, 93] \cup [355, 377] \subseteq V_{nc}.$$

Decycling number of Generalized Kautz Digraphs

$$Y_1 = [94, 98].$$

$$R(Y_1) = [360, 1] \subseteq V_{nc}.$$

$$[0, 98] \cup [355, 377] \subseteq V_{nc}.$$

Decycling number of Generalized Kautz Digraphs

$$Y_1 = [94, 98].$$

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$$[0, 98] \cup [355, 377] \subseteq V_{nc}.$$

Step 2.

$$A_2 = [99, 151].$$

$$[0, 151] \cup [355, 357] \subseteq V_{nc}.$$

Decycling number of Generalized Kautz Digraphs

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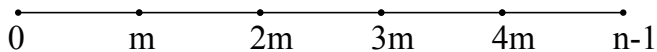
$$[0, 151] \cup [355, 357] \subseteq V_{nc}.$$

$$S_2 = [152, 188].$$

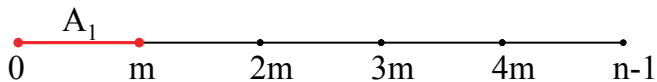
$$R(S_2) = [0, 147] \subseteq V_{nc}.$$

$$[0, 188] \cup [355, 377] \subseteq V_{nc}.$$

Decycling number of Generalized Kautz Digraphs

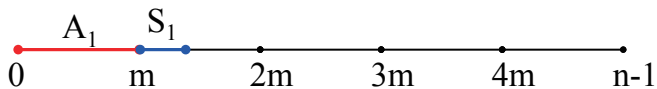


Decycling number of Generalized Kautz Digraphs



$$A_1 \subseteq V_{nc}.$$

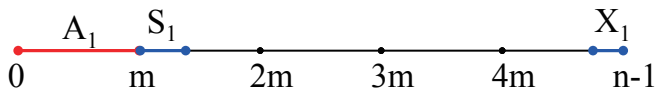
Decycling number of Generalized Kautz Digraphs



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Decycling number of Generalized Kautz Digraphs

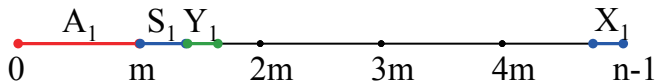


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Decycling number of Generalized Kautz Digraphs



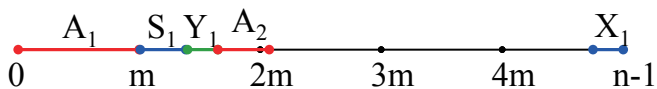
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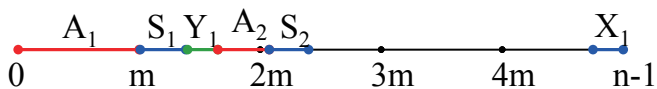
$$Y_1 \subseteq V_{nc}.$$

Decycling number of Generalized Kautz Digraphs



$$A_2 \subseteq V_{nc}.$$

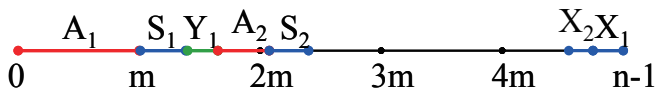
Decycling number of Generalized Kautz Digraphs



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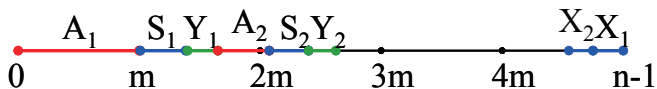


$$A_2 \subseteq V_{nc}.$$

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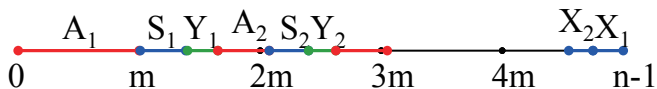
$$A_2 \subseteq V_{nc}.$$

$$S_2 \subseteq V_{nc}.$$

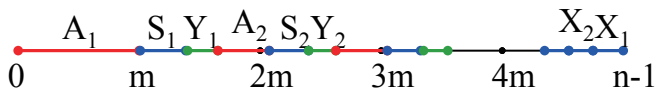
$$X_2 \subseteq V_{nc}.$$

$$Y_2 \subseteq V_{nc}.$$

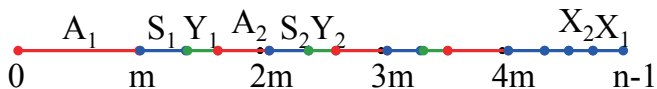
Decycling number of Generalized Kautz Digraphs



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Decycling number of Generalized Kautz Digraphs



Total decycling Problem

Definition (Total Decycling Problem)

Given a directed/undirected graph $G = (V, E)$, find sets $V' \subseteq V$ and $E' \subseteq E$ such that $G \setminus (V' \cup E')$ is acyclic and $a|V'| + b|E'|$ is minimum, where a, b are the weights of vertex and edge respectively.

- **Definition:** $\nabla_T(G) :=$ the total decycling number (minimum weight of total decycling set) of G .
- For convenience, we can assume $a = 1$ and $b = r$.

Consider Regular Graphs

- Consider $\frac{b}{a} = r$.

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- Consider $\frac{b}{a} = r$.
- Example: G : 4-regular with n -vertices.
Let $S_1 \subseteq V(G)$ and $S_2 \subseteq E(G)$: total decycling set.
 $2n - 4|S_1| - |S_2| \leq n - |S_1| - 1$.
 $|S_1| + \frac{1}{3}|S_2| \geq \frac{n+1}{3}$.

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$$|S_1| + \frac{1}{3}|S_2| \geq \frac{n+1}{3}.$$
- $r = \frac{1}{3}$.
Means: A vertex and an edge have the same weight.
Select: Vertex or Edge.

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 $2n - 4|S_1| - |S_2| \leq n - |S_1| - 1$.
 $|S_1| + \frac{1}{3}|S_2| \geq \frac{n+1}{3}$.
- $r = \frac{1}{3}$.
Means: A vertex and an edge have the same weight.
Select: Vertex or Edge.
- $r = \frac{1}{4}$.
Means: The weight of an edge $<$ The weight of a vertex.
Priority Select: **Edge**.

Consider Regular Graphs

- Consider $\frac{b}{a} = r$.
- Example: G : 4-regular with n -vertices.
Let $S_1 \subseteq V(G)$ and $S_2 \subseteq E(G)$: total decycling set.
 $2n - 4|S_1| - |S_2| \leq n - |S_1| - 1$.
 $|S_1| + \frac{1}{3}|S_2| \geq \frac{n+1}{3}$.
- $r = \frac{1}{3}$.
Means: A vertex and an edge have the same weight.
Select: Vertex or Edge.
- $r = \frac{1}{4}$.
Means: The weight of an edge $<$ The weight of a vertex.
Priority Select: **Edge**.
- $r = \frac{1}{2}$.
Means: The weight of an edge $>$ The weight of a vertex.
Priority Select: **Vertex**.

Consider Bound of $C_m \square C_n$

- Consider $\frac{1}{3} < r < 1$.
- $C_m \square C_n$: 4-regular.

Let $S_1 \subseteq V(G)$ and $S_2 \subseteq E(G)$: total decycling set.

$$\lceil \frac{mn+2}{3} \rceil = \nabla(C_m \square C_n) \geq \nabla_T(C_m \square C_n) > \frac{mn+1}{3}.$$

Consider Bound of $P_m \square P_n$

- Consider $\frac{1}{3} < r < 1$.
- $P_m \square P_n$: almost 4-regular.

Let $S_1 \subseteq V(G)$ and $S_2 \subseteq E(G)$: total decycling set.

$$\left\lceil \frac{(m-1)(n-1)+1}{3} \right\rceil + 1 \geq \nabla(C_m \square C_n) \geq \nabla_T(P_m \square P_n) > \frac{(m-1)(n-1)}{3}.$$

Objective Work

- For a planar graph G , $\nabla(G) \stackrel{?}{\leq} 2c(G)$ (Jones' conjecture 2002).
- For a planar graph G , $\nabla(G) \stackrel{?}{\leq} |V(G)|/2$ (Albertson and Berman 1979).
- For a bipartite planar graph G , $\nabla(G) \stackrel{?}{\leq} 3|V(G)|/8$ (Albertson and Berman 1979).
- We have $\lceil \frac{(m-1)(n-1)+1}{3} \rceil \leq \nabla(P_m \square P_n) \leq \lceil \frac{(m-1)(n-1)+1}{3} \rceil + 1$. Find the exact value of $\nabla(P_m \square P_n)$.
- Find the lower bound of directed graphs.

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Thank you for your attention!