Linear Algebra (II) H. W. (III)

2011,06,08

(10 points each)

1. Prove Cauchy-Schwarz Inequality.
(If $\vec{x}, \vec{y} \in \mathbb{R}^n$, then $| < \vec{x}, \vec{y} > | \leq ||\vec{x}|| \cdot ||\vec{y}||$.)

2. Let V be a vector subspace of $\mathbb{R}^n$. Prove that the following statements are true.

- **(a)** $V^{\perp}$ is a subspace of $\mathbb{R}^n$.
- **(b)** $V \cap V^{\perp} = \{\vec{0}\}$.
- **(c)** If $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r\}$ forms a basis of V and $\{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_{n-r}\}$ forms a basis of $V^{\perp}$, then $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r, \vec{w}_1, \vec{w}_2, \ldots, \vec{w}_{n-r}\}$ forms a basis of $\mathbb{R}^n$.

3. Let V be a subspace of $\mathbb{R}^m$. Prove that given $\vec{b} \in \mathbb{R}^m$, there is a unique $\vec{p} \in V$ such that $\vec{b} - \vec{p} \in V^{\perp}$.

4. Find the projection of the vector $(5, 3, -7)$ onto the subspace spanned by $(1, 3, 3)$ and $(2, -2, 1)$.

5. Let A be an $m \times n$ matrix and $rank(Col(A)) = n$. Prove that $A^T A$ is nonsingular.

6. Find the least-squares solution of the system

$$\begin{bmatrix} 1 & -3 \\ 2 & 6 \\ 7 & -3 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 2 \\ 1 \end{bmatrix}.$$

7. Apply the Gram-Schmidt process to the set of three vectors: $\vec{u}_1 = (1, 2, 1)$, $\vec{u}_2 = (1, 0, 1)$ and $\vec{u}_3 = (3, 1, 0)$, i.e. to find three orthogonal vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ such that $Span(\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}) = Span(\{\vec{u}_1, \vec{u}_2, \vec{u}_3\})$.

8. Prove that every orthogonal set of vectors in an inner product space is linear independent.

9. Prove that $(Col(A))^{\perp} = Null(A^{T})$.

10. In $\mathbb{R}^3$, find the distance from the point $(1, 0, 2)$ to the subspace spanned by $(1, 2, -1)$ and $(3, -2, 1)$.

(Bonus) 20 points

Prove that if A is a real symmetric matrix of order n, then there exists an othogonal matrix P (over $\mathbb{R}$) such that $P^T AP$ is a diagonal matrix.