Week 16

June 8, 10.

Note The last class of this semester is June, 15 th.
There is no class on the 17 th.

(*) The 3rd test will be held on the 22nd of June.

(**) You shall know you grades on the 24 th of June before noon. please go over to the assistant to check.

Gram - Schmidt Orthogonalization Process

Part II, knowing the basis for the subspace.

Theorem Let $\mathbb{B}_1 = \{\vec{u}_1, \vec{u}_2, \cdots, \vec{u}_r\}$ be a basis for $W \leq \mathbb{R}^m$.

Then, there exists an orthonormal basis $\mathbb{B} = \{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_r\}$ for W such that $\vec{v}_i$ is a linear combination of $\vec{u}_1, \vec{u}_2, \cdots, \vec{u}_i$ for each $i = 1, 2, \cdots, r$.

Proof. By induction on r.

For $r = 1$, let $\vec{v}_1 = \vec{u}_1 / \|\vec{u}_1\|$ and we have the proof.
(assertion is true for $r=1$.)

Assume the assertion is true for $r = k$ and let $(k \geq 2)$

$$\vec{w}_k = \vec{u}_k - \langle \vec{v}_1, \vec{u}_k \rangle \vec{v}_1 - \langle \vec{v}_2, \vec{u}_k \rangle \vec{v}_2 - \cdots - \langle \vec{v}_{k-1}, \vec{u}_k \rangle \vec{v}_{k-1}.$$

Now, consider

$$\vec{w}_{k+1} = \vec{u}_{k+1} - \langle \vec{v}_1, \vec{u}_{k+1} \rangle \vec{v}_1 - \cdots - \langle \vec{v}_k, \vec{u}_{k+1} \rangle v_k .$$

For each $1 \leq j < k+1$,

$$\langle \vec{v}_{k+1}, \vec{v}_j \rangle = \langle \vec{u}_{k+1}, \vec{v}_j \rangle - \langle \vec{v}_j, \vec{u}_{k+1} \rangle \langle \vec{v}_j, \vec{v}_j \rangle = 0.$$

Hence $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\}$ where $\vec{v}_{k+1} = \vec{w}_{k+1} / \|\vec{w}_{k+1}\|$. is an orthonormal ~~basis~~ set of vectors.

Moreover,

$\vec{w}_{k+1}$ ($\vec{v}_{k+1}$) is a linear combination of vectors

$\vec{v}_1, \vec{v}_2, \dots, \vec{v}_{k+1}$ and thus $\vec{u}_1, \vec{u}_2, \dots, \vec{u}_{k+1}$.

This concludes the proof. ∎

<u>Example</u> $B_1 = \{ \overset{\vec{u}_1}{(2,0,2,1)}, \overset{\vec{u}_2}{(0,0,4,1)}, \overset{\vec{u}_3}{(8,0,3,5)} \}$

$$\vec{v}_1 = \frac{1}{3}(2,0,2,1) \quad (?)$$

$$\vec{w}_2 = (0,0,4,1) - 3\left(\frac{2}{3}, 0, \frac{2}{3}, \frac{1}{3}\right) = (-2,0,2,0).$$

$$\vec{v}_2 = \vec{w}_2 / \|\vec{w}_2\| = \left(-\frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2}, 0\right).$$

$$\vec{w}_3 = \vec{u}_3 - \langle \vec{v}_1, \vec{u}_3 \rangle \vec{v}_1 - \langle \vec{v}_2, \vec{u}_3 \rangle \vec{v}_2$$

$$= \left(-\frac{1}{2}, 0, -\frac{1}{2}, 2\right).$$

$$\vec{v}_3 = \frac{\sqrt{2}}{6}(-1,0,-1,4).$$

(Check yourself!)

<u>Example</u> Find an O.N. basis for $\mathbb{R}^3$ which contains $\frac{1}{\sqrt{3}}(1,1,1)$ and $\frac{1}{\sqrt{2}}(1,-1,0)$.

<u>Sol.</u> Find $\vec{v}_3$ <u>only</u>, $\vec{v}_3 = \frac{1}{\sqrt{6}}(-1,-1,2)$.

Orthogonal Matrices

A matrix P (over $\mathbb{R}$) is orthogonal if $P^T = P^{-1}$.

(Note)

$$P^T P = \begin{bmatrix} \vec{v}_1 \\ \vec{v}_2 \\ \vdots \\ \vec{v}_n \end{bmatrix} \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{bmatrix}$$

$$= \begin{bmatrix} \langle \vec{v}_i, \vec{v}_j \rangle \end{bmatrix}_{n\times n} = I_n$$

$\Longrightarrow \langle \vec{v}_i, \vec{v}_j \rangle = 0$ if $i \neq j$ and

$\langle \vec{v}_i, \vec{v}_i \rangle = 1$ for $i = 1, 2, \cdots, n$.

(Over $\mathbb{C}$)

$$P^* P = I$$

(Take transpose and conjugate.)

Definition

① Conjugate of A, $\bar{A} = [\bar{a}_{ij}]$, where $A = [a_{ij}]$, $a_{ij} \in \mathbb{C}$.

② Conjugate transpose of A, $A^* = [\bar{a}_{ij}]^T$.

Theorem All eigenvalues of a real symmetric matrix A are real.

Proof. Let λ be an eigenvalue of A. $\quad (A : \mathbb{C}^n \to \mathbb{C}^n)$

Then $A\vec{x} = \lambda \vec{x}$, $\vec{x} \neq \vec{0}$ is an eigenvector associate with λ.

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad \vec{x}^* A \vec{x} = \lambda \vec{x}^* \vec{x}$$

$$\vec{x}^* A^* \vec{x} = \lambda \vec{x}^* \vec{x} = \vec{x}^* A \vec{x} = (\vec{x}^* A \vec{x})^*.$$

This implies that $\vec{x}^* A \vec{x}$ is real and thus

$$\lambda = \frac{\vec{x}^* A \vec{x}}{\vec{x}^* \vec{x}} \text{ is real.} \qquad \blacksquare$$

Note Hermitian matrices

A matrix H (over F) is Hermitian if and only if $H^* = H$.

Theorem $\forall$ Hermitian matrix H, $\exists$ invertible matrix P such that $P^{-1} H P$ is a diagonal matrix.

We prove the case when A is real symmetric.

(If A is real symmetric, then A is Hermitian.)

<u>Theorem</u> If A is a real symmetric matrix of order n, then there exists an orthogonal matrix P (over $\mathbb{R}$) such that $P^T A P$ is a diagonal matrix.

<u>Proof.</u> By induction on n and it is clear that "$n=1$" is true. Assume that the assertion is true when $n=k$ and let A be a real symmetric matrix of order $k+1$.

Let $\lambda_1 \in \mathbb{R}$ be an eigenvalue of A with corresponding eigenvector $\vec{v}_1$. By Gram-Schmidt process, we have an orthonormal basis $\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_{k+1}\}$ for $\mathbb{R}^{k+1}$. Now, let $P_1 = [\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_{k+1}]$.

Then $P_1^T A P_1 = [\vec{v}_i^T A \vec{v}_j]_{(k+1)\times(k+1)}$. By the fact that $A\vec{x}_1 = \lambda_1 \vec{x}_1$ we have

$$P_1^T A P_1 = \left[\begin{array}{c|ccc} \lambda_1 & 0 & 0 \cdots & 0 \\ \hline 0 & & & \\ 0 & & & \\ \vdots & & A_2 & \\ 0 & & & \end{array}\right].$$

(*) $\vec{v}_1^T A \vec{v}_j = (\vec{v}_j^T A \vec{v}_1)^T = (\vec{v}_j^T (\lambda_1 \vec{v}_1))^T = \lambda_1 (\underline{\vec{v}_j^T \vec{v}_1})^T = 0 .\ (j \neq 1)$

$\vec{v}_i^T A \vec{v}_1 = \vec{v}_i^T (\lambda \vec{v}_1) = \lambda \underline{\vec{v}_i^T \vec{v}_1} = 0 \quad (i \neq 1).$

(**) A_2 is a real symmetric matrix of order k.

By induction hypothesis, $\exists\ Q$ (orthogonal matrix of order k), such that
$$Q^T A_2 Q = \begin{bmatrix} \lambda_2 & & 0 \\ & \ddots & \\ 0 & & \lambda_{k+1} \end{bmatrix} .$$

($\lambda_2, \cdots, \lambda_{k+1}$ are eigenvalues of A_2.)

Now, let $P_2 = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & & & & \\ 0 & & Q & & \\ \vdots & & & & \\ 0 & & & & \end{bmatrix}$ and $P = P_1 P_2$.

Then,
$$\begin{aligned} P^T A P &= (P_1 P_2)^T A (P_1 P_2) \\ &= P_2^T (P_1^T A P_1) P_2 \\ &= P_2^T \left[\begin{array}{c|c} \lambda_1 & \vec{0}^T \\ \hline \vec{0} & A_2 \end{array}\right] P_2 \\ &= \left[\begin{array}{c|c} 1 & \vec{0}^T \\ \hline \vec{0} & Q^T \end{array}\right] \left[\begin{array}{c|c} \lambda_1 & \vec{0}^T \\ \hline \vec{0} & A_2 \end{array}\right] \left[\begin{array}{c|c} 1 & \vec{0}^T \\ \hline \vec{0} & Q \end{array}\right] \\ &= \left[\begin{array}{c|c} \lambda_1 & \vec{0}^T \\ \hline \vec{0} & Q^T A_2 Q \end{array}\right] = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_{k+1} \end{bmatrix} . \end{aligned}$$

∎

(Note) The proof for Hermitian matrices is similar.

Theorem If λ_1 and λ_2 are distinct eigenvalues of A (real symmetric) with associated eigenvectors $\vec{v}_1$ and $\vec{v}_2$, respectively, then $\vec{v}_1 \perp \vec{v}_2$.

(*)

Proof. Suppose not. Let $\langle \vec{v}_1, \vec{v}_2 \rangle \neq 0$.

Since $A\vec{v}_1 = \lambda_1 \vec{v}_1$ and $A\vec{v}_2 = \lambda_2 \vec{v}_2$,

$$\vec{v}_1^T A \vec{v}_2 = \vec{v}_1^T \lambda_2 \vec{v}_2 = \lambda_2 \vec{v}_1^T \vec{v}_2 \text{ ; and}$$

$$(\vec{v}_1^T A \vec{v}_2)^T = \vec{v}_2^T A \vec{v}_1 = \lambda_1 \vec{v}_2^T \vec{v}_1 .$$

$$(\lambda_1 \vec{v}_2^T \vec{v}_1)^T = \lambda_1 \vec{v}_1^T \vec{v}_2 = \lambda_2 \vec{v}_1^T \vec{v}_2 ,$$

This implies that $\lambda_1 = \lambda_2 \rightarrow\leftarrow$, ∎