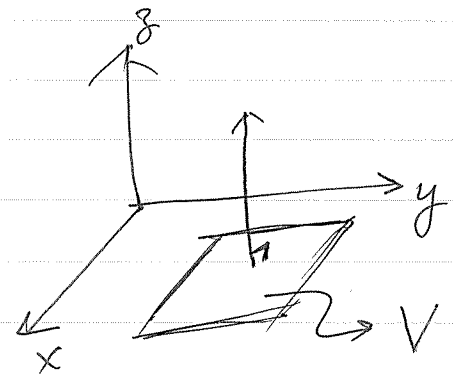
Orthogonal Complements

A vector $\vec{u} \in \mathbb{R}^m$ is orthogonal to a subspace V of $\mathbb{R}^m$, denoted $\vec{u} \perp V$, if $\vec{u} \perp \vec{v}$ $\forall \vec{v} \in V$, i.e. $\langle \vec{u}, \vec{v} \rangle = 0$ $\forall \vec{v} \in V$.

Example



$V = \mathrm{Span}(\{(1,0,0),(0,1,0)\})$, $\vec{u} = (0,0,c)$, $c \in \mathbb{R}$.

Example If $V = \mathbb{R}^m$ and $\vec{u} \in \mathbb{R}^m$ s.t. $\vec{u} \perp V$, then $\vec{u} = \vec{0}$.

Fact If $V = \mathrm{Span}(\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\})$, then
" $\vec{u} \perp V \iff \vec{u} \perp \vec{v}_i$ $\forall i = 1, 2, \dots, k$. "

Proof. ($\Rightarrow$) Clearly true.

($\Leftarrow$) $\forall \quad \vec{v} \in V, \quad \vec{v} = \sum_{i=1}^{k} c_i \vec{v}_i$.

$$\langle \vec{u}, \vec{v} \rangle = \langle \vec{u}, \sum_{i=1}^{k} c_i \vec{v}_i \rangle = \sum_{i=1}^{k} c_i \langle \vec{u}, \vec{v}_i \rangle = 0. \quad \blacksquare$$

Definition (Orthogonal complement of V)

The set $V^{\perp} = \{\vec{u} \in \mathbb{R}^n \mid \vec{u} \perp V\}$ is called the orthogonal complement of V. (Note that $V^{\perp}$ is also a subspace of $\mathbb{R}^n$.)

Theorem Let V be a vector subspace of $\mathbb{R}^n$. Then the following statements are true.

(a) $V^{\perp}$ is a subspace of $\mathbb{R}^n$.

(b) If $\dim V = r$, then $\dim V^{\perp} = n - r$.

(c) $V \cap V^{\perp} = \{\vec{0}\}$.

(d) If $\{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_r\}$ forms a basis of V and $\{\vec{w}_1, \vec{w}_2, \cdots, \vec{w}_{n-r}\}$ forms a basis of $V^{\perp}$, then $\{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_r, \vec{w}_1, \vec{w}_2, \cdots, \vec{w}_{n-r}\}$ form a basis of $\mathbb{R}^n$.

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Proof.

(a) Let $\vec{u}_1, \vec{u}_2 \in V^\perp$. Then $\vec{u}_1 + \lambda \vec{u}_2 \in V^\perp$, $\forall \lambda \in F$. $(\mathbb{R})$

(b) follows from (d).

(c) If $\vec{u} \in V \cap V^\perp$, then $\langle \vec{u}, \vec{u} \rangle = 0$ and thus $\vec{u} = \vec{0}$.

(d) It suffices to prove that $\{\vec{v}_1, \cdots, \vec{v}_r, \vec{w}_1, \cdots, \vec{w}_{n-r}\}$ is an independent set.

Let $\lambda_1 \vec{v}_1 + \cdots + \lambda_r \vec{v}_r + \mu_1 \vec{w}_1 + \cdots + \mu_{n-r} \vec{w}_{n-r} = \vec{0}$.

Then $\underbrace{\lambda_1 \vec{v}_1 + \cdots + \lambda_r \vec{v}_r}_{\in V} = -(\mu_1 \vec{w}_1 + \cdots + \mu_{n-r} \vec{w}_{n-r}) \in V^\perp$.

This implies that $\lambda_1 \vec{v}_1 + \cdots + \lambda_r \vec{v}_r = \vec{0}$ and

$\mu_1 \vec{w}_1 + \cdots + \mu_{n-r} \vec{w}_{n-r} = \vec{0}$.

Hence, $\lambda_1 = \lambda_2 = \cdots = \lambda_r = 0 = \mu_1 = \mu_2 = \cdots = \mu_{n-r}$. ∎

(*) Let A be an $m \times n$ matrix. Then the column space $C(A)$ of A is a subspace of $\mathbb{R}^m$.

$A = [\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n]$

(**) $C(A)^\perp = Null(A^T)$.

proof. Let $\vec{v} \in Null(A^T)$. Then $\vec{v} \in \mathbb{R}^m$ and $A^T \vec{v} = \vec{0}$.

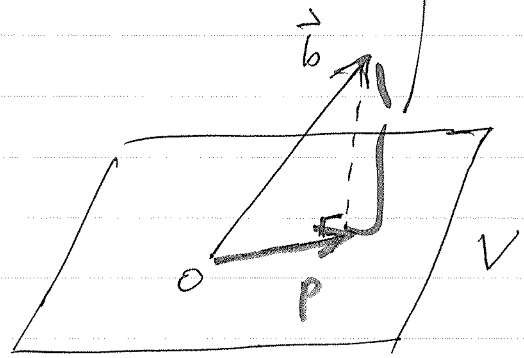
This implies that $\langle \vec{v}_i, \vec{v} \rangle = 0$ $\forall i \in \{1, 2, \cdots, n\}$ $\vec{v} \in C(A)^\perp$

Projections

垂直距離(几何觀点)

$\mathbb{R}^3$

(Example)



$\vec{b} \notin V$ ($\vec{b}$ is not a linear combination of the vectors in V.)

$\Downarrow$

If we consider V as the column space of A,

then $A\vec{x} = \vec{b}$ has no solutions !?

So, what is the "best possible" replacement $\vec{p}$ of $\vec{b}$ such (close to $\vec{b}$)

that $A\vec{x} = \vec{p}$ has a solution?

Answer: $\vec{p}$ is a projection of $\vec{b}$ on V.

V is a subspace of $\mathbb{R}^m$.

<u>Theorem</u> Given $\vec{b} \in \mathbb{R}^m$, there is a unique $\vec{p} \in V$ such that $\vec{b} - \vec{p} \in V^\perp$.

<u>Proof</u>. Let $\{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n\}$ be a basis ~~of~~ for V and $\{\vec{w}_1, \vec{w}_2, \cdots, \vec{w}_{m-n}\}$ be a basis ~~of~~ for $V^\perp$. Then $\{\vec{v}_1, \cdots, \vec{v}_n, \vec{w}_1, \cdots, \vec{w}_{m-n}\}$ is a basis for $\mathbb{R}^m$.

Let $\vec{b} = \sum_{i=1}^{n} \alpha_i \vec{v}_i + \sum_{j=1}^{m-n} \beta_j \vec{w}_j$.

$= \vec{p} + \vec{q}$.

This implies that $\vec{q} \in V^\perp$ and $\vec{q} = \vec{b} - \vec{p}$. (Existence)

Suppose that $\exists\ \vec{p}_1 \in V$ and $\vec{b} - \vec{p}_1 \in V^\perp$.

Then $(\vec{b} - \vec{p}) - (\vec{b} - \vec{p}_1) \in V^\perp$ ($V^\perp$ is a vector subspace)

$$\Downarrow$$

$$\vec{p}_1 - \vec{p} \in V \cap V^\perp.$$

$$\Downarrow$$

$$\Rightarrow \vec{p}_1 - \vec{p} = \vec{0} \quad \text{i.e } \vec{p}_1 = \vec{p}. \qquad \blacksquare$$

<u>Definition</u> (Projection)

Let V be a subspace of $\mathbb{R}^m$ and $\vec{b} \in \mathbb{R}^m$. Then the projection of $\vec{b}$ onto V is the unique vector $\vec{p}$ such that $\vec{b} - \vec{p}$

Note $\vec{p}$ is a projection of $\vec{b}$ onto V iff

$\vec{p}$ is in V and $\vec{b}-\vec{p}$ is in $V^{\perp}$.

($\vec{p} \in V$ and $\vec{b}-\vec{p} \in V^{\perp}$).

(*) If $V = \mathbb{R}^m$, then $V^{\perp} = \{\vec{0}\}$ and thus $\vec{b} = \vec{p}$.

Fact

The projection of $\vec{b}$ onto V ($\vec{p}$) is the vector in V that is closest to $\vec{b}$.

Proof. Consider another vector $\vec{p}'$ in V.

Then $\vec{p} - \vec{p}' \in V$.

Since $\vec{b} - \vec{p} \in V^{\perp}$,

$$\underbrace{\|\vec{p} - \vec{p}'\|^2}_{\geq 0} + \|\vec{b} - \vec{p}\|^2 = \|\vec{b} - \vec{p}'\|^2$$

$$\Rightarrow \|\vec{b} - \vec{p}'\|^2 \geq \|\vec{b} - \vec{p}\|^2.$$

∎

How to find the projection ?

<u>Example</u> Let V be the subspace spanned by $(1,1,1)$.

Find the projection of $(1,1,2)$ onto V and onto $V^{\perp}$ respectively.

Let $A = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

please check if the calculation is correct or not.

$$A^T = [1 \;\; 1 \;\; 1]$$

$$\text{Null}(A^T) = \left\{ \underset{\underset{\vec{x}}{\shortparallel}}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}} \;\middle|\; A^T \vec{x} = \vec{0} \right\}$$

Hence $x_1 + x_2 + x_3 = 0$, Solutions can be written

as $\alpha(1,0,-1) + \beta(0,1,-1)$, i.e.

$\{(1,0,-1), (0,1,-1)\}$ is a basis of $\text{Null}(A^T)$.

$\{(1,1,1), (1,0,-1), (0,1,-1)\}$ is a basis of $\mathbb{R}^3$.

$(1,1,2) = \alpha(1,1,1) + \beta(1,0,-1) + \gamma(0,1,-1)$.

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & -1 & -1 & 2 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & -2 & -1 & 1 \end{array}\right] \to \left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & +1 & -1 & 0 \\ 0 & 1 & \frac{1}{2} & -\frac{1}{2} \end{array}\right)$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \end{array}\right. \quad \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \end{array}\right. \quad \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{4}{3} \end{array}\right.$$

$\alpha = \frac{4}{3}, \ \beta = -\frac{1}{3}, \ \gamma = -\frac{1}{3}$

$\frac{4}{3}(1,1,1)$ is the projection of $(1,1,2)$ onto V.

$-\frac{1}{3}(1,0,-1) - \frac{1}{3}(0,1,-1)$ is the projection of $(1,1,2)$ onto $V^{\perp}$.

<u>Example</u> $V = \mathrm{Span}(\{(2,0,-1),(1,1,0)\})$

$\vec{b} = (7,-1,5)$ Find $\vec{p}$, $\vec{b} - \vec{p}$.

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 2 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & \frac{1}{2} \end{bmatrix}$$

$$\left.\begin{matrix} x_3 = 2, \ x_1 = \frac{1}{2}\cdot 2 = 1 \\ x_3 = 2, \ x_2 = -1 \end{matrix}\right\} \Rightarrow \underline{\underline{(1,-1,2)}}$$

$$\mathrm{Null}(A^T) = \mathrm{Span}(\{\overset{(1,-1,2)}{\cancel{(1,0,2),(0,-1,2)}}\})$$

5 min. later

$$\vec{b} = (7,-1,5) = \underbrace{1\cdot(2,0,-1) + 2(1,1,0)}_{} + 3(1,-1,2)$$

$$\Rightarrow \vec{p} = (4,2,1)$$

So, what is the "General process"?

Notice that $\vec{p} = A\vec{x}'$ for some $\vec{x}'$ and

$$A^T(\vec{b}-\vec{p}) = \begin{bmatrix} \langle \vec{v}_1, \vec{b}-\vec{p} \rangle \\ \langle \vec{v}_2, \vec{b}-\vec{p} \rangle \\ \vdots \\ \langle \vec{v}_n, \vec{b}-\vec{p} \rangle \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ | \\ 0 \end{bmatrix}$$

$$A^T\vec{b} = A^T\vec{p} = A^TA\vec{x}'$$

Fill in this part yourself!

Review. Example in 8

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad A^T = \begin{bmatrix} 2 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 5 & 2 \\ 2 & 2 \end{bmatrix} \qquad A^T\vec{b} = \begin{bmatrix} 2 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 7 \\ -1 \\ 5 \end{bmatrix} = \begin{bmatrix} 9 \\ 6 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 5 & 2 & 9 \\ 2 & 2 & 6 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 5 & 2 & 9 \\ & & \end{array}\right] \rightsquigarrow \vec{x}' = \begin{bmatrix} 1 \\ \end{bmatrix} \quad \vec{p} = A\vec{x}'.$$