Week 10

May 18th

Inner-product space

Definition

An inner-product space is a linear space endowed with an additional algebraic operation, called an inner product.

Definition

Let V be a vector space over F. An inner product on V is a mapping $\varphi : V \times V \to F$ satisfies the following postulates:

(1) $\varphi(\vec{x}, \vec{x}) > 0$ whenever $\vec{x} \neq \vec{0}$.

(2) $\varphi(\vec{x}, \vec{y}) = \overline{\varphi(\vec{y}, \vec{x})}$ $\forall\ \vec{x}, \vec{y} \in V$ where $\bar{t}$ is the conjugate of $t \in \mathbb{C}$.

(3) $\varphi(\vec{x}+\vec{y}, \vec{z}) = \varphi(\vec{x}, \vec{z}) + \varphi(\vec{y}, \vec{z})$.

(4) $\varphi(\alpha\vec{x}, \vec{y}) = \alpha\, \varphi(\vec{x}, \vec{y})$ where $\alpha \in F$.

For convenience, we denote $\varphi(\vec{x}, \vec{y})$ by $\langle \vec{x}, \vec{y} \rangle$.

Or, $\vec{x} \cdot \vec{y}$ (dot product)

Examples

1. In $\mathbb{R}^n$, if $\vec{x} = (x_1, x_2, \cdots, x_n)$ and $\vec{y} = (y_1, y_2, \cdots, y_n)$, then $\langle \vec{x}, \vec{y} \rangle = \sum_{i=1}^{n} x_i y_i$ is an inner product, it is also known as <u>the</u> <u>standard</u> inner product.

2. In $\mathbb{C}^n$, the standard inner product is defined as

$$\langle \vec{x}, \vec{y} \rangle = \sum_{i=1}^{n} x_i \overline{y_i} .$$

3. We may consider $\vec{x} = (x_1, x_2, \cdots, x_n)$ as a real column vector, (so is $\vec{y}$) $\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$. Therefore $\langle \vec{x}, \vec{y} \rangle = \underbrace{\vec{x}^T \vec{y}}_{\text{a scalar}}$. $(= \vec{y}^T \vec{x} \ ?)$

4. Define $A^H = [b_{ij}]_{n \times n}$ where $b_{ij} = \overline{a_{ji}}$.

(Hermitian of A)

$$\begin{bmatrix} 1 & 1+i \\ 1-2i & 2 \end{bmatrix}^H = \begin{bmatrix} 1 & 1+2i \\ 1-i & 2 \end{bmatrix}$$ (See it ?)

5. In $\mathbb{C}^n$, $\langle \vec{x}, \vec{y} \rangle = \sum_{i=1}^{n} x_i \overline{y_i} = \underline{\vec{y}^H \vec{x}}$.

(Not $\vec{x}^H \vec{y}$!)

6. $C[a,b]$: The set of continuous functions defined on $[a,b]$.

$C[a,b]$ is a vector space (over $\mathbb{R}$ or $\mathbb{C}$ or F).

Define $\langle f,g\rangle = \int_a^b f(t)g(t)\,dt$. Then, we have an inner product.

(If the functions on mapping into $\mathbb{C}$, then
$\langle f,g\rangle = \int_a^b f(t)\overline{g(t)}\,dt$ is an inner product.)

7. More examples?

The norm in an inner-product space

Definition The magnitude (or length or norm) of a vector $\vec{x}$ in an inner-product space can be defind as

$$\|\vec{x}\| = \langle \vec{x},\vec{x}\rangle^{\frac{1}{2}}. \quad = \left(\vec{x}^H\vec{x}\right)^{1/2}$$

Note : In $\mathbb{R}^n$, $\|\vec{x}\| = \underline{|\vec{x}|}$ (length).

$$\| \quad \left(\sum_{i=1}^{n} x_i^2\right)^{1/2}.$$

In $\mathbb{C}^n$, $\|\vec{x}\| = \left(\sum_{i=1}^{n} x_i \overline{x_i}\right)^{1/2}$.

$\vec{u} = (2+i, 4-3i)$

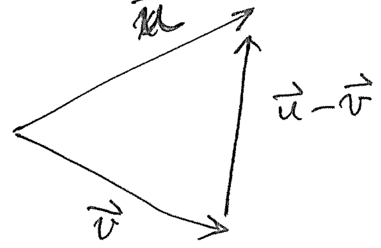
$\|\vec{u}\| = (2+i)(2-i) + (4-3i)(4+3i) = 5 + 25 = 30$.

Now, we come back to deal with vectors in $\mathbb{R}^n$.

Theorem

Let $\vec{u}, \vec{v} \in \mathbb{R}^n$. Then $\vec{u}$ and $\vec{v}$ are perpendicular (orthogonal) if and only if $\langle \vec{u}, \vec{v} \rangle = 0$.

Proof.



($\Longrightarrow$) If $\vec{u} \perp \vec{v}$, then

$\|\vec{u}-\vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2$ (Pythagorean Theorem).

$$\begin{aligned} \|\vec{u}-\vec{v}\|^2 &= \langle \vec{u}-\vec{v}, \vec{u}-\vec{v} \rangle \\ &= \langle \vec{u}, \vec{u} \rangle + \langle \vec{v}, \vec{v} \rangle - 2\langle \vec{u}, \vec{v} \rangle \\ &= \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\langle \vec{u}, \vec{v} \rangle \end{aligned} \Rightarrow \langle \vec{u}, \vec{v} \rangle = 0.$$

($\Longleftarrow$) Easy.

In fact, if $\vec{u}$ and $\vec{v}$ has an angle θ,

then we have

$$\|\vec{u}-\vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\|\vec{u}\|\|\vec{v}\|\cos\theta \quad \text{(Law of Cosine)}$$

$$\Rightarrow \quad -2\langle \vec{u},\vec{v}\rangle = -2\|\vec{u}\|\|\vec{v}\|\cos\theta.$$

$$\Rightarrow \quad \cos\theta = \frac{\langle \vec{u},\vec{v}\rangle}{\|\vec{u}\|\,\|\vec{v}\|}. \quad \text{———} (**)$$

<u>Definition</u> A set $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_k$ of vectors in a vector space V is called "orthogonal" if $\langle \vec{v}_i, \vec{v}_j\rangle = 0$ whenever $1 \le i \ne j \le k$.

<u>**Theorem**</u> If $T = \{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_k\}$ is an orthogonal set of nonzero vectors in V, then $T = \{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_k\}$ is an independent set in V.

<u>Proof.</u> Let $\sum_{i=1}^{k} c_i \vec{v}_i = \vec{0}$.

$\forall\, 1 \le j \le k$, consider $\langle \vec{v}_j, \sum_{i=1}^{k} c_i \vec{v}_i\rangle$. (This is equal to 0.)

Since T is an orthogonal set, $\langle \vec{v}_j, \sum_{i=1}^{k} c_i \vec{v}_i\rangle = \sum_{i=1}^{k} c_i \langle \vec{v}_j, \vec{v}_i\rangle$

$= c_j \langle \vec{v}_j, \vec{v}_j\rangle = 0$. This implies that $c_j = 0$ since $\vec{v}_j \ne \vec{0}$. ∎

<u>Theorem</u> (Cauchy-Schwarz Inequality)

If $\vec{x}, \vec{y} \in \mathbb{R}^n$, then $|\langle \vec{x}, \vec{y} \rangle| \le \|\vec{x}\| \cdot \|\vec{y}\|$.

($(x_1y_1 + x_2y_2 + \cdots + x_ny_n)^2 \le (x_1^2 + x_2^2 + \cdots + x_n^2)(y_1^2 + y_2^2 + \cdots + y_n^2)$)

Normally we see this ↗.

<u>Proof.</u>

$\forall\ \alpha \in F$

$$\|\vec{x} - \alpha \vec{y}\|^2 = \|\vec{x}\|^2 - 2\alpha\langle \vec{x}, \vec{y} \rangle + \alpha^2 \|\vec{y}\|^2 \ge 0$$

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Let $\alpha = \dfrac{\langle \vec{x}, \vec{y} \rangle}{\|\vec{y}\|^2}$.

$$\|\vec{x}\|^2 - 2 \cdot \frac{\langle \vec{x}, \vec{y} \rangle^2}{\|\vec{y}\|^2} + \langle \vec{x}, \vec{y} \rangle^2 / \|\vec{y}\|^2 \ge 0$$

$$\|\vec{x}\|^2 \cdot \|\vec{y}\|^2 \ge \langle \vec{x}, \vec{y} \rangle^2$$

$$\Rightarrow \quad \|\vec{x}\| \cdot \|\vec{y}\| \ge |\langle \vec{x}, \vec{y} \rangle|. \quad \blacksquare$$

Moreover, equality holds if and only if one of the vectors $\vec{x}, \vec{y}$ is a scalar multiple of the other.

<u>Theorem</u> (Triangle Inequality)

For any two vectors $\vec{x}$ and $\vec{y}$ in $\mathbb{R}^n$,

$$\|\vec{x}+\vec{y}\| \le \|\vec{x}\| + \|\vec{y}\|.$$

<u>Proof.</u>
$$\|\vec{x}+\vec{y}\|^2 = \langle \vec{x}+\vec{y}, \vec{x}+\vec{y} \rangle$$
$$= \|\vec{x}\|^2 + 2\langle \vec{x}, \vec{y} \rangle + \|\vec{y}\|^2$$
$$\le \|\vec{x}\|^2 + 2\|\vec{x}\|\|\vec{y}\| + \|\vec{y}\|^2$$
$$= \left(\|\vec{x}\| + \|\vec{y}\|\right)^2. \qquad \blacksquare$$

Do you know how to prove the Law of Cosine ?