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$$f(A) = f(\lambda)I + A^* \cdot (A - \lambda I)$$

$$= (A - \lambda I) \left[\text{adj}(A - \lambda I) + A^* \right]$$

$$= (A - \lambda I) \left(\underbrace{B_k \lambda^k + B_{k-1} \lambda^{k-1} + \dots + B_0}_{\text{where } k \leq n-1} \right)$$

$$= -B_k \lambda^{k+1} + (B_k A - B_{k-1}) \lambda^k + \dots + (B_1 A - B_0) \lambda + B_0 A.$$

$$f(A) = B_0 A \quad (\text{why?})$$

$$\text{and } B_k = 0, B_k A - B_{k-1} = 0, \dots, B_1 A - B_0 = 0.$$

This implies that $B_k = B_{k-1} = \dots = B_0 = 0$ and thus

$$f(A) = 0. \quad \square$$

(*) Observation

If $f(x) = f_1(x) f_2(x)$ and $f(x) = 0$, then $f_1(x) = 0$ or $f_2(x) = 0$.

(Is this correct?)

(**) Let $f_1(x) = a_m x^m + a_{m-1} x^{m-1} + \dots + a_1 x + a_0$ and

$f_2(x) = b_n x^n + b_{n-1} x^{n-1} + \dots + b_1 x + b_0$ where $a_i, b_j \in \mathbb{R}$, $a_m \neq 0, b_n \neq 0$.

$$f_1(x) f_2(x) = a_m b_n x^{m+n} + (a_m b_{n-1} + b_n a_{m-1}) x^{m+n-1} + \dots = 0$$

$$a_m b_n = 0 \Rightarrow a_m = 0 \text{ or } b_n = 0 \quad (?) \Rightarrow$$

③

(***) For each square matrix A , it is interesting to find a polynomial (nonzero) with smallest degree such that $f(A) = 0$.

Definition

or...
 $\mathbb{R}$ or $\mathbb{C}$

Let A be an $n \times n$ matrix with entries in a field F .

The unique monic polynomial $m(x)$ of smallest degree such that $m(A) = 0$ is called the minimal polynomial of A .

Theorem Let $f(x)$ be the characteristic polynomial of A .
the minimal poly. of A

Then $m(x) \mid f(x)$.

Proof. Suppose not. Let $f(x) = m(x)q(x) + r(x)$ where

$\deg(r(x)) < \deg(m(x))$ and $r(x) \neq 0$. By Cayley-Hamilton

theorem $f(A) = m(A)q(A) + r(A) = 0$. Since $m(A) = 0$,

$r(A) = 0 \rightarrow \leftarrow$.



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Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

Then, $f(\lambda) = (1-\lambda)^2 \lambda$ and $m(\lambda) = (1-\lambda)\lambda$.

An application to solving systems of differential equations

Denote a vector function $\vec{x}(t) = (x_1(t), x_2(t), \dots, x_n(t))$. Then,

$\vec{y}' = \vec{x}'(t) = (x'_1(t), x'_2(t), \dots, x'_n(t))$. Now, assume that

$$x'_i(t) = \sum_{j=1}^n a_{ij} x_j(t), \quad i = 1, 2, \dots, n.$$

We have a system of differential equations and

$$\vec{x}'(t) = A \vec{x}(t) \quad \text{where} \quad A = [a_{ij}]_{n \times n}.$$

($\vec{y}' = A \vec{x}'$ in short.)

Theorem Let A be an $n \times n$ matrix, $\vec{v}$ a nonzero n -vector,

and λ a scalar. Then, the function $\vec{x}(t) = e^{\lambda t} \vec{v}$ is a

solution of $\vec{x}'(t) = A \vec{x}(t)$ if and only if λ is an eigenvalue

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Example

$$\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 2x_1(t) + 2x_2(t) \\ x_1(t) + 3x_2(t) \end{bmatrix}$$

$$\frac{d\vec{x}}{dt} = A \vec{x} \quad \text{where} \quad A = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}$$

$$\frac{d\vec{x}}{dt} = \begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{bmatrix}$$

$$\frac{dx_1}{dt} = 2x_1 + 2x_2$$

$$\frac{dx_2}{dt} = x_1 + 3x_2$$

$$\Rightarrow \begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Solution

$$\vec{x}^{(1)}(t) = \begin{bmatrix} -2e^t \\ e^t \end{bmatrix} = e^t \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

← Special solutions

$$\vec{x}^{(2)}(t) = \begin{bmatrix} e^{4t} \\ e^{4t} \end{bmatrix} = e^{4t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

You may check that $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ is an eigenvector asso. with $\lambda=1$
and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector asso. with $\lambda=4$.

Proof. (Main Theorem)

($\Rightarrow$) Since $\vec{x}(t) = e^{\lambda t} \vec{v}$ is a solution of $\vec{x}'(t) = A \vec{x}(t)$,
 $\lambda e^{\lambda t} \vec{v} = A \cdot e^{\lambda t} \vec{v}$. This implies that

$$A \vec{v} = \lambda \vec{v}.$$

($\Leftarrow$) If $\vec{v}$ is an eigenvector of A asso. with λ , then

$$A \vec{v} = \lambda \vec{v}.$$

Now, $\frac{d}{dt} (e^{\lambda t} \vec{v}) = \lambda e^{\lambda t} \vec{v} = e^{\lambda t} \lambda \vec{v} = e^{\lambda t} A \vec{v} = A (e^{\lambda t} \vec{v}).$

Example $\vec{x}'(t) = A \vec{x}(t)$, $A = \begin{bmatrix} 2 & -5 & 5 \\ 0 & -3 & -1 \\ 0 & -1 & 3 \end{bmatrix}.$

Eigenvalues are 2 (multiplicity 2) and 4 .

Eigenvectors are $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, and $\begin{bmatrix} 5 \\ -1 \\ 1 \end{bmatrix}.$

Solutions $\vec{x}(t) = e^{2t} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, e^{2t} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, e^{4t} \begin{bmatrix} 5 \\ -1 \\ 1 \end{bmatrix}.$

General Solutions (?)

$$\vec{x}(t) = c_1 e^{2t} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_3 e^{4t} \begin{bmatrix} 5 \\ -1 \\ 1 \end{bmatrix}.$$

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Theorem Let A be an $n \times n$ matrix that has n independent eigenvectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$. Let $\lambda_1, \lambda_2, \dots, \lambda_n$ be the corresponding eigenvalues. Then the solutions

$$\vec{x}^{(1)}(t) = e^{\lambda_1 t} \vec{v}_1, \quad \vec{x}^{(2)}(t) = e^{\lambda_2 t} \vec{v}_2, \quad \dots, \quad \vec{x}^{(n)}(t) = e^{\lambda_n t} \vec{v}_n$$

form a basis for the solution space of $\frac{d\vec{x}}{dt} = A\vec{x}$, i.e. the general solution is

$$\begin{aligned} \vec{x}(t) &= c_1 e^{\lambda_1 t} \vec{v}_1 + c_2 e^{\lambda_2 t} \vec{v}_2 + \dots + c_n e^{\lambda_n t} \vec{v}_n \\ &= \sum_{i=1}^n c_i e^{\lambda_i t} \vec{v}_i. \end{aligned}$$

Proof. Routine check.

$$m(\lambda) = (\lambda - 2)(\lambda - 4) \quad (?)$$

$$= \lambda^2 - 6\lambda + 8$$

$$\begin{aligned} m(A) &= \begin{bmatrix} 2 & -5 & 5 \\ 0 & -3 & -1 \\ 0 & -1 & 3 \end{bmatrix}^2 - 6 \begin{bmatrix} 2 & -5 & 5 \\ 0 & -3 & -1 \\ 0 & -1 & 3 \end{bmatrix} + 8 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 4 & 0 & 0 \\ \dots & \dots & \dots \end{bmatrix} \neq 0 \end{aligned}$$

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Note If $\vec{x}^{(1)}(t), \vec{x}^{(2)}(t), \dots, \vec{x}^{(m)}(t)$ are solutions of $\vec{x}'(t) = A\vec{x}(t)$,

then $\sum_{i=1}^m c_i \vec{x}^{(i)}(t)$ is also a solution of $\vec{x}'(t) = A\vec{x}(t)$.

Proof.
$$\frac{d \left(\sum_{i=1}^m c_i \vec{x}^{(i)}(t) \right)}{dt} = \sum_{i=1}^m c_i \frac{d \vec{x}^{(i)}(t)}{dt}$$

$$= \sum_{i=1}^m c_i A \vec{x}^{(i)}(t) = A \left(\sum_{i=1}^m c_i \vec{x}^{(i)}(t) \right). \quad \blacksquare$$

Solving "Homogeneous differential equations".

Let $y = f(t)$ be a function of t .

Equation : $y^{(m)} + a_{n-1}y^{(n-1)} + \dots + a_1y' + a_0y = 0$

Observation $y^{(m)} = -(a_0x_1 + \dots + a_{n-1}x_{n-1})$

Define $\vec{x}(t) = (y, y', y'', \dots, y^{(m-1)})$

$$= (x_1(t), x_2(t), x_3(t), \dots, x_n(t))$$

$$\vec{x}'(t) = (x_1'(t), x_2'(t), \dots, x_n'(t)) = A(x_1(t), x_2(t), \dots, x_n(t))$$

$$= \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ & & \vdots & & & \\ 0 & 0 & 0 & \dots & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}$$

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(*) Finding eigenvalues of A gives the solutions.

Example $y'' - y = 0$

$$y'' = y, \quad A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\det(A - \lambda I) = 0 \Rightarrow \lambda = \pm 1. \quad \text{eigenvectors} \begin{cases} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \end{cases}$$

$$\text{Solutions are } e^t \begin{bmatrix} 1 \\ 1 \end{bmatrix}, e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\begin{matrix} \parallel & \parallel \\ \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} & \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \end{matrix} \quad x_1(t) = y.$$

$$y = e^t \text{ or } y = e^{-t}$$

$$\text{General solutions are } c_1 e^t + c_2 e^{-t}.$$

Example $y''' + y'' - 2y' = 0$

$$a_2 = 1, a_1 = -2, a_0 = 0$$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} \quad \det(A - \lambda I) = \det \begin{bmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 0 & 2 & -1-\lambda \end{bmatrix}$$

$$= (-\lambda)(-\lambda)(-1-\lambda) - 2(-\lambda)$$

$$= \lambda^2(1-\lambda) + 2\lambda$$

$$= \lambda(\lambda - 2)(-\lambda - 1) = -\lambda(\lambda + 1)(\lambda - 2)$$

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Example $y'' + y = 0$

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \lambda = \pm i$$

Solutions : e^{it}, e^{-it}

$$c_1 e^{it} + c_2 e^{-it}$$

$$= c_1 (\cos t + i \sin t) + c_2 (\cos t - i \sin t).$$

(*) Eigenvalues may be complex numbers.

How about the system of differential equations

$$\begin{cases} \frac{d^2 x_1}{dt^2} = -5x_1 - 2x_2 \\ \frac{d^2 x_2}{dt^2} = 2x_1 - 2x_2. \end{cases} \quad ?$$

Solution

$$\begin{cases} x_1 = a_1 \cos \omega_1 t + a_2 \cos \omega_2 t \\ x_2 = b_1 \cos \omega_1 t + b_2 \cos \omega_2 t \end{cases} \quad (?)$$

Find $\frac{d^2 x_1}{dt^2}$ and $\frac{d^2 x_2}{dt^2}$ and

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Substitute x_1, x_2 , we have.

$$\omega_1^2 \cos \omega_1 t \begin{bmatrix} a_1 \\ b_1 \end{bmatrix} + \omega_2^2 \cos \omega_2 t \begin{bmatrix} a_2 \\ b_2 \end{bmatrix} = \cos \omega_1 t \begin{bmatrix} 5 & -2 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} a_1 \\ b_1 \end{bmatrix} + \cos \omega_2 t \begin{bmatrix} 5 & -2 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} a_2 \\ b_2 \end{bmatrix}$$

$$\omega_1^2 \begin{bmatrix} a_1 \\ b_1 \end{bmatrix} = \begin{bmatrix} 5 & -2 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} a_1 \\ b_1 \end{bmatrix}, \quad \omega_2^2 \begin{bmatrix} a_2 \\ b_2 \end{bmatrix} = \begin{bmatrix} 5 & -2 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} a_2 \\ b_2 \end{bmatrix}.$$

Hence $\begin{bmatrix} a_1 \\ b_1 \end{bmatrix}$ and $\begin{bmatrix} a_2 \\ b_2 \end{bmatrix}$ are eigenvectors of $\begin{bmatrix} 5 & -2 \\ -2 & 2 \end{bmatrix}$ asso. with eigenvalues ω_1^2 and ω_2^2 .

Eigenvalues of $\begin{bmatrix} 5 & -2 \\ -2 & 2 \end{bmatrix}$ are $\lambda_1 = 1$ and $\lambda_2 = 6$.

Hence $\omega_1 = 1$ and $\omega_2 = \sqrt{6}$

Eigenvectors are $\begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ i.e. $x_2 = 2x_1$, $a_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ for $\lambda_1 = 1$

and for $\lambda_2 = 6$, $b_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

Now, if we have initial conditions say

$x_1 = -1, x_2 = 2$ we can solve $a_1 = \frac{3}{5}$ and $b_2 = \frac{4}{5}$,
(get)
 $a_2 = -\frac{8}{5}$ and $b_1 = \frac{6}{5}$.
(at some t)
 $t = 0$

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$$\begin{cases} -1 = a_1 + a_2 \\ 2 = b_1 + b_2 \\ \cancel{2a_1 + 2a_2} \quad a_2 = -2b_2 \\ b_1 = 2a_1 \end{cases}$$

$$\Rightarrow \begin{cases} -1 = a_1 + a_2 \\ 2 = 2a_1 - \frac{1}{2}a_2 \end{cases}$$

$$\Rightarrow -\frac{1}{2} = \frac{1}{2}a_1 + \frac{1}{2}a_2$$

$$a_1 = -1 - a_2$$

$$\frac{3}{2} = \frac{5}{2}a_1 \Rightarrow \boxed{a_1 = \frac{3}{5}}$$

Week 10 May, 4; 6.

Note If λ is an eigenvalue of A and $\vec{v}$ is the eigenvector asso. with λ , then

$$A^k \vec{v} = \lambda^k \vec{v}$$

||

$$A^{k-1}(A\vec{v}) = \lambda A^{k-1}\vec{v} = \dots = \lambda^k \vec{v}.$$

(*) Let $\lambda_1, \lambda_2, \dots, \lambda_m$ be n eigenvalues of A with asso. eigenvectors $\vec{v}_1^{(1)}, \vec{v}_2^{(2)}, \dots, \vec{v}_m^{(m)}$. If $\{\vec{v}_1^{(1)}, \vec{v}_2^{(2)}, \dots, \vec{v}_m^{(m)}\}$ forms a basis for $\mathbb{R}^n$ (or n -dim. vector space), then

$$\forall \vec{x} \in \mathbb{R}^n, \quad A^k \vec{x} = \sum_{j=1}^n c_j \lambda_j^k \vec{v}_j^{(j)}.$$

||
 $\sum_{i=1}^n c_i \vec{v}_i^{(i)}$

(Hint: A is a linear map from $\mathbb{R}^n$ into $\mathbb{R}^n$.)

Application: Dynamical Systems

$$\vec{x}^{(k)} = A \vec{x}^{(k-1)}, \quad k=1, 2, \dots.$$