Week 8 April 20, 22

(Review)

<u>Theorem</u> If A and B are similar, then they have the same (multi)-set of eigenvalues.

<u>Proof.</u> Since $A \sim B$, $\exists$ invertible matrix P, s.t. $P^{-1}AP = B$. Now, consider

$$
\begin{aligned}
&\det(B-\lambda I) \\
&= \det(P^{-1}AP - \lambda I) \\
&= \det(P^{-1}AP - \lambda P^{-1}P) \\
&= \det(P^{-1}(A-\lambda I)P) \\
&= \det(P^{-1})\det(A-\lambda I)\det(P) \\
&= \det(A-\lambda I)\det(P)\det(P^{-1}) \\
&= \det(A-\lambda I)\det(PP^{-1}) \\
&= \det(A-\lambda I).
\end{aligned}
$$

■

By using the above fact, we can prove that $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ is not diagonalizable.

Suppose not. Let $P = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$.

Then
$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} x & y \\ z & w \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad (?)$$

$$x + z = x \Rightarrow z = 0$$
$$\Rightarrow P \text{ is not invertible.}$$
$$y + w = y \Rightarrow w = 0$$

∎

Another fact

$\lambda = 1$,

$$\dim(\text{Null}(A - I)) = \dim\left(\text{Null}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right)$$

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$ $y = 0$ and x is free.

$$E_A(\lambda) = E_A(1)$$

$$\dim(E_A(1)) = 1.$$

In A,

$\lambda = 1$ is of algebraic multiplicity "2".

Geometric multiplicity

Whether a matrix ($n \times n$) is diagonalizable or not depends on the relationship between algebraic and geometric multiplicity of eigenvalues.

Lemma Let $\det(A-\lambda I) = (-1)^n (\lambda-\lambda_1)^{s_1}(\lambda-\lambda_2)^{s_2}\cdots(\lambda-\lambda_t)^{s_t}$

and $\dim(E_A(\lambda_i)) = g_i$ $\forall\ i=1,2,\cdots,t$. Then $g_i \le s_i$, $\forall\ i=1,2,\cdots$

Proof. Assume that $\dim(E_A(\lambda_i)) = g_i$. Then, there exists a basis $B_i = \{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_{g_i}\}$ of $E_A(\lambda_i)$, moreover, $A\vec{v}_j = \lambda_i \vec{v}_j$ for $j=1,2,\cdots,g_i$.

Since $E_A(\lambda_i)$ is a subspace of $\mathbb{R}^n$ (or $\mathbb{C}^n$), B_i can be extended to a basis $\mathbb{B}$ of $\mathbb{R}^n$ such that $\mathbb{B} = \{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_{g_i}\}$. Now,

A is similar to

$$\tilde{A} = \left[\begin{array}{c|c} \begin{matrix}\lambda_i & & O \\ & \lambda_i & \\ O & & \ddots\ \lambda_i\end{matrix} & * \\ \hline O & \end{array}\right].$$

$$\left(A\begin{bmatrix}\vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_{g_i} & \cdots & \vec{v}_n\end{bmatrix} = \begin{bmatrix}\lambda_i\vec{v}_1 & \lambda_i\vec{v}_2 & \cdots & \lambda_i\vec{v}_{g_i} & A\vec{v}_{g_i+1} & \cdots & A\vec{v}_n\end{bmatrix}\right.$$

$$\left. = \begin{bmatrix}\vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n\end{bmatrix}\left[\begin{array}{c|c} \begin{matrix}\lambda_i & & O \\ & \lambda_i & \\ & & \ddots\ \lambda_i \\ O & & \end{matrix} & \begin{matrix}c_1 \\ c_2 \\ \vdots \\ \\ c_n\end{matrix}\ \ * \end{array}\right] \right)$$

$A\vec{v}_{g_i+1} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n$

This implies that $(\lambda-\lambda_i)^{g_i} \mid \det(\tilde{A}-\lambda I)$. ■

$\Downarrow$

$g_i \le s_i$, $i=1,2,\cdots,t$

<u>Theorem</u> A is diagonalizable if and only if the geometric multiplicity of each eigenvalue of A is equal to the algebraic multiplicity.

<u>Proof</u>. ($\Rightarrow$) Observe that if A is similar to B, then A and B have the same <u>multiset</u> of <u>eigenvalues</u> (spectrum). Moreover, for each eigenvalue λ of A (so is B), $\dim E_A(\lambda) = \dim E_B(\lambda)$ (?)

Now, since for each diagonal matrix D whose geometric and algebraic multiplicity of every eigenvalue of D are equal, a diagonalizable matrix satisfies the same condition.

($\Leftarrow$) It suffices to prove that A has n independent eigenvectors correspond to its eigenvalues. Let $\mathbb{B}_i = \{\vec{v}_{i,1}, \vec{v}_{i,2}, \cdots, \vec{v}_{i,g_i}\}$ be a basis of $E_A(\lambda_i)$, $i=1,2,\cdots,t$. Since $g_i = d_i$, $i=1,2,\cdots,t$, $\mathbb{B} = \bigcup_{i=1}^{t} \mathbb{B}_i$ has n vectors. Now, we claim $\mathbb{B}$ is an independent set. We start with $n=2$. (By induction.)

Let $\underbrace{c_{1,1}\vec{v}_{1,1} + c_{1,2}\vec{v}_{1,2} + \cdots + c_{1,g_1}\vec{v}_{1,g_1}}_{\vec{\alpha}} + \underbrace{c_{2,1}\vec{v}_{2,1} + c_{2,2}\vec{v}_{2,2} + \cdots + c_{2,g_2}\vec{v}_{2,g_2}}_{\vec{\beta}} = \vec{0}$

First, if $\vec{\alpha}$ (or $\vec{\beta}$) is equal to $\vec{0}$, then all coefficients are equal to 0. (?) This implies that $B_1 \cup B_2$ is a l. independent set of g_1+g_2 vectors. On the other hand, $\vec{\alpha} \neq \vec{0}$ and $\vec{\beta} \neq \vec{0}$. By definition, $\vec{\alpha} \in E_A(\lambda_1)$ and $\vec{\beta} \in E_A(\lambda_2)$. But, now, $\vec{\alpha} = -\vec{\beta}$. Therefore $\vec{\alpha} \in E_A(\lambda_2)$, i.e., $A\vec{\alpha} = \lambda_2\vec{\alpha}$ and $A\vec{\alpha} = \lambda_1\vec{\alpha}$. So, $\lambda_1 = \lambda_2$, a contradiction. Hence, we have proved the case $n=2$

Assume that the assertion is true for $n=k$, consider $\mathbb{B}_1 \cup \mathbb{B}_2 \cup \cdots \cup \mathbb{B}_k \cup \mathbb{B}_{k+1}$. Since $\bigcup_{i=1}^{k} \mathbb{B}_i$ is an independent set, we may let $\mathbb{B}' = \bigcup_{i=1}^{k} \mathbb{B}_i$ and the proof follows by a similar argument of $\mathbb{B}' \cup \mathbb{B}_{k+1}$. (In fact, we use the idea that

$$\mathrm{Span}(\mathbb{B}') \cap \mathrm{Span}(\mathbb{B}_{k+1}) = \{\vec{0}\}. \quad (?)$$

Note that if $\underset{\neq \vec{0}}{\vec{\alpha}} \in \mathbb{B}'$, then $\vec{\alpha} \in E_A(\lambda_i)$ for some $i \in \{1, 2, \cdots, k\}$.

■

Let $A = \begin{bmatrix} 11 & -6i \\ 4i & 1 \end{bmatrix}$.

Then, the characteristic poly. of A is $\lambda^2 - 12\lambda - 13$.

Let $\lambda^2 - 12\lambda - 13 = f(\lambda)$.

$f(A) = A^2 - 12A - 13I = \cdots = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

<u>Another example</u>

Let $A = \begin{bmatrix} 5 & 2 & 1 \\ 1 & 1 & 7 \\ 3 & 0 & 11 \end{bmatrix}$.

Then, $p(\lambda; A) = f(\lambda) = -\lambda^3 + 17\lambda^2 - 66\lambda + 72$.

$f(A) = \cdots = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

<u>Theorem</u> (Cayley-Hamilton)

Let $f(\lambda)$ be the characteristic polynomial of $\underset{}{\overset{n\times n \text{ matrix}}{A}}$.

Then, $f(A) = O^{n\times n}$ ($n\times n$ matrix with all entries "0".)

<u>Proof.</u> Will be given later (after 2nd test).

Finding the inverse of a matrix A.

Example $\quad A = \begin{bmatrix} 5 & 2 & 1 \\ 1 & 1 & 7 \\ 3 & 0 & 11 \end{bmatrix}$

$$\det(A - \lambda I) = -\lambda^3 + 17\lambda^2 - 66\lambda + 72.$$

By Cayley-Hamilton's Theorem

$$-A^3 + 17A^2 - 66A + 72I = O^{3\times 3}.$$

$$72I = A^3 - 17A^2 + 66A$$

$$= A(A^2 - 17A + 66I)$$

$$A^{-1} = \frac{1}{72}(A^2 - 17A + 66I) \quad (\text{See it?})$$

(*) 所以,如果 A 有逆矩陣,在代入 A 之後就會得到 $I = A\cdot(A^{-1})$, A^{-1} 於是成為 A 的多項式形式。

What if A is not invertible? Can we use f(A) to determine that "A has no inverse"?

(*) $A \mid f(A)$.

More examples

$P = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ is invertible.

$$\det \begin{bmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{bmatrix} = (1-\lambda)^2 = \lambda^2 - 2\lambda + 1$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^2 - 2\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} + I = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

$$I = 2\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^2$$

$$= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\left(2I - \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}.$$

$$P^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}.$$

$$P = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad f(\lambda) = (1-\lambda)(-\lambda) = \lambda^2 - \lambda$$

$$f(A) = A^2 - A = 0$$

$\underbrace{f(A)}_{=0} = A(A-I)$. Now, if A is invertible,

then $A - I = 0$, $A = I$ $\rightarrow\leftarrow$.

Definition (Quadratic Form)

A quadratic form of A (defined by A) is a function $\vec{x} \longmapsto \vec{x}^T A \vec{x}$ from $\mathbb{R}^n$ ($\mathbb{C}^n$) into $\mathbb{R}$ (or $\mathbb{C}$).

(A is an $n \times n$ matrix.)

(*) Let A and B are real matrices (in $\mathbb{R}^{n \times n}$) such that $B = \frac{1}{2}(A+$

Then A and B have identical quadratic form.

(**) If $\vec{x}$ is an eigenvector of A and $\vec{x}^T\vec{x} = 1$, then

$$\lambda = \vec{x}^T A \vec{x}.$$

If $\vec{x}$ is an (real) eigenvector asso. with λ, i.e. $A\vec{x} = \lambda$

then $\vec{x}/\|\vec{x}\|$ is also an eigenvector asso. with λ

where

$$\|\vec{x}\| = \left(\sum_{i=1}^{n} x_i^2\right)^{1/2}.$$

$\vec{x} \in \mathbb{C}^{n \times 1}$, $\|\vec{x}\| = (\vec{x} \cdot \bar{\vec{x}})^{1/2}$.

$$\begin{aligned}
\vec{x}^T B \vec{x} &= \vec{x}^T \left(\tfrac{1}{2}(A + A^T)\right) \vec{x} \\
&= \tfrac{1}{2}(\vec{x}^T A \vec{x} + \vec{x}^T A^T \vec{x}) \\
&= \tfrac{1}{2}(\vec{x}^T A \vec{x}) + \tfrac{1}{2}\vec{x}^T A^T \vec{x} \\
&= \tfrac{1}{2}(\vec{x}^T A \vec{x}) + \tfrac{1}{2}(\vec{x}^T A \vec{x})^T \\
&= \tfrac{1}{2}(\vec{x}^T A \vec{x}) + \tfrac{1}{2}(\vec{x}^T A \vec{x}) \\
&= \vec{x}^T A \vec{x} \quad \blacksquare
\end{aligned}$$