Week 7 – Continued (April, 15th)

Theorem

If A is a symmetric real matrix, then

(a) all of the eigenvalues of A are real; and

(b) A is diagonalizable.

Proof. (a) Let $\vec{x}^* = \overline{(\vec{x}^T)}$. (Take the conjucate of the transpose of $\vec{x}$.)

Let $\vec{x} = (x_1, x_2, \cdots, x_n)$.

Then $\overline{(\vec{x}^T)} = [\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_n]$. This implies that

$$\vec{x}^*\vec{x} = \sum_{i=1}^{n} x_i \bar{x}_i = \sum_{i=1}^{n} |x_i|^2 \text{ (a real \#.)}$$

$$A\vec{x} = \lambda\vec{x}$$

$$\vec{x}^* A \vec{x} = \lambda \vec{x}^* \vec{x} = \lambda \sum_{i=1}^{n} |x_i|^2$$

$\vec{x}^* A \vec{x}$ is a real number, since $(\vec{x}^* A \vec{x})^*$

$$= \vec{x}^* A^* \vec{x} = \vec{x}^* A \vec{x}.$$

This implies that $\lambda = \dfrac{\vec{x}^* A \vec{x}}{\vec{x}^* \vec{x}}$ is real. ∎

(b) Can you do this yourself?

Example

$$A = \begin{bmatrix} 1 & -2 & 2 \\ -2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$$ (Symmetric Real).

$$p(\lambda; A) = \det(A - \lambda I) = (\lambda-3)^2(\lambda+3) = 0$$

$\lambda = 3$ (multiplicity 2), $\lambda = -3$

$$A - 3I = \begin{bmatrix} -2 & -2 & 2 \\ -2 & -2 & 2 \\ 2 & 2 & -2 \end{bmatrix} \xrightarrow{\text{Reduced}} \begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\mathrm{Null}(A - 3I) = \{(x_1, x_2, x_3) \mid x_1 + x_2 = x_3\}$$

Choose two vectors $(1,-1,0)$, $(1,0,1)$.
(independent)

(*) Both of them are eigenvectors of A asso. with $\lambda = 3$.

$$A + 3I = \begin{bmatrix} 4 & -2 & 2 \\ -2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix} \xrightarrow{\text{Reduced}} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\mathrm{Null}(A + 3I) = \{(x_1, x_2, x_3) \mid x_1 + x_3 = 0,\ x_2 + x_3 = 0\},$$

Choose one vector $(-1,-1,1)$ (eigenvector of A asso. with $\lambda = -3$.)

Since $(1,-1,0)$, $(1,0,1)$ and $(-1,-1,1)$ are three independent eigenvectors associated with eigenvalues $3, -3$, A is diagonalizable.

(*) $E_A(3)$ has dimension 2. $\Rightarrow$ The algebraic multiplicity is equal to the geometric multiplicity of $\lambda = 3$, so is that for $\lambda = -3$.

Theorem.

Let A be an $n \times n$ matrix and let $\lambda_1, \lambda_2, \cdots, \lambda_t$ be the distinct eigenvalues and $\det(A - \lambda I) = (-1)^n \prod_{i=1}^{t} (\lambda - \lambda_i)^{s_i}$. Then A is diagonalizable if and only if $\dim E_A(\lambda_i) = s_i$, $i = 1, 2, \cdots, t$, i.e., the geometric multiplicity is equal to the algebraic multiplicity for each eigenvalue λ_i, $i = 1, 2, \cdots, t$