Week 6 April, 8th.

(•) Let λ be an eigenvalue of a square matrix A. Then
$(n \times n)$

$\{\vec{x} \mid A\vec{x} = \lambda\vec{x}\}$ is a vector subspace of $\mathbb{R}^n$ (or $\mathbb{C}^n$).

(•) Note here that we shall consider $\mathbb{C}^n$ in this chapter.

Consider $A = \begin{bmatrix} 1 & 1 \\ -2 & 3 \end{bmatrix}$.

$\det\left(\begin{bmatrix} 1-\lambda & 1 \\ -2 & 3-\lambda \end{bmatrix}\right) = 0 \Rightarrow \lambda^2 - 4\lambda + 5 = 0 \Rightarrow \lambda = 2 \pm i$.

For $\lambda = 2+i$. Let $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ be the eigenvector of A associated with λ. Then $A\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$$\begin{cases} x_1 + x_2 = (2+i)x_1 \\ -2x_1 + 3x_2 = (2+i)x_2 \end{cases}$$

$$\begin{cases} (1+i)x_1 - x_2 = 0 \\ -2x_1 + (1-i)x_2 = 0 \end{cases}$$

$\det\left(\begin{bmatrix} 1+i & -1 \\ -2 & 1-i \end{bmatrix}\right) = 0 \Rightarrow$ Two equations are the same.

Let $x_1 = 1$. Then $x_2 = 1+i$.

$\begin{bmatrix} 1 \\ 1+i \end{bmatrix}$ is an eigenvector of A associated with $\lambda = 2+i$

<u>Theorem</u> (Important)

Any set of eigenvectors corresponding to distinct eigenvalues of a matrix A is linearly independent.

<u>Proof.</u> Let $\{\lambda_1, \lambda_2, \cdots, \lambda_t\}$ be a set of distinct eigenvalues of A and $A\vec{x}_1 = \lambda_1\vec{x}_1, \cdots, A\vec{x}_t = \lambda_t\vec{x}_t$. We claim that $\{\vec{x}_1, \vec{x}_2, \cdots, \vec{x}_t\}$ is linear independent.

Suppose not. Then, there exists an index $j < t$ such that $\vec{x}_{j+1} = \sum_{i=1}^{j} c_i\vec{x}_i$ and $\{\vec{x}_1, \vec{x}_2, \cdots, \vec{x}_j\}$ is l. indep. Hence $A\vec{x}_{j+1} = A\left(\sum_{i=1}^{j} c_i\vec{x}_i\right)$.

$$\lambda_{j+1}\vec{x}_{j+1} = \sum_{i=1}^{j} c_i\lambda_i\vec{x}_i$$

$$\parallel$$

$$\lambda_{j+1}\left(\sum_{i=1}^{j} c_i\vec{x}_i\right)$$

$$\Rightarrow \sum_{i=1}^{j} c_i(\lambda_i - \lambda_{j+1})\vec{x}_i = \vec{0}$$

Since $\{\vec{x}_1, \vec{x}_2, \cdots, \vec{x}_j\}$ is linearly independent,

$$c_i(\lambda_i - \lambda_{j+1}) = 0 \quad \forall\, i = 1, 2, \cdots, j.$$

$$\Rightarrow c_i = 0 \qquad (\lambda_i \neq \lambda_{j+1})$$

$$\Rightarrow \vec{x}_{j+1} = \vec{0} \rightarrow\leftarrow$$ (Eigenvectors are nonzero vectors!)

$$p(\lambda; A) = \det\left(\begin{bmatrix} A - \lambda I \end{bmatrix}_{n \times n}\right)$$

$$= \pm(\lambda - \lambda_1)(\lambda - \lambda_2)\cdots(\lambda - \lambda_n)$$

$$= \pm(\lambda - \lambda_1)^{a_1}(\lambda - \lambda_2)^{a_2}\cdots(\lambda - \lambda_t)^{a_t}$$

(∘) $a_i : i = 1, 2, \cdots, t$

↳ algebraic multiplicity of λ_i (A)

$$E(\lambda_i) = \{\vec{x} \mid A\vec{x} = \lambda_i \vec{x}\}$$

$\dim(E(\lambda_i)) = d_i$

↳ geometric multiplicity of λ_i (A)

(∘) The algebraic multiplicity (resp. geometric multiplicity) of $\underline{\lambda}$ of $\underline{A}$ is defined as above.

(*) $\boxed{d_i \le a_i \quad \forall\, i = 1, 2, \cdots, t.}$ Can you prove it?

Definition (Eigenvalues of l. transformations)
(operators)

Let L be a l. transf. mapping V into itself.

If $L(\vec{v}) = \lambda \vec{v}$ and $\vec{v} \neq \vec{0}$, then we call λ an eigenvalue and $\vec{v}$ an eigenvector of L.

(*) $[A]_{n \times n} : \mathbb{C}^n \to \mathbb{C}^n$.

(*) Let $V = C^2[-\pi, \pi]$ (twice-differentiable).

$\sin t \in V$ (sine function) and $\underset{\sim}{L}(\vec{v}) = \vec{v}'' = -\vec{v}$ if $\vec{v} = \sin t$.

Hence "-1" is an eigenvalue and $\sin t$ (sine function) is an eigenvector of L.

We shall talk about this later.

Review that <u>A is similar to B if there exits an invertible matrix P such that $B = P^{-1}AP$.</u> Finding P is not easy at all.

Definition (Diagonalizable)

A matrix A is diagonalizable if $\exists$ a diagonal matrix D such that A is similar to D, i.e., $\exists P$ (invertible) s.t. $P^{-1}AP = D$ or $PDP^{-1} = A$.

Example Is $\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$ diagonalizable?

Step 1. Find the eigenvalues of $\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$.

$$\det\left(\begin{bmatrix} 1-\lambda & 1 \\ 0 & 2-\lambda \end{bmatrix}\right) = 0 \Rightarrow \lambda = 1, 2.$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} = 2 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector of $\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$ associated with $\lambda = 2$.

$$\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

↳ eigenvector asso. with $\lambda = 1$

$$\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} \overset{\vec{v}_1}{1} & \overset{\vec{v}_2}{1} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} A\vec{v}_1 & A\vec{v}_2 \end{bmatrix} = \begin{bmatrix} \lambda_1\vec{v}_1 & \lambda_2\vec{v}_2 \end{bmatrix} = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

Why

$$\begin{bmatrix} 1 & 1 \\ \cdots \end{bmatrix}^{-1}\begin{bmatrix} 1 & 1 \\ \cdots \end{bmatrix}\begin{bmatrix} 1 & 1 \\ \cdots \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \cdots \end{bmatrix}$$

Week 7 April 13, 15.

<u>Definition</u> An $n \times n$ matrix A is said to be diagonalizable if there is a matrix (diagonalizing matrix) such that $P^{-1}AP = \Lambda$ is a diagonal matrix.

<u>Theorem</u> An $n \times n$ matrix A is diagonalizable if and only if it has n independent eigenvectors.

<u>Proof.</u> ($\Rightarrow$)

Let $P = [\vec{w}_1\ \vec{w}_2 \cdots \vec{w}_n]$ and $\Lambda = \begin{bmatrix} d_1 & & & 0 \\ & d_2 & & \\ & & \ddots & \\ 0 & & & d_n \end{bmatrix}$.

Since A is diagonalizable, $\exists P$, s.t. $P^{-1}AP = \Lambda$.

Then $AP = P\Lambda$. This implies that

$$\begin{bmatrix} A\vec{w}_1 & A\vec{w}_2 & \cdots & A\vec{w}_n \end{bmatrix} = \begin{bmatrix} d_1\vec{w}_1 & d_2\vec{w}_2 & \cdots & d_n\vec{w}_n \end{bmatrix}.$$

By the fact that P is invertible, $\{\vec{w}_1, \vec{w}_2, \cdots, \vec{w}_n\}$ is a l. indep. set. Moreover, $A\vec{w}_i = d_i\vec{w}_i$, $i = 1, 2, \cdots, n$.

Thus, $\vec{w}_1, \vec{w}_2, \cdots, \vec{w}_n$ are independent eigenvectors of A and $d_1, d_2, \cdots, d_n$ are the associated eigenvalues.

($\Leftarrow$)

Let $\{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n\}$ be a set of l. indep. eigenvectors of A with associated eigenvalues $\mu_1, \mu_2, \cdots, \mu_n$.

Then
$$A \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{bmatrix} = \begin{bmatrix} A\vec{v}_1 & A\vec{v}_2 & \cdots & A\vec{v}_n \end{bmatrix} = \begin{bmatrix} \mu_1\vec{v}_1 & \mu_2\vec{v}_2 & \cdots & \mu_n\vec{v}_n \end{bmatrix}$$

$$= \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{bmatrix} \begin{bmatrix} \mu_1 & & & 0 \\ & \mu_2 & & \\ & & \ddots & \\ 0 & & & \mu_n \end{bmatrix}.$$
The proof follows by letting

$$P = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{bmatrix}.$$ ∎

<u>Corollary</u> If A is an $n \times n$ matrix with n distinct eigenvalues, then A is diagonalizable.

<u>Proof.</u> Corresponding eigenvectors are l. indep. ∎

<u>Application</u> Find A^k for large k.

(~~If~~ A is diagonalizable!)

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}$$

$$p(\lambda; A) = \det\left(\begin{bmatrix} 1-\lambda & 0 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 2 & 1-\lambda \end{bmatrix}\right)$$

$$= (1-\lambda)^3 - (1-\lambda) = (1-\lambda)\left[(1-\lambda)^2 - 1\right]$$

$$= (1-\lambda)(1-\lambda-1)(1-\lambda+1) = 0$$

$\lambda = 0, 1, 2.$ Eigenvectors are $(-1, 0, 1), (-2, 1, 0), (1, 0, 1).$

$$P = \begin{bmatrix} -1 & -2 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad P^{-1} = \begin{bmatrix} -\frac{1}{2} & -1 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 1 & \frac{1}{2} \end{bmatrix}$$

$$A^{100} = ?$$

$$P^{-1}AP = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$A = P \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} P^{-1}$$

$$A^{100} = \left(P \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} P^{-1}\right)^{100} = P \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}^{100} P^{-1}$$

Result 1

Let A be an $n \times n$ matrix with eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$.

Then $\sum_{i=1}^{n} \lambda_i = \mathrm{trace}(A) = \sum_{i=1}^{n} a_{ii}$ (1) and $\prod_{i=1}^{n} \lambda_i = \det A$ (2).

Proof. By definition, $\det(A - \lambda I) = (-1)^n \prod_{i=1}^{n} (\lambda - \lambda_i)$.

By letting $\lambda = 0$, then we have (2) $\det A = (-1)^n (-1)^n \prod_{i=1}^{n} \lambda_i$.

By $(-1)^n \prod_{i=1}^{n} (\lambda - \lambda_i)$ we have

$$(-1)^n \left[\lambda^n - (\lambda_1 + \lambda_2 + \cdots + \lambda_n)\lambda^{n-1} + (\lambda_1\lambda_2 + \cdots + \lambda_{n-1}\lambda_n)\lambda^{n-2} + \cdots \right]$$

$(-1)^n \cdot (-1) \cdot (\lambda_1 + \lambda_2 + \cdots + \lambda_n)$ is the coefficient of λ^{n-1} in the expansion.

Now, in $\det[A - \lambda I] = \det \begin{bmatrix} a_{11}-\lambda & & * \\ & a_{22}-\lambda & \\ & & \ddots & \\ ** & & & a_{nn}-\lambda \end{bmatrix}$,

the coefficient of λ^{n-1} is $\sum_{i=1}^{n} (-1)^{n-1} a_{ii} = (-1)^{n-1} \sum_{i=1}^{n} a_{ii}$.

Hence we have $\sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} a_{ii} = \mathrm{trace}(A)$. ▨

Result 2

If A and B are similar, then A and B have the same "multiset" of eigenvalues.

Proof.

$B\vec{x} = \lambda\vec{x}$ $\quad (B = P^{-1}AP)$.

$(P^{-1}AP)\vec{x} = \lambda\vec{x}$

$AP\vec{x} = \lambda P\vec{x}$

$A(\underline{P\vec{x}}) = \lambda(\underline{P\vec{x}})$

Theorem (Result 3)

If A is a real matrix and has a complex eigenvalue $\overset{a+bi}{\overset{\parallel}{\lambda}}$, then the conjugate $\bar{\lambda} = a-bi$ $(a, b \in \mathbb{R})$ is also an eigenvalue of A.

Proof. $A\vec{x} = \lambda\vec{x}$

$\overline{A\vec{x}} = \overline{\lambda\vec{x}}$ $\quad A\bar{\vec{x}} = \bar{\lambda}\bar{\vec{x}}$. ▨

(Result 4) Exercise

If A is a symmetrix (real) matrix, then all eigenvalues are real. Moreover, A is diagonalize.

not necessary

(Eigenvectors $(\vec{x}, \vec{y})$ corresponding to different eigenvalues $(\lambda_1 \neq \lambda_2)$ are orthogonal,

i.e. $\vec{x}^T\vec{y} = 0$.)