$$[\vec{v}]_{\mathbb{B}} = (c_1, c_2, \cdots, c_n) \in \mathbb{R}^n$$

$(\mathbb{B} = \{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n\}$ is a basis of V over $\mathbb{R}$.)

Let $\mathbb{C} = \{\vec{u}_1, \vec{u}_2, \cdots, \vec{u}_n\}$ be another basis of V.

Then $\vec{v}_i = \sum_{j=1}^{n} a_{ji} \vec{u}_j$, $i = 1, 2, \cdots, n$.

$$[\vec{v}]_{\mathbb{C}} = [c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n]_{\mathbb{C}}$$

$$= \sum_{i=1}^{n} c_i [\vec{v}_1]_{\mathbb{C}}$$

$$= \underbrace{\begin{bmatrix} [\vec{v}_1]_{\mathbb{C}} & [\vec{v}_2]_{\mathbb{C}} & \cdots & [\vec{v}_n]_{\mathbb{C}} \end{bmatrix}}_{P \text{ invertible}} \underbrace{\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}}_{[\vec{v}]_{\mathbb{B}}}$$

Example $\mathbb{B} = \{(1,3), (-1,7)\}$ and $\mathbb{C} = \{(-1,2), (2,1)\}$

(Bases of $\mathbb{R}^2$)

$$[(0,10)]_{\mathbb{B}} = (1,1), \quad [(0,10)]_{\mathbb{C}} = (4,2)$$

$$(1,3) = 1\cdot(-1,2) + 1\cdot(2,1)$$

$$(-1,7) = 3\cdot(-1,2) + 1\cdot(2,1)$$

$$P = \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix}$$

$$[0,10]_C = P\cdot [0,10]_B$$

$$= \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} = \underline{\underline{(4,2)}}.$$

$$\left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right]$$

$$\rightsquigarrow \left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & -2 & -1 & 1 \end{array}\right]$$

$$\rightsquigarrow \left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & 1 & \frac{1}{2} & -\frac{1}{2} \end{array}\right]$$

$$\rightsquigarrow \left[\begin{array}{cc|cc} 1 & 0 & -\frac{1}{2} & +\frac{3}{2} \\ 0 & 1 & \frac{1}{2} & -\frac{1}{2} \end{array}\right]$$

$$\frac{1}{2}\begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -1 & 3 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Longrightarrow P^{-1} = \frac{1}{2}\begin{bmatrix} -1 & 3 \\ 1 & -1 \end{bmatrix}$$

$$P^{-1}\begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Let $\mathbb{B} = \{\vec{u}_1, \vec{u}_2, \cdots, \vec{u}_n\}$ and
$\mathbb{C} = \{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n\}$.

$P = [[\vec{u}_1]_{\mathbb{C}}, [\vec{u}_2]_{\mathbb{C}}, \cdots, [\vec{u}_n]_{\mathbb{C}}]$ is called the transition matrix from $\mathbb{B}$ to $\mathbb{C}$ and

$$\forall \vec{x} \in V,\ [\vec{x}]_{\mathbb{B}} = P[\vec{x}]_{\mathbb{C}}.$$

$\Longrightarrow$ P^{-1} is the transition matrix from $\mathbb{C}$ to $\mathbb{B}$.

$$[\vec{x}]_{\mathbb{C}} = P^{-1}[\vec{x}]_{\mathbb{B}}.$$

$$T: V \longrightarrow W \quad \text{(linear transformation)}.$$

Review that $T(V)$ is a vector subspace of W.

Let $\underline{\mathbb{B}}$ and $\underline{\mathbb{C}}$ be bases of V and $T(V)$ respectively. (As above).

$$A = \left[[T(\vec{u}_1)]_{\mathbb{C}} [T(\vec{u}_2)]_{\mathbb{C}} \cdots [T(\vec{u}_n)]_{\mathbb{C}} \right]$$

$$\forall \vec{x} \in V,\quad [T(\vec{x})]_{\mathbb{C}} = A[\vec{x}]_{\mathbb{B}}.$$

<u>Proof</u>. See next page.

$\forall \ \vec{x} \in V$,

$$\vec{x} = \sum_{i=1}^{n} a_i \vec{u}_i \qquad \left([\vec{x}]_{\mathbb{B}} = (a_1, a_2, \cdots, a_n) \right)$$

$$[T(\vec{x})]_{\mathbb{C}} = \left[T\left(\sum_{i=1}^{n} a_i \vec{u}_i\right)\right]_{\mathbb{C}} = \left[\sum_{i=1}^{n} a_i T(\vec{u}_i)\right]_{\mathbb{C}} = \sum a_i [T(\vec{u}_i)]_{\mathbb{C}}$$

$$= \begin{bmatrix} [T(\vec{u}_1)]_{\mathbb{C}} & [T(\vec{u}_2)]_{\mathbb{C}} & \cdots & [T(\vec{u}_n)]_{\mathbb{C}} \end{bmatrix} [\vec{x}]_{\mathbb{B}}$$

$$= A \, [\vec{x}]_{\mathbb{B}} .$$

$\blacksquare$

We can view "coordinate change" as a linear transformation from V into itself (with different bases), $\mathbb{B}$ and $\mathbb{C}$

i.e. $V_{(\mathbb{B})} \longrightarrow V_{(\mathbb{C})}$.

$T([\vec{x}]_{\mathbb{B}}) = [\vec{x}]_{\mathbb{C}}$.

$T : V \longrightarrow V$

① Using $\mathbb{B}$ as a basis of T, we have

$[T(\vec{x})]_{\mathbb{B}} = A[\vec{x}]_{\mathbb{B}}$. $\quad A = \left[[T(\vec{u}_1)]_{\mathbb{B}} \; [T(\vec{u}_2)]_{\mathbb{B}} \cdots \right]$

② Using $\mathbb{C}$ as a basis of T, we have

$[T(\vec{x})]_{\mathbb{C}} = B[\vec{x}]_{\mathbb{C}}$.

③ From $\mathbb{B}$ to $\mathbb{C}$ we have

$[\vec{x}]_{\mathbb{C}} = P[\vec{x}]_{\mathbb{B}}$

④ Combining ① ~ ③

$B[\vec{x}]_{\mathbb{C}} \overset{②}{=} [T(\vec{x})]_{\mathbb{C}} \overset{③}{=} P[T(\vec{x})]_{\mathbb{B}}$

$\overset{①}{=} PA[\vec{x}]_{\mathbb{B}}$

$= PAP^{-1}[\vec{x}]_{\mathbb{C}} \quad \Rightarrow B = PAP^{-1}$.

coordinate-change

$B \sim A$

Similar!

How to check whether given two matrices A and B are similar or not?

(*) If $A \sim B$, then $\exists$ <u>invertible</u> matrix P such that $AP = PB$ (since $A = PBP^{-1}$).

(**) If P is not required to be invertible, then it is easy (why?).

(***) Before we use the idea of eigenvalues, this problem is related to solve the homogeneous system involving n^2 equations in n^2 unknown if P is an $n \times n$ matrix.

- $\vec{x} \longmapsto A\vec{x}$ (linear transformation) and $\boxed{PQP^{-1} = A.}$

Let the columns of P be the basis $\mathbb{B}$. Then

$$\vec{x} = P[\vec{x}]_{\mathbb{B}} \Rightarrow [\vec{x}]_{\mathbb{B}} = P^{-1}\vec{x}$$

$$[A\vec{x}]_{\mathbb{B}} = P^{-1}A\vec{x} = QP^{-1}\vec{x} = Q[\vec{x}]_{\mathbb{B}}.$$

matrix for l. transf. $\vec{x} \mapsto A\vec{x}$.