Week 3 3/16, 3/18

Theorem Let T be a linear transformation from a vector space V (over F) of dimension n into a vector space W (over F) of dimension m.

Prove that the following statements are true:

(1) $\mathrm{Ker}(T)$ is a vector subspace of V.

(2) $\mathrm{Range}(T)$ is a vector subspace of W.

(3) $\dim \mathrm{Ker}(T) + \dim \mathrm{Rang}(T) = n$.

Proof. Let $\vec{u}, \vec{v} \in \mathrm{Ker}(T)$ and $\lambda \in F$.

Then $T(\vec{u} + \lambda\vec{v}) = T(\vec{u}) + T(\lambda\vec{v}) = T(\vec{u}) + \lambda T(\vec{v}) = \vec{0}$.

This concludes the proof of (1).

Let $\vec{x}, \vec{y} \in \mathrm{Range}(T)$ and $\lambda \in F$. By definition,

$\vec{x} = T(\vec{x}')$, $\vec{y} = T(\vec{y}')$ for some $\vec{x}', \vec{y}' \in V$.

$\vec{x} + \lambda\vec{y} = T(\vec{x}') + \lambda T(\vec{y}') = T(\vec{x}' + \lambda\vec{y}')$. Since

$\vec{x}' + \lambda\vec{y}' \in V$, the proof of (2) follows.

Now, we prove (3).

Let $\dim(\mathrm{Range}(T)) = k$ and $\{\vec{y}_1, \vec{y}_2, \cdots, \vec{y}_k\}$ be a basis.

Let $\dim \mathrm{Ker}(T) = h$ and $\{\vec{x}_1, \vec{x}_2, \cdots, \vec{x}_h\}$ be a basis.

Since $\vec{y}_i \in \text{Range}(T)$, let $\vec{u}_i \in V$ and $T(\vec{u}_i) = \vec{y}_i$, $i = 1, 2, \cdots, k$.

Now, consider $\mathbb{B} = \{\vec{x}_1, \vec{x}_2, \cdots, \vec{x}_h, \vec{u}_1, \vec{u}_2, \cdots, \vec{u}_k\}$.

We claim $\mathbb{B}$ is a basis of V and then $\underline{\underline{h + k = n}}$ which concludes the proof.

~~Case~~ Step 1 $\text{Span}(\mathbb{B}) = V$.

Let $\vec{v} \in V$. $T(\vec{v}) \in \text{Range}(T)$, $T(\vec{v}) = \alpha_1 \vec{y}_1 + \alpha_2 \vec{y}_2 + \cdots + \alpha_k \vec{y}_k$

$= \alpha_1 T(\vec{u}_1) + \alpha_2 T(\vec{u}_2) + \cdots + \alpha_k T(\vec{u}_k)$

$= T(\alpha_1 \vec{u}_1 + \alpha_2 \vec{u}_2 + \cdots + \alpha_k \vec{u}_k)$

This implies that $T\left(\vec{v} - \left(\sum_{i=1}^{k} \alpha_i \vec{u}_i\right)\right) = \vec{0}$

$\vec{v} - \left(\sum_{i=1}^{k} \alpha_i \vec{u}_i\right) \in \text{Ker}(T)$

$\vec{v} - \left(\sum_{i=1}^{k} \alpha_i \vec{u}_i\right) = \beta_1 \vec{x}_1 + \beta_2 \vec{x}_2 + \cdots + \beta_h \vec{x}_h$.

Hence $\vec{v} = \alpha_1 \vec{u}_1 + \cdots + \alpha_k \vec{u}_k + \beta_1 \vec{x}_1 + \cdots + \beta_h \vec{x}_h$ and

we have $\text{Span}(\mathbb{B}) \supseteq V$. Since $\text{Span}(\mathbb{B})$ is a subspace of V,

$\text{Span}(\mathbb{B}) \subseteq V$ and thus $\text{Span}(\mathbb{B}) = V$.

~~Case~~ Step 2 $\mathbb{B}$ is an independent set.

Assume that $\alpha_1\vec{u}_1+\cdots+\alpha_k\vec{u}_k+\beta_1\vec{x}_1+\cdots+\beta_h\vec{x}_h=\vec{0}$.

Then $T(\alpha_1\vec{u}_1+\cdots+\beta_h\vec{x}_h)=T(\vec{0})=\vec{0}$.

This implies that $\alpha_1T(\vec{u}_1)+\alpha_2T(\vec{u}_2)+\cdots+\alpha_kT(\vec{u}_k)=\vec{0}$.

Since $\{\vec{y}_1,\vec{y}_2,\cdots,\vec{y}_k\}$ is linear independent, we have

$\alpha_1=\alpha_2=\cdots=\alpha_k=0$. Thus

$$\beta_1\vec{x}_1+\cdots+\beta_h\vec{x}_h=\vec{0}.$$

By the fact that $\{\vec{x}_1,\cdots,\vec{x}_h\}$ is l. independent,

$$\beta_1=\beta_2=\cdots=\beta_h=0. \qquad \blacksquare$$

($\mathbb{B}$ is a basis of V!)

Review

Let $A=[a_{ij}]_{m\times n}$ be an $m\times n$ real matrix.

① $A:\mathbb{R}^n\longrightarrow\mathbb{R}^m$

② The rows (resp. columns) of A span a row (resp. column) space with dimension row (resp. column) **rank**.

③ A is a linear transformation.

④ Null space of $A = \{ \vec{x} \in \mathbb{R}^n \mid A\vec{x} = \vec{0} \}$.

dim. of Null space is nullity of A, Null(A).

⑤ Range space of $A = \{ \vec{y} \in \mathbb{R}^m \mid \vec{y} = A\vec{x}$ for some $\vec{x} \in \mathbb{R}^n \}$.

Range(A) has dimension (column rank).

⑥ $A \rightarrow \tilde{A}$ (row-reduced echelon form)

What can we see from $\tilde{A}$?

(1) The number of nonzero rows : row rank !

(2) The number of columns with pivot "1" : column rank

(3) The number of columns without pivot "1" : nullity (A)!
(free variables!)

(*) ⑦ dim Range(A) + dim Null(A) = n.

Rank(A) + Nullity(A) = n.

(*) ⑧ rank (A) = row rank of A = column rank of A.

(Look at $\tilde{A}$!)

Vector Spaces of Infinite Dimension

① $\mathbb{R}^{\infty} = \{(a_1, a_2, \cdots, a_n, \cdots) \mid a_i \in \mathbb{R}\}$.

② $C[a,b] = \{f \mid f$ is continuous on $[a,b]\}$.

$C'[a,b] = \{f \mid f'$ is continuous on $[a,b]\}$ is a vector subspace of $C[a,b]$.

③ $\mathbb{P}[a,b] = \{f \mid f$ is a polynomial$\}$.

$\mathbb{P}[a,b] \leq C'[a,b] \leq C[a,b]$

$\mathbb{P}_t[a,b] = \{f \mid f$ is a polynomial of degree at most $t\}$.

$\mathbb{P}_t[a,b] \leq \mathbb{P}[a,b]$.

④ $\mathbb{R}^{m\times n} = \{A \mid A$ is an $m\times n$ real matrix$\}$.

$\mathbb{R}^{m\times n}$ is a vector space of dimension mn.

$\{E_{(i,j)} \mid E_{(i,j)} = [d_{k,h}]_{m\times n}$, $d_{k,h} = 1$ if and only if $(k,h) = (i,j)$ $\}$

↑ (0,1)-matrix

is a trivial basis of the vector space $\mathbb{R}^{m\times n}$.

$\mathbb{B} = \{\vec{u_1}, \vec{u_2}, \cdots, \vec{u_n}\}$ is basis of V.

$$\forall \vec{x} \in V, \ \vec{x} = \sum_{i=1}^{n} a_i \vec{u_i}.$$

(*) $(a_1, a_2, \cdots, a_n)$ is called the coordinate vector of $\vec{x}$ relative to $\mathbb{B}$, denoted by $[\vec{x}]_{\mathbb{B}}$.

(**) If we use a distinct basis of V, then the coordinate vector of $\vec{x}$ is going to be "different".

Example: $\mathbb{B} = \{(1,0,0), (0,1,0), (0,0,1)\}$.

$$[(3,12,13)]_{\mathbb{B}} = (3,12,13).$$

$$\mathbb{C} = \{(1,3,1), (2,1,4), (3,-2,3)\}$$

Chebyshev polynomials

$\mathbb{B}$: $\underline{T_0(x) = 1}$, $\underline{T_1(x) = x}$, $\underline{T_2(x) = 2x^2 - 1}$, $\underline{T_3(x) = 4x^3 - 3x}$.

Form a basis for $\mathbb{P}_3 = \{ax^3 + bx^2 + cx + d \mid a,b,c,d \in \mathbb{R}\}$.

<u>Note</u>

$\mathbb{P}_3$ has a natural basis

$\mathbb{C}$: $p_0(x) = 1$, $p_1(x) = x$, $p_2(x) = x^2$ and
$p_3(x) = x^3$.

Check:

$T_0 = p_0$, $T_1 = p_1$, $T_2 = 2p_2 - p_0$, $T_3 = 4p_3 - 3p_1$

$$P = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

$\mathbb{B} =$ $\mathbb{C} =$ 8

(*) Suppose that $\{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n\}$ and $\{\vec{w}_1, \vec{w}_2, \cdots, \vec{w}_m\}$ are bases for V and W, respectively. Let $T: V \to W$ be a l. transf. and let A be the matrix for T w.r. to chosen bases. If $\vec{x}$ is the coordinate vector of $\vec{v}$ w.r. to $\mathbb{B}$, then $A\vec{x}$ is the coordinate vector of $T(\vec{v})$ w.r.t. $\mathbb{C}$.

If $V = W$, then $m = n$ and we have the idea of bases change!

Definition (Matrix for T)

The $m \times n$ matrix whose jth column is the coordinate vector of $T(\vec{v}_j)$ w.r. to. $\mathbb{C}$ is called the matrix for T w.r. to $\mathbb{B}$ and $\mathbb{C}$.

Example $T: \mathbb{R}^3 \to \mathbb{R}^2$

$$T(x_1, x_2, x_3) = (x_1 - x_2,\ x_2 + x_3)$$

$\mathbb{B} = \{\vec{v}_1 = (2,0,1),\ \vec{v}_2 = (0,2,2),$ ~~and~~ $\vec{v}_3 = (0,2,3)\}$.

$\mathbb{C} = \{\vec{w}_1 = (1,2),\ \vec{w}_2 = (0,1)\}$.

$T(\vec{v}_2) = (-2,4) = -2\vec{w}_1 + 8\vec{w}_2$

Matrix for T

$$A_T = \begin{bmatrix} 2 & -2 & -2 \\ -3 & 8 & 9 \end{bmatrix}$$

$$\vec{v} = (-6, 6, 7)$$
$$= -3\vec{v}_1 - \vec{v}_2 + 4\vec{v}_3$$

$$\vec{v}_{\mathbb{B}} = (-3, -1, 4)$$

$$\begin{bmatrix} 2 & -2 & -2 \\ -3 & 8 & 9 \end{bmatrix} \begin{bmatrix} -3 \\ -1 \\ 4 \end{bmatrix} = \begin{bmatrix} -12 \\ 37 \end{bmatrix} = T(\vec{v})_{\mathbb{C}}$$

$$T(\vec{v}) = -12\vec{w}_1 + 37\vec{w}_2 = (-12, 13)$$

$$T(-6, 6, 7) = (-6-6, 6+7) = (-12, 13)$$

$\{(1,0), (1,1)\}_{\mathbb{B}}$ a basis of $\mathbb{R}^2$.

$$(2,3) = (-1)(1,0) + \cancel{}\, 3\cdot(1,1)$$

$$(1,0)_{\mathbb{C}} = 1\cdot(1,0) + 0\cdot(1,1)$$

$$(0,1)_{\mathbb{C}} = -1\,(1,0) + 1\cdot(1,1)$$

$$\vec{x} = (2,3)_{\mathbb{B}}$$

$$\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & -1 \\ & 1 \end{bmatrix}$$