Week 15 (12,24)

行列式 (Determinant)

從排列的概念談起：

<u>Definition</u> A permutation of a set S is a 1-1 onto mapping from S onto S.

(*) 廣義而言，S 不一定要是有限集合。

(**) 以下考慮的 S 都是有限集合。

For convenience, we let $S = \{1, 2, \cdots, n\}$
$= [1, n]$

Then we have

<u>Proposition 1</u> There are $n!$ permutations from $[1, n]$ onto $[1, n]$.

<u>Notations</u>

① A permutation α can be expressed as two arrays:

$$\begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ \alpha(1) & \alpha(2) & \alpha(3) & \cdots & \alpha(n) \end{pmatrix}.$$

② A permutation α can be written as the product of disjoint "cycles".

e.g. 1. $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 4 & 1 & 6 & 5 \end{pmatrix}$ is a permutation from $[1,6]$ onto $[1,6]$.

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 4 & 1 & 6 & 5 \end{pmatrix} = (1\,2\,3\,4)(5\,6).$$

Note here that $(1\,2\,3\,4)$ and $(5\,6)$ are also permutations ! (of $[1,6]$)

$$(1\,2\,3\,4) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 4 & 1 & 5 & 6 \end{pmatrix} \text{ and}$$

$$(5\,6) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 6 & 5 \end{pmatrix}.$$

__Definition__ (Transposition)

A transposition is a 2-cycle.

e.g. 2. $(5\,6)$ is a transposition of $[1,6]$.

__Proposition 2__ Any cycle can be written as a product of transpositions and the number of transpositions is either even or odd but not both.

e.g. 3 $(1\,2\,3\,4) = (1\,4)(1\,3)(1\,2)$

（想成一个函数是由三个函数所合成！）

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}.$$

More precisely, an even (odd) cycle contains an odd (even) number of transpositions.

Theorem 3 A permutation can be expressed as a product of transpositions. If the number of transpositions is even (odd), then it is an even (odd) permutation.

e.g. 4 $(1\,2\,3\,4)$ is an odd permutation ~~of~~ $[1,4]$.

$(1\,2)(3\,4)$ is an even permutation of $[1,4]$.

Theorem 4 Let $S=[1,n]$. Then there are $n!$ permutations of S, half of them are even (odd respectively)

e.g. 5 $S=[1,3]$.

Odd permutations : $(1\,2), (1\,3), (2\,3)$.

Even permutations : $(1\,2\,3), (1\,3\,2)$, e (identity) e.g. 6 $S = [1,4]$

Odd permutations: $(12), (13), (14), (23), (24), (34),$
$(1234), (1324), (1423), (1243)$
$(1342), (1432).$

Even permutations: $(12)(34), (13)(24), (14)(23),$
$(123), (132), (134), (143), (234), (243),$
$(124), (142), e.$

Notation $\qquad (O_n, E_n) \qquad$ (Odd, Even)

We use S_n to denote the set of all permutations in $[1,n]$. Then $|S_n| = n!$, $|O_n| = n!/2$, $|E_n| = n!/2$.

Definition (Determinant)

Let A be an $n \times n$ matrix, $A = [a_{ij}]$.

Then $\underline{\det A} = \sum_{\alpha \in S_n} \delta_\alpha a_{1\alpha(1)} a_{2\alpha(2)} \cdots a_{n\alpha(n)}$ where

$$\delta_\alpha = \begin{cases} 1, & \text{if } \alpha \text{ is even;} \\ -1, & \text{if } \alpha \text{ is odd.} \end{cases}$$

e.g. 7. $n = 2$. $\alpha = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$ or $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} = \underset{\text{even}}{e}$ or $\underset{\text{odd}}{(12)}$

$\underline{\det A} = a_{11}a_{22} - a_{12}a_{21}.$

$n=3$

α : $\underbrace{(12), (13), (23)}_{\text{odd}}$, $\underbrace{(123), (132), e}_{\text{even}}$.

$$\det A = a_{12}a_{23}a_{31} + a_{13}a_{32}a_{21} + a_{11}a_{22}a_{33}$$
$$- a_{12}a_{21}a_{33} - a_{13}a_{31}a_{22} - a_{11}a_{23}a_{32}$$

$$\det\left(\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\right)$$

How about $n=4$?

Week 16 12,29; 12,31

Permutations (Continued)

Definition (Review)

Permutations

$\swarrow$ $\searrow$

odd permutations   even permutations

(Fact 1) There are $n!$ permutations from $[1,n]$ onto $[1,n]$.
(of $[1,n]$)

(Fact 2) There are $\frac{n!}{2}$ odd permutations and $\frac{n!}{2}$ even permutations.

It suffices to prove that $|O_n| = |E_n|$.

Proof. Let O_n (E_n) be the set odd (even) permutations of $[1,n]$.

Define $\varphi : O_n \to E_n$ by $\varphi(\pi) = \pi \circ (1\,2)$.

Then φ is 1-1 and onto.

(1-1) $\varphi(\pi_1) = \varphi(\pi_2) \Rightarrow \pi_1 \circ (1\,2) = \pi_2 \circ (1\,2)$

$$\Rightarrow (\pi_1 \circ (1\,2)) \circ (1\,2) = (\pi_2 \circ (1\,2)) \circ (1\,2)$$

$$\Rightarrow \pi_1 = \pi_2 .$$

(onto) $\forall\ \pi \in E_n$, $\pi \circ (1\,2) \in O_n$, and $\varphi(\pi \circ (1\,2)) = \pi$. ▨

How to determine the parity of a permutation?

(Method 1)

A permutation can be written as the composition of disjoint cycles and a cycle of odd (even) length is an even (odd) permutation.

Therefore, if a permutation is written as the composition which has an odd number of even cycles, then it is an odd permutation.

e.g.

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 1 & 3 & 5 & 4 & 7 & 6 \end{pmatrix}$$

$$= \underline{(1\,2)} \circ (3) \circ \underline{(4\,5)} \circ \underline{(6\,7)} \quad \text{is odd.}$$

(Method 2)

A permutation $\begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ j_1 & j_2 & j_3 & \cdots & j_n \end{pmatrix}$ is even (odd resp.) if $\sum_{i=1}^{n}$ (number of larger integers precedes j_i) is even (odd resp.).

e.g.

even $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 1 & 3 & 4 & 5 & 2 \end{pmatrix}$: $0+1+1+1+1+4=8$,

odd $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 3 & 1 & 4 & 5 & 2 \end{pmatrix}$: $0+1+2+1+1+4=9$.

Theorem 1 $\det(A) = \det(A^T)$.

Proof: It suffices to prove that the inverse permutation of an even (odd) permutation is also an even (odd) permutation.

Since $\alpha \circ \alpha^{-1} = e$ (identity) which is even, α and α^{-1} have the same parity. !

The following facts can be checked directly.

(1) Row operations E_1, E_2, E_3

E_3 → use $\alpha\vec{r}_i + \vec{r}_j$ for $\vec{r}_j$

E_2 → change two rows

E_1 → multiply a constant to a row

(2) $\det(A) = 0$ if

(i) one row is $\vec{0}$.

(ii) one column is $\vec{0}$,

(iii) one row is a multiple of the other, i.e., $\vec{r}_i = c\vec{r}_j$.

(iv) one column is a multiple of the other, i.e., $\vec{c}_i = \alpha \vec{c}_j$.

⋮

详细证明 (Week 16 Continued)

Thm. $\det(A) = \det(A^T)$ for each $n\times n$ matrix A.

Proof. Let $A = [a_{ij}]_{n\times n}$ and $A^T = [b_{ij}]_{n\times n}$. Then $b_{ij} = a_{ji}$ for $1 \le i, j \le n$. By definition

$$\det(A^T) = \sum_{\alpha \in S_n} \delta_\alpha \cdot b_{1\alpha(1)} \cdot b_{2\alpha(2)} \cdots b_{n\alpha(n)}$$ where

$$\delta_\alpha = \begin{cases} 1, & \text{if } \alpha \text{ is an even permutation; and} \\ -1 & \text{if } \alpha \text{ is an odd permutation.} \end{cases}$$

Since $b_{ij} = a_{ji}$, $\det(A^T) = \sum_{\alpha \in S_n} \delta_\alpha\, a_{\alpha(1)1}\, a_{\alpha(2)2} \cdots a_{\alpha(n)n}$

$$= \sum_{\alpha \in S_n} \delta_\alpha \cdot a_{1\alpha^{-1}(1)} \cdot a_{2\alpha^{-1}(2)} \cdots a_{n\alpha^{-1}(n)}$$

$$= \sum_{\alpha \in S_n} \delta_{\alpha^{-1}}\, a_{1\alpha^{-1}(1)} \cdot a_{2\alpha^{-1}(2)} \cdots a_{n\alpha^{-1}(n)} \quad (\alpha \circ \alpha^{-1} = e \Rightarrow \delta_\alpha = \delta_{\alpha^{-1}}.)$$

$$= \sum_{\alpha^{-1} \in S_n} \delta_{\alpha^{-1}}\, a_{1\alpha^{-1}(1)}\, a_{2\alpha^{-1}(2)} \cdots a_{n\alpha^{-1}(n)}$$

$= \det(A)$. ■

(α, α^{-1} 有1-1对應的関係 而且 $\{\alpha \mid \alpha \in S_n\} = \{\alpha^{-1} \mid \alpha \in S_n\}$.)

⑤

Apply $E_3 : k\vec{r}_i + \vec{r}_j \longrightarrow \vec{r}_j$ （不会改变行列式值）

$$A \xrightarrow[(i,j)]{E_3} \begin{bmatrix} \vec{r}_1 \\ \vdots \\ \vec{r}_i \\ \vdots \\ k\vec{r}_i + \vec{r}_j \\ \vdots \end{bmatrix}$$

把第 j 列改成 $k\vec{r}_i + \vec{r}_j$

$$= \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ & & & \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & & & \\ ka_{i1}+a_{j1} & ka_{i2}+a_{j2} & \cdots & ka_{in}+a_{jn} \\ \vdots & & & \end{bmatrix} = \tilde{A}$$

$\det(\tilde{A})$

$$= \sum_{\alpha \in S_n} \delta_\alpha a_{1\alpha(1)} a_{2\alpha(2)} \cdots a_{(j-1)\alpha(j-1)} \boxed{b_{j\alpha(j)}} a_{(j+1)\alpha(j+1)} \cdots a_{n\alpha(n)}$$

$$b_{j\alpha(j)} = ka_{i\alpha(j)} + a_{j\alpha(j)}$$

$$= \sum_{\alpha \in S_n} \delta_\alpha a_{1\alpha(1)} \cdots a_{n\alpha(n)} + k \sum_{\alpha \in S_n} \delta_\alpha a_{1\alpha(1)} a_{2\alpha(2)} \cdots \boxed{a_{i\alpha(i)}} \cdots \boxed{a_{i\alpha(j)}} \cdots a_{n\alpha(n)}$$

来自同一列

$$= \det(A) + k \underbrace{\det\left(\begin{bmatrix} \vec{r}_1 \\ \vdots \\ \vec{r}_i \\ \vdots \\ \vec{r}_i \\ \vdots \\ \vec{r}_n \end{bmatrix}\right)}_{=0} = \det(A).$$

■

Review

(*) 有三个 Row operations E_1, E_2, E_3，它们对 determinant 的影响分别为

$$A \xrightarrow{E_1} \tilde{A}, \quad \det(\tilde{A}) = k\det(A).$$

(在某一列乘上 k)

$$A \xrightarrow{E_2} \tilde{\tilde{A}}, \quad \det(\tilde{\tilde{A}}) = -\det(A).$$

(交换两列)

$$A \xrightarrow{E_3} \tilde{\tilde{\tilde{A}}}, \quad \det(\tilde{\tilde{\tilde{A}}}) = \det(A).$$

(*) 有三个 Column Operations F_1, F_2, F_3，它们的影响对应为 $\det(A') = k\det(A)$，$\det(A'') = -\det(A)$ 及 $\det(A''') = \det(A)$.

(註) 列运算的结果可以直接用定義来证明，而行运算则可以利用 $\det(A^T) = \det(A)$ 来协助说明。

<u>Lemma</u> $\det\left(\begin{bmatrix} A & O \\ -I_n & B \end{bmatrix}_{2n\times 2n}\right) = \det(A)\det(B).$

<u>Proof.</u> By definition of determinant.

<u>Theorem</u> $\det(AB) = \det(A)\det(B).$

<u>Proof.</u> By using column operations $\underset{\sim\sim}{(F_3)}$, we can transform $\begin{bmatrix} A & O \\ -I_n & B \end{bmatrix}$ to $\begin{bmatrix} A & AB \\ -I_n & O \end{bmatrix}$. (why?) (行列式值不变)

Now, $\det\left(\begin{bmatrix} A & AB \\ -I_n & O \end{bmatrix}\right) = (-1)^n \det\left(\begin{bmatrix} A & AB \\ I_n & O \end{bmatrix}\right)$

$= (-1)^n \cdot (-1)^n \det\left(\begin{bmatrix} AB & A \\ O & I_n \end{bmatrix}\right)$

$= \det(AB)\det(I_n) = \det(AB).$

By Lemma (above), $\det(AB) = \det(A)\det(B).$ ∎

Corollary $\forall n \in \mathbb{N}, \det(A^n) = (\det(A))^n$.

Example

(1) $\det\left(\begin{bmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{bmatrix}\right) = (c-a)(c-b)(b-a)$.

(2) $$\det\left(A = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ a_1 & a_2 & a_3 & \cdots & a_n \\ a_1^2 & a_2^2 & a_3^2 & \cdots & a_n^2 \\ & & \cdots & & \\ a_1^{n-1} & a_2^{n-1} & a_3^{n-1} & \cdots & a_n^{n-1} \end{bmatrix} \right)$$

$$= \prod_{n \geq j > i \geq 1} (a_j - a_i).$$

Proof. 令 $\det(A) = f(a_1, a_2, \cdots, a_n)$，则此多项式的次数为 $1+2+\cdots+(n-1) = \frac{n(n-1)}{2}$. 由于当 $a_i = a_j$ 时，$f(a_1, a_2, \cdots, a_n) = 0$，$(a_j - a_i) \mid f(a_1, \cdots, a_n)$，因此，

$$\prod_{n \geq j > i \geq 1} (a_j - a_i) \mid f(a_1, \cdots, a_n).$$

因为左右两式次数相等，而且领导係数相同，所以只差 $\pm$ 号。因为 $a_2 a_3^2 \cdots a_n^{n-1}$ 这一项是正号，而 $\prod_{n \geq j > i \geq 1} (a_j - a_i)$ 的这一项也是正号，所以相等。(?) ∎

Theorem If $\det(A)=0$, then A^{-1} does not exist.

Proof. Suppose not. Let $B=A^{-1}$, i.e., $AB=I_n$.

Then $\det(AB)=\det(A)\det(B)=0$. But, $\det(I_n)=1$. ∎

觀察：

(1) $\det(A)=0 \Leftrightarrow \operatorname{rank}(A)<n$.

(2) $\det(A)\neq 0 \Leftrightarrow A$ is invertible. (A^{-1} 存在).

結論 Let A be a $n\times n$ matrix. Then A^{-1} exists if and only if $\det(A)\neq 0$.