Week 9 11,17; 11,19

Linear Transformation (Continued)

Example 8

A linear transf. $T : \mathbb{R}^n \to \mathbb{R}^m$

$T(x_1, x_2, \cdots, x_n) = (y_1, y_2, \cdots, y_m)$ can be defined

by

$$
\begin{aligned}
y_1 &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\
y_2 &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\
&\vdots \\
y_m &= a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n
\end{aligned}
$$

符合 Linear 的概念

$$
\Rightarrow \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ - & - & & - \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \Rightarrow T(\vec{x}) = A\vec{x}.
$$

A : Standard matrix for T

Example 9

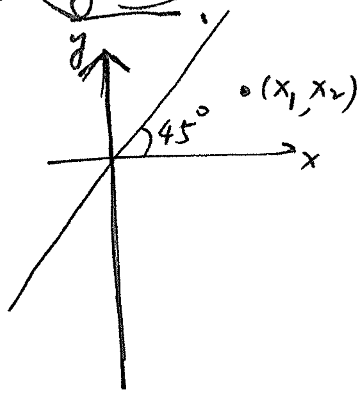
Describe the following mappings:

(1) $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ 旋轉90°(逆時針)

(2) $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ $(x_1, x_2) \longmapsto (x_1, x_1 + x_2)$

(3) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $(x_1, x_2) \longmapsto (x_2, x_1)$

对稱於: $y = x$



(4) $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ $(x_1, x_2) \longrightarrow (x_1, -x_2)$

对稱於 x軸

(5) $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$: 投影到 x軸上 $(x_1, x_2) \longrightarrow (x_1, 0)$

就一般線性空間(向量空間)的對應而言，

如果 $T: V \to W$ 為一個 l. transf.，則

此對應可以由 $\left[T(\vec{b_1}), T(\vec{b_2}), \cdots, T(\vec{b_n})\right]$ 這矩陣

來表示，其中 $\mathbb{B} = \{\vec{b_1}, \vec{b_2}, \cdots, \vec{b_n}\}$ 為 V 的一個基底。
(basis)

(∵ $\forall\ \vec{v} \in V,\ \vec{v} = \lambda_1\vec{b_1} + \lambda_2\vec{b_2} + \cdots + \lambda_n\vec{b_n}$,

$$T(\vec{v}) = T(\lambda_1\vec{b_1} + \lambda_2\vec{b_2} + \cdots + \lambda_n\vec{b_n})$$

$$= \lambda_1 T(\vec{b_1}) + \lambda_2 T(\vec{b_2}) + \cdots + \lambda_n T(\vec{b_n})$$

$$= \underbrace{\begin{bmatrix} T(\vec{b_1}) & T(\vec{b_2}) & \cdots & T(\vec{b_n}) \end{bmatrix}}\begin{bmatrix}\lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n\end{bmatrix}$$

⇑

Standard matrix for T (with respect to $\mathbb{B}$).

Hence, $T: R^n \to R^n$ 可以由

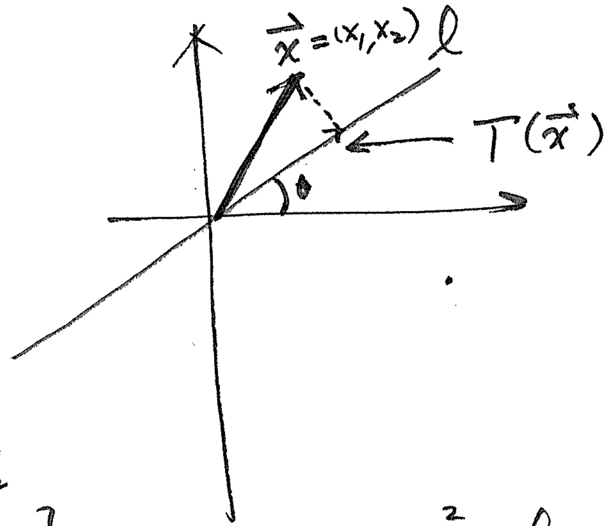
$\begin{bmatrix} T(\vec{e_1}) & T(\vec{e_2}) & \cdots & T(\vec{e_n}) \end{bmatrix}$ 來表示。

Example 10

Let ℓ be the line in the xy-plane that passes through the origin and makes an angle θ with the positive x-axis, where $0 \le \theta \le \pi$.

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear mapping that maps each vector into orthogonal projection on ℓ.

Find the standard matrix for T.

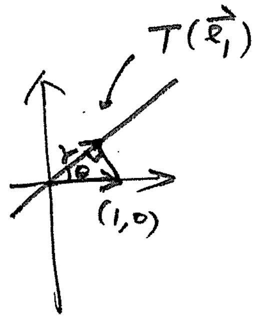


Sol. Since $\{\overset{\vec{e}_1}{(1,0)}, \overset{\vec{e}_2}{(0,1)}\}$ is a basis of $\mathbb{R}^2$, the l. transf. is determined by $A = [\,T(\vec{e}_1), T(\vec{e}_2)\,]$

$$T(\vec{e}_1) = \begin{bmatrix} r\cos\theta \\ r\sin\theta \end{bmatrix}$$

← 極座標

$$= \begin{bmatrix} \cos^2\theta \\ \sin\theta\cos\theta \end{bmatrix}$$



$$T(\vec{e}_2) = \begin{bmatrix} r' \cos\theta \\ r' \sin\theta \end{bmatrix}$$

$$= \begin{bmatrix} \sin\theta \cos\theta \\ \sin^2\theta \end{bmatrix}$$

$$A = \begin{bmatrix} \cos^2\theta & \cos\theta \sin\theta \\ \cos\theta \sin\theta & \sin^2\theta \end{bmatrix}$$

Review that a function $f: A \to B$ is surjective if $f(A) = B$ and injective if f is 1-1 or "$\forall\ x, y \in A,\ f(x) = f(y) \Rightarrow x = y$".

Theorem Let A be an $m \times n$ matrix. Then $T(\vec{x}) = A\vec{x}$ is surjective if and only if the columns of A span $\mathbb{R}^m$ or the set of column vectors forms a basis of $\mathbb{R}^m$.

Proof.

($\Rightarrow$) Since T is surjective, $\forall\, \vec{y} \in \mathbb{R}^m$, $\exists\, \vec{x} \in \mathbb{R}^n$ such that $T(\vec{x}) = \vec{y}$, i.e., $A\vec{x} = \vec{y}$. By the fact that

$$A\vec{x} = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$= x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots + x_n\vec{a}_n,$$

$$\vec{y} \in \mathrm{Span}(\{\vec{a}_1, \vec{a}_2, \cdots, \vec{a}_n\}).$$

($\Leftarrow$) The above statements can be reversed. ∎

Theorem $T : \mathbb{R}^n \to \mathbb{R}^m$ is injective if and only if $\mathrm{Ker}(T) = \{\vec{0}\}$.

Proof. ($\Leftarrow$) Let $\vec{x}, \vec{y} \in \mathbb{R}^n$ s.t. $T(\vec{x}) = T(\vec{y})$.

Then $T(\vec{x} - \vec{y}) = \vec{0} \Rightarrow \vec{x} - \vec{y} \in \mathrm{Ker}(T)$,

$\vec{x} - \vec{y} = \vec{0} \Rightarrow \vec{x} = \vec{y}$. T is injective.

($\Rightarrow$) Trivial, since more than one vectors are mapping into $\vec{0}$ if $\mathrm{Ker}(T) \neq \{\vec{0}\}$. ∎

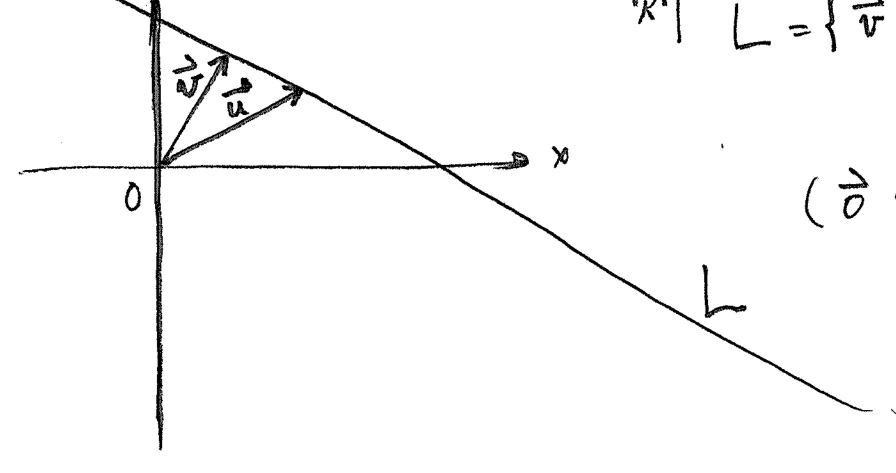
Let $\vec{v} \in \mathbb{R}^n$. Then $L = \{\lambda\vec{v} \mid \lambda \in \mathbb{R}\}$ is a vector space (subspace) contained in $\mathbb{R}^n$.

(*) L is in fact a line through origin.
(通过原点的直线).

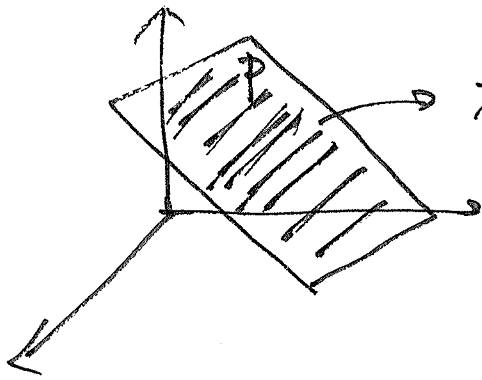
Let $\vec{v}_1, \vec{v}_2 \in \mathbb{R}^n$. Then $P = \{\lambda_1\vec{v}_1 + \lambda_2\vec{v}_2 \mid \lambda_i \in \mathbb{R}, i=1,2\}$ is a vector space (subspace) contained in $\mathbb{R}^n$.

(*) P is a plane through origin.
(通过原点由 $\vec{v}_1$ 与 $\vec{v}_2$ 所决定的平面.)

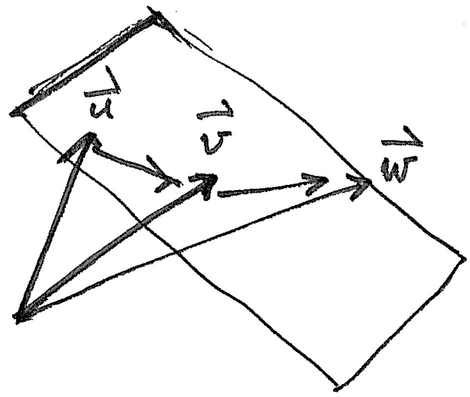
如果直线,或平面不通过原点则不会形成 v. space.
但是我们可以写出它们的一般形式。



给定向量 $\vec{v}, \vec{u}$,
则 $L = \{\vec{v} + t(\vec{u} - \vec{v}) \mid t \in \mathbb{R}\}$.
($\vec{0} \notin L$?)



$$ax + by + cz = d$$
$$(d \neq 0)$$



$$P = \left\{ \vec{u} + \lambda_1(\vec{v} - \vec{u}) + \lambda_2(\vec{w} - \vec{v}) \;\middle|\; \lambda_1, \lambda_2 \in \mathbb{R} \right\}.$$

Exercise：如何判断两条直线平行（或相同），两个平面平行（或相同）。

（用向量形式）