Week 7

11.3; 11.5

Vector Space (Linear Space) 向量空間，線性空间

$(V_F, +, \cdot)$ (佈於一个 Field F)

V_F 中的元素為向量 (vector)

F 中的元素是純量 (scalar)

"+" 為向量的加法

"·" 是純量乘上向量的運算 "$\cdot$" : $F \times V_F \longrightarrow V_F$.

滿足以下 10 个條件：

(1) ~ (5) $\langle V_F, + \rangle$ 為一交換群 (abelian group)

(6) $\forall\ \lambda \in F,\ \vec{v} \in V_F,\ \lambda\vec{v} \in V_F.$

(7) $\forall\ \lambda \in F,\ \vec{u}, \vec{v} \in V_F,\ \lambda\cdot(\vec{u}+\vec{v}) = \lambda\cdot\vec{u} + \lambda\cdot\vec{v}.$

(8) $\forall\ \mu, \lambda \in F,\ \vec{v} \in V_F,\ (\mu+\lambda)\cdot\vec{v} = \mu\cdot\vec{v} + \lambda\cdot\vec{v}.$

(9) $\forall\ \mu, \lambda \in F,\ \mu\cdot(\lambda\cdot\vec{v}) = (\mu\lambda)\cdot\vec{v}.$

F 中的乘法

(10) $\forall\ \vec{v} \in V_F,\ 1\cdot\vec{v} = \vec{v}.$

(Fact 1) $0 \cdot \vec{v} = \vec{0}$.

<u>Proof.</u> $(0+0)\cdot \vec{v} = 0\vec{v} + 0\cdot\vec{v}$

$$0\cdot\vec{v} = 0\cdot\vec{v} + 0\cdot\vec{v}$$

$\exists\ \vec{u}$, $\vec{u} + 0\cdot\vec{v} = \vec{0}$.

$$\vec{0} = \vec{u} + 0\cdot\vec{v} = \vec{u} + (0\cdot\vec{v} + 0\cdot\vec{v}) = (\vec{u} + 0\cdot\vec{v}) + 0\cdot\vec{v} = 0\cdot\vec{v}. \quad \blacksquare$$

<u>Examples of vector spaces</u>

1. R^n is a vector space over R.
2. Let F be a field, $F[x]$ be the set of polynomials with coefficients in F. Then $F[x]$ is a vector space over F.
3. Let $F_n[x]$ be the set of polynomials in $F[x]$ with maximum degree n. Then $F_n[x]$ is a vector space over F.
4. F 本身也可以看成是佈於 F 的向量空間，也就是說 F 中的元素是向量也是純量。
5. 例2的 F 可以選擇 GF(2)，則多項式的係數不是 0 就是 1；每个多項式也可以看成是 (0,1)-向量。

6.例：我们用 $\mathbb{R}_{m\times n}$ 表示所有填入实数的 $m\times n$ 矩阵，同时定义在 $A=[a_{ij}]_{m\times n}$ 时，$\forall\ \lambda\in\mathbb{R}$，$\lambda A=[\lambda a_{ij}]_{m\times n}$

则 $\langle \mathbb{R}_{m\times n},+,\cdot\rangle$ 为佈於 $\mathbb{R}$ 的向量空间。

7.例：$A:\mathbb{R}^n\to\mathbb{R}^m \quad (A\in\mathbb{R}_{m\times n})$

则 $\mathrm{Ker}(A)=\{\vec{x}\mid \vec{x}\in\mathbb{R}^n,\ A\vec{x}=\vec{0}\}$ 为佈於 $\mathbb{R}$ 的向量空间，由於 $\mathrm{Ker}(A)\subseteq\mathbb{R}^n$，而且採用相同的運算，因此，$\mathrm{Ker}(A)$ 又稱为是 $\mathbb{R}^n$ 的向量子空间 (Vector Subspace).

8.例：令 $S=\{f:\mathbb{R}\to\mathbb{R}\mid f$ 可微，而且 $f'(0)=0\}$.

则 S 为佈於 $\mathbb{R}$ 的向量空间，其中

$$(f+g)(x)=f(x)+g(x),$$

$$(\lambda f)(x)=\lambda f(x).$$

例：$\mathbb{R}^{\mathbb{R}}=\{f:\mathbb{R}\to\mathbb{R}\}$ 本身即为佈於 $\mathbb{R}$ 的向量空间。

(*) 讨论一组向量是否线性相依或独立可以扩展至向量是来自向量空间，而不仅是 $\mathbb{R}^n$。

例如：

$\left\{ \begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}, \begin{bmatrix} 4 & 2 \\ -1 & 2 \end{bmatrix}, \begin{bmatrix} -1 & 7 \\ 7 & 19 \end{bmatrix} \right\}$ 是线性相依，

$\left\{ \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\}$ 是线性独立。

探讨向量空间的维度 (Dimension).

Problem How many vectors of $\mathbb{R}^n$ do we need to span all the vectors of $\mathbb{R}^n$?

Question Can we find $n-1$ vectors $\vec{u}_1, \vec{u}_2, \cdots, \vec{u}_{n-1}$ such that $\text{Span}(\{\vec{u}_1, \vec{u}_2, \cdots, \vec{u}_{n-1}\}) = \mathbb{R}^n$?

Answer: 直觉地我们回答 "No"，但是，如何证明这件事？

Theorem 1 If $m > n$ and $\{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_m\}$ is a set of vectors in $\mathbb{R}^n$, then $\{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_m\}$ is linearly dependent.

Corollary If S is a minimal set such that $\mathrm{Span}(S) = \mathbb{R}^n$, then S is of size at most n.

Proof. First, we observe that if $S_1 \subseteq S_2$, then $\mathrm{Span}(S_1) \subseteq \mathrm{Span}(S_2)$. Now, if $|S| > n$, then S is linear dependent. Therefore, there exists a vector $\vec{v} \in S$ such $\vec{v}$ is a linear combination of $S \setminus \{\vec{v}\}$. This implies that $\mathrm{Span}(S) \subseteq \mathrm{Span}(S \setminus \{\vec{v}\})$. By observation, we conclude that $\mathrm{Span}(S) = \mathrm{Span}(S \setminus \{\vec{v}\}) = \mathbb{R}^n$. This contradicts to the fact that S is minimal. ∎

Lemma 2 If S is a minimal set such that $\mathrm{Span}(S) = \mathbb{R}^n$, then S is linearly independent.

Proof. For otherwise we can find a smaller S' such that $\mathrm{Span}(S') = \mathrm{Span}(S) = \mathbb{R}^n$.

Lemma 3 If S is a set of vectors such that $\mathrm{Span}(S) = \mathbb{R}^n$, then $|S| \geq n$.

Proof. Suppose not. Let S be a minimal one. Then S is a linearly independent set in $\mathbb{R}^n$ and $|S| = m < n$. Let $S = \{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_m\}$, $\vec{v}_i \in \mathbb{R}^n$.

Consider $A = \begin{bmatrix} \vec{v}_1, \vec{v}_2, \cdots, \vec{v}_m \end{bmatrix}$. Since $m < n$, A is row equivalent to a matrix $\tilde{A}$ which has a zero rows, say the last row. Then, it is not difficult to see that $A\vec{x} = \vec{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}_{n \times 1}$ has no solution,

i.e. $\vec{e}_n \notin \mathrm{Span}(S)$. This concludes the proof. ■

$$\left[\begin{array}{cccc|c} & & & & 0 \\ & * & & & 0 \\ & & & & \vdots \\ & & & & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{array}\right]$$

0-rows (after finding its row reduced echelon form).

Lemma 4 If S is l. indep. and $\mathrm{Span}(S) \subsetneq \mathbb{R}^n$ and $\vec{v}$ is not in $\mathrm{Span}(S)$, then $S \cup \{\vec{v}\}$ is l. indep.

proof. Let $S = \{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_m\}$. Assume that $S \cup \{\vec{v}\}$ is l. dependent. Then, there exist $\lambda, \lambda_1, \lambda_2, \cdots, \lambda_m$ (not all zero) such that

$$\lambda \vec{v} + \lambda_1 \vec{v}_1 + \cdots + \lambda_m \vec{v}_m = \vec{0}.$$

Now, if $\lambda \neq 0$, then $\vec{v} = \sum_{i=1}^{m} \frac{\lambda_i}{\lambda} \vec{v}_i$ and thus $\vec{v} \in \mathrm{Span}(S)$, $\rightarrow\leftarrow$.

On the other hand, if $\lambda = 0$, then $\lambda_1 \vec{v}_1 + \cdots + \lambda_m \vec{v}_m = \vec{0}$. By the assumption that S is l. independent, $\lambda_1 = \lambda_2 = \cdots = \lambda_m = 0$, not what we assume. Hence, we have the proof. ▨

<u>Theorem</u> Any n l. independent vectors in $\mathbb{R}^n$ form a set S which spans $\mathbb{R}^n$, i.e., $\mathrm{Span}(S) = \mathbb{R}^n$.

<u>Proof.</u> Suppose not. $\exists \vec{v} \in \mathbb{R}^n$ s.t. $\vec{v} \notin \mathrm{Span}(S)$. By Lemma 4, $S \cup \{\vec{v}\}$ is l. independent. But, ~~not~~ $S \cup \{\vec{v}\}$ has $n+1$ vectors, by Theorem 1, $S \cup \{\vec{v}\}$ is l. dependent. $\rightarrow\leftarrow$. This concludes the proof. ∎

Basis of a vector space

<u>Definition</u>

A set of l. independent vectors S is said to be a basis of a vector space V if $\mathrm{Span}(S) = V$.

e.g. 1. $\{(0,1), (1,0)\}$ is a basis of $\mathbb{R}^2$.

e.g. 2. $\{1, x, x^2, \cdots\}$ is a basis of $\mathbb{R}[x]$ (實係數多項式。

<u>Question</u> Let V_F be a vector space with two bases S_1 and S_2. Is $|S_1| = |S_2|$?