

Let $M(n)$ denotes the largest number of mutually orthogonal Latin squares of order n .
(Proposition: $M(n) \leq n-1$.)

Good!
for 3, 18

Claim that $M(n) \geq n-2 \Rightarrow M(n) = n-1$.

(pf)

After proper permutations, we can make the first row of each $(n-2)$ mutually orthogonal Latin

squares be

0	1	...	n-1
---	---	-----	-----

, and denote these Latin squares by $A^{(1)}, A^{(2)}, \dots, A^{(n-2)}$.

<table border="1" style="display: inline-table;"><tr><td>0</td><td>1</td><td>...</td><td>n-1</td></tr><tr><td colspan="4" style="height: 80px;"></td></tr></table>	0	1	...	n-1					<table border="1" style="display: inline-table;"><tr><td>0</td><td>1</td><td>...</td><td>n-1</td></tr><tr><td colspan="4" style="height: 80px;"></td></tr></table>	0	1	...	n-1					...	<table border="1" style="display: inline-table;"><tr><td>0</td><td>1</td><td>...</td><td>n-1</td></tr><tr><td colspan="4" style="height: 80px;"></td></tr></table>	0	1	...	n-1				
0	1	...	n-1																								
0	1	...	n-1																								
0	1	...	n-1																								
$A^{(1)}$	$A^{(2)}$		$A^{(n-2)}$																								

Now, create a square L of order n by $\{L_{ij}\} = \begin{cases} \{j\}, & \text{if } i = 0 \\ \{0, 1, \dots, n-1\} \setminus \left(\bigcup_{k=1}^{n-2} \{A_{ij}^{(k)}\} \cup \{j\} \right), & \text{if } i \neq 0 \end{cases}$

(1) (Check that L is a Latin Square)

(i) If there exists ~~the same elements in a column, say~~, two x's in the j-th column, then, the j-th column must be lack of some element, say, y .

Observe the j-th column of each $A^{(k)}$. Clearly that y appears in the ~~(n-1)~~ ^{times} entries in the j-th column of $A^{(1)}, A^{(2)}, \dots, A^{(n-2)}$.

By Pigeonhole principle, two y's appears in the same j-th column, and it is a contradiction.

	j	
	x	}
	x	

(n-1) entries, lack of y

(ii) If there exists ~~the same elements in a row, say~~, two x's in the i-th row, then, the i-th row must be lack of some element, say, y .

Observe the i-th row of each $A^{(k)}$. The y-th entries ^{times} cannot be y , and y appears in the ~~(n-1)~~ ^{times} entries in the i-th row of $A^{(1)}, A^{(2)}, \dots, A^{(n-2)}$.

By Pigeonhole principle, two y's appears in the same j-th column, and it is a contradiction.

				y
x	x			

(n-1) entries, lack of y

(2) (Check that $L \perp A^{(k)}$, for $k = 1, 2, \dots, n-2$)

If there exists $A^{(s)}$ such that the 2-finger's rule does not hold, i.e., there exists

	$\bar{j}_1$		$\bar{j}_2$	
$\bar{i}_1$			k_1	
$\bar{i}_2$		k_1		

 $A^{(s)}$

	$\bar{j}_1$		$\bar{j}_2$	
$\bar{i}_1$			k_2	
$\bar{i}_2$		k_2		

 L

, where $0 \leq s \leq n-2, 0 \leq k_1, k_2 \leq n-1, k_1 \neq k_2$.

Consider

Pick out the all k_1 's in $A^{(s)}$, and observe the ~~same~~^{corresponding} entries in the other $(n-3)$ $A^{(k)}$'s, $k \neq s$.

	$\bar{j}_1$	k_1		$\bar{j}_2$	k_2			
$\bar{i}_1$				k_1				
					k_1			
$\bar{i}_2$	k_1							

$A^{(s)}$

Since $A^{(k)} \perp A^{(s)}$, $k \neq s$, k_2 appears in exactly one of these

entries in each $A^{(k)}$. However,

(i) k_2 cannot appear in the two entries where k_2 has appeared in L .

(ii) k_2 cannot appear in the first row because it must be k_1 .

(iii) k_2 cannot appear in the k_2 -th column.

Hence there are $(n-4)$ entries that can be k_2 in the $(n-3)$ $A^{(k)}$'s, $k \neq s$. By Pigeonhole

principle, k_2 appears in the same entry in two $A^{(k)}$'s, and it is a contradiction.

(Otherwise $A^{(k_1)} \neq A^{(k_2)}$)

for some $k \neq s$