Discrete Math, Test(II)

2009.5.13

1. Prove that in any subset of {1, 2, …, 200} with 101 elements there exist two elements which are not relatively prime. (15 points)

2. Prove that in any real sequence with 101 elements, there exists a monotonic subsequence with 11 elements. (15 points)

3. Prove that there are 576 distinct latin squares of order 4. (15 points)

4. Let A be an $n \times n$ (0, 1) – matrix such that each row (and each column) has exactly two 1's. Prove that A can be written as a sum of 2 permutation matrices. (A permutation matrix is a matrix such that each row and each column has exactly one "1".) (15 points)

5. Construct two orthogonal latin squares of order 4 and order 5, respectively. (20 points)

6. Find a 2 – (7, 3, 1) design and a 2 – (9, 3, 1) design. (20 points)

7. Find a 2 – (13, 4, 1) design and a 2 – (16, 4, 1) design. (20 points)

8. Let (X, B) be a balanced incomplete block design. Prove that $|X| \leq |B|$. (Hint: Use the incidence matrix of (X, B).) (20 points)

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1. Prove that in any subset of {1, 2, …, 200} with 101 elements there exist two elements which are not relatively prime. (15 points)

Solution:

令 $\{a_1, ..., a_{101}\}$爲所選取的元素.

Case 1: 如果存在兩個偶數,done！

Case 2: 如果恰存在一偶數,則所有的奇數均選取,因此很容易看出存在兩個數不互質.

2. Prove that in any real sequence with 101 elements, there exists a monotonic subsequence with 11 elements. (15 points)

3. Prove that there are 576 distinct latin squares of order 4. (15 points)

Solution:

4	$\times 4!$	$\times 3!$
Standard latin square 的個數	column permutation	row permutation

4. Let A be an $n \times n$ (0, 1) – matrix such that each row (and each column) has exactly two 1's. Prove that A can be written as a sum of 2 permutation matrices. (A permutation matrix is a matrix such that each row and each column has exactly one "1".) (15 points)

5. Construct two orthogonal latin squares of order 4 and order 5, respectively. (20 points)

Solution:

1	2	3	4
2	1	4	3
3	4	1	2
4	3	2	1

1	2	3	4
4	3	2	1
2	1	4	3
3	4	1	2

1	2	3	4	5
5	1	2	3	4
4	5	1	2	3
3	4	5	1	2
2	3	4	5	1

1	2	3	4	5
2	3	4	5	1
3	4	5	1	2
4	5	1	2	3
5	1	2	3	4

6. Find a 2 – (7, 3, 1) design and a 2 – (9, 3, 1) design. (20 points)

7. Find a 2 – (13, 4, 1) design and a 2 – (16, 4, 1) design. (20 points)

Solution:

(1) {1,2, 3} {4, 5, 6} {7, 8,9} {1, 4, 7} {2, 5, 8} {3, 6, 9} {1, 5, 9} {2, 6, 7} {3, 4, 8} {3, 5, 7} {2, 4, 9} {1, 6, 8}

(2)

S:

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

Let A_1, A_2, A_3 be MOLS of order 4. Let

$R_i = \{(i-1)4+1, (i-1)4+2, \cdots, i4\}$

$C_i = \{i, 4+i, \cdots, 12+i\}$

$B_{r,m} = \{x \mid x \text{ is the label in } S \text{ of a position in which } A_r \text{ has entry } m\}$

Then $\{R_i, C_i, B_{r,m} \mid 1 \le i \le 4, 1 \le r \le 3, 1 \le m \le 4\}$ is a 2 – (16, 4, 1) design.

Ex: $R_1 = \{1, 2, 3, 4\}$, $C_1 = \{1, 5, 9, 13\}$, $B_{1,1} = \{1, 8, 10, 15\}$ where A_1:第五題的第二個 4*4 latin square.

8. Let (X, B) be a balanced incomplete block design. Prove that $|X| \leq |B|$. (Hint: Use the incidence matrix of (X, B).) (20 points)

2.
Let the sequence be $(a_1, a_2, ..., a_{101})$ and m_i be the length of the longest increasing subsequence that starts with a_i. If $m_i \geq 11$ for some i, then we done. Suppose not. Then by pigeonhole principle there exists m_{i_1}, m_{i_2}, m_{i_3}, ..., $m_{i_{11}}$ such that $m_{i_1} = m_{i_2} = ... = m_{i_{11}}$. Then we have a decreasing sequence $(a_{i_1}$, a_{i_2}, ..., $a_{i_{11}})$.

4.
Let rows of A indexed by some elements and columns indexed by some sets such that $A_{ij} = 1$ if set j contains element i, and $A_{ij} = 0$ otherwise. Then we claim these sets satisfies Hall's condition. To prove this suppose otherwise, then we have sets $j_1, j_2, ...j_k$ such that $|\bigcup_{i=1}^{k} j_i| < k$. Consider the number of 1's in these columns. There must be $2k$ 1's in these columns by the assumption, however $|\bigcup_{i=1}^{k} j_i| < k$ shows that there are at most $2(k-1)$ 1's, a contradiction.
Thus we can find a SDR of all sets which is, in fact, a permutation matrix P if we presents it as what we did to A, since all elements of that SDR are distinct. It's clear $A - P$ is still a permutation matrix, therefore we are done.

6.
Let X =$\{1, 2, 3, 4, 5, 6, 7\}$ be points, and
$B = \{\{1, 2, 4\}, \{2, 3, 5\}, \{3, 4, 6\}, \{4, 5, 7\}, \{5, 6, 1\}, \{6, 7, 2\}, \{7, 1, 3\}\}$, then (X, B) is a 2-(7,3,1) design.

Let X =$\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ be points, and $B =$
$\{\{1, 2, 3\}, \{4, 5, 6\}, \{7, 8, 9\}, \{1, 4, 7\}, \{2, 5, 8\}, \{3, 6, 9\}, \{1, 5, 9\}, \{3, 4, 8\}, \{2, 6, 7\}, \{1, 6, 8\}, \{2, 4, 9\}, \{3, 5, 7\}\}$
then (X, B) is a 2-(9,3,1) design.

8.
Since $r(k-1) = \lambda(v-1)$ and $v \neq k$(for we only consider incomplete designs), we have $r \neq \lambda$. Thus $det(AA^t) = (r + (v-1)\lambda)(r-\lambda)^{v-1} \neq 0$, therefore $rank(AA^t) = v$. This also implies $det(A)$ is not zero, therefore
$rank(A) = min(b, v)$
Thus $v = rank(AA^t) \leq min(rank(A), rank(A^t)) = min(b, v)$ and then $b \geq v$.