

the "heartamard product" of F and G . Show that if F and G are rational, then so is $F \heartsuit G$. Moreover, if F is rational and G is algebraic, then $F \heartsuit G$ is algebraic.

- 6.13. a. [3-] Let $k \in \mathbb{P}$, and define $\eta = \sum_{n \geq 0} \binom{kn}{n} x^n$. Example 6.2.7 shows that η is a root of the polynomial

$$P(y) = k^k x y^k - (y-1)[(k-1)y+1]^{k-1}.$$

Find (as fractional series) the other $k-1$ roots of the polynomial $P(y)$.

Deduce that $P(y)$ is irreducible (as a polynomial over $\mathbb{C}(x)$).

- b. [3-] Find the discriminant of $P(y)$.

- 6.14. [3-] Define $f: \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Q}$ by

$$f(i-1, j) - 2f(i, j) + f(i+1, j-1) = 0 \quad (6.53)$$

for all $(i, j) \in \mathbb{N} \times \mathbb{N} - \{(0, 0)\}$, with the initial conditions $f(0, 0) = 1$ and $f(i, j) = 0$ if $i < 0$ or $j < 0$. Thus $f(i, 0) = 2^{-i}$, $f(0, 1) = \frac{1}{4}$, $f(1, 1) = \frac{1}{4}$, etc. Find the generating function $F(x, y) = \sum_{i, j \geq 0} f(i, j) x^i y^j$.

- 6.15. [2+] Let $f, g, h \in K[[x]]$ with $h(0) = 0$. Find a polynomial $P(f, g, h, x)$ so that

$$\text{diag} \frac{1}{1 - sf(st) - tg(st) - h(st)} = \frac{1}{\sqrt{P}},$$

where diag is in the variable x .

- 6.16. [5-] Let $f(n)$ be the number of paths from $(0, 0)$ to (n, n) using the steps $(1, 0)$, $(0, 1)$, and $(1, 1)$; and let $g(n)$ be the number of paths from $(0, 0)$ to (n, n) using any elements of $\mathbb{N}^2 - \{(0, 0)\}$ as steps. It is immediate from equations (6.27) and (6.30) that $g(n) = 2^{n-1} f(n)$, $n > 0$. Is there a combinatorial proof?

- 6.17. a. [2+] Let S be a subset of $\mathbb{N} \times \mathbb{N} - \{(0, 0)\}$ such that (i) every element of S has the form (n, n) , $(n+1, n)$, or $(n, n+1)$, and (ii) $(n, n+1) \in S$ if and only if $(n+1, n) \in S$. Let $g(n)$ be the number of paths from $(0, 0)$ to (n, n) using steps from S . Let $h(n)$ be the number of such paths that never go above the line $y = x$. (Let $g(0) = h(0) = 1$.) Define $G(x) = \sum_{n \geq 0} g(n)x^n$, $H(x) = \sum_{n \geq 0} h(n)x^n$, and $K(x) = \sum_{(n, n) \in S} x^n$. Show that

$$H(x) = \frac{2}{1 - K(x) + G(x)^{-1}}.$$

- b. [2-] Compute $H(x)$ explicitly when $S = \{(0, 1), (1, 0), (1, 1)\}$ and deduce that in this case $h(n)$ is the Schröder number r_n , thus confirming Exercise 6.39(j).

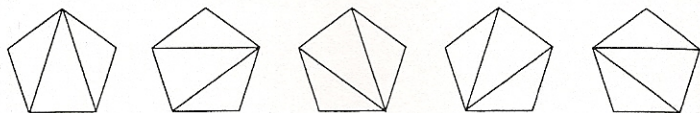
- c. [3-] Give a combinatorial proof that when $S = \{(0, 1), (1, 0), (1, 1)\}$ and $n \geq 2$, then $h(n)$ is twice the number of ways to dissect a convex $(n+2)$ -gon with any number of diagonals that don't intersect in their interiors.

- 6.18. [3] Let S be a subset of $\mathbb{N} \times \mathbb{N} - \{(0, 0)\}$ such that $\sum_{(m, n) \in S} x^m y^n$ is rational, e.g., S is finite or cofinite. Let $f(n)$ be the number of lattice paths from $(0, 0)$ to (n, n) with steps from S that never go above the line $y = x$. Show that $\sum_{n \geq 0} f(n)x^n$ is algebraic.

- ✓ 6.19. [1]-[3+] Show that the Catalan numbers $C_n = \frac{1}{n+1} \binom{2n}{n}$ count the number of elements of the 66 sets S_i , $(a) \leq i \leq (\text{nnn})$, given below. We illustrate the elements of each S_i for $n = 3$, hoping that these illustrations will make any

undefined terminology clear. (The terms used in (vv)–(yy) are defined in Chapter 7.) Ideally S_i and S_j should be proved to have the same cardinality by exhibiting a simple, elegant bijection $\phi_{ij} : S_i \rightarrow S_j$ (so 4290 bijections in all). In some cases the sets S_i and S_j will actually coincide, but their descriptions will differ.

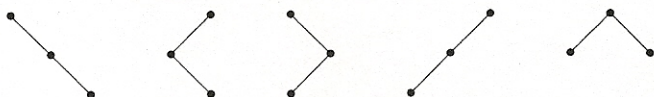
- a. Triangulations of a convex $(n + 2)$ -gon into n triangles by $n - 1$ diagonals that do not intersect in their interiors:



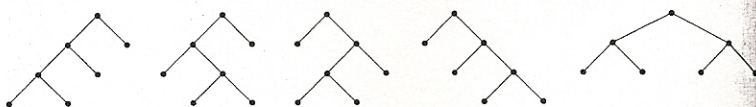
- b. Binary parenthesizations of a string of $n + 1$ letters:

$$(xx \cdot x)x \quad x(xx \cdot x) \quad (x \cdot xx)x \quad x(x \cdot xx) \quad xx \cdot xx$$

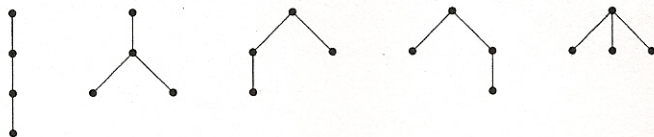
- c. Binary trees with n vertices:



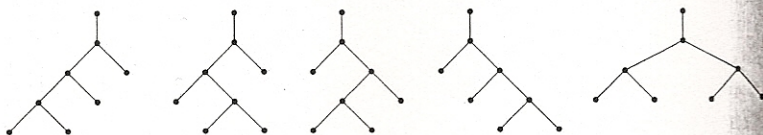
- d. Plane binary trees with $2n + 1$ vertices (or $n + 1$ endpoints):



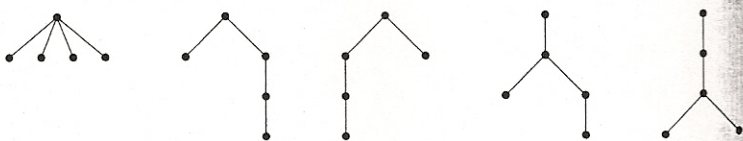
- e. Plane trees with $n + 1$ vertices:



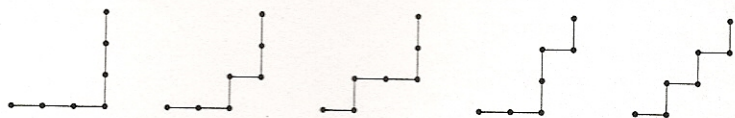
- f. Planted (i.e., root has degree one) trivalent plane trees with $2n + 2$ vertices:



- g. Plane trees with $n + 2$ vertices such that the rightmost path of each subtree of the root has even length:



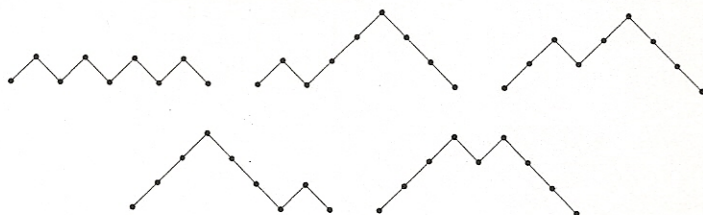
- h. Lattice paths from $(0, 0)$ to (n, n) with steps $(0, 1)$ or $(1, 0)$, never rising above the line $y = x$:



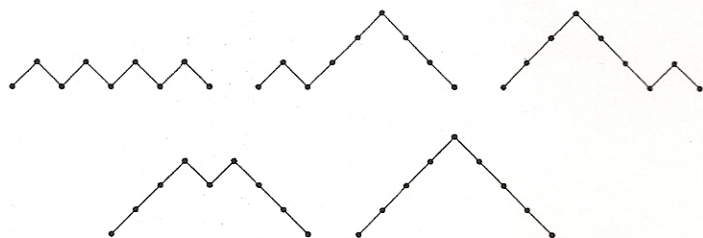
- i. Dyck paths from $(0, 0)$ to $(2n, 0)$, i.e., lattice paths with steps $(1, 1)$ and $(1, -1)$, never falling below the x -axis:



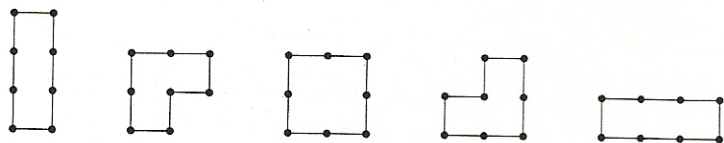
- j. Dyck paths (as defined in (i)) from $(0, 0)$ to $(2n + 2, 0)$ such that any maximal sequence of consecutive steps $(1, -1)$ ending on the x -axis has odd length:



- k. Dyck paths (as defined in (i)) from $(0, 0)$ to $(2n + 2, 0)$ with no peaks at height two:

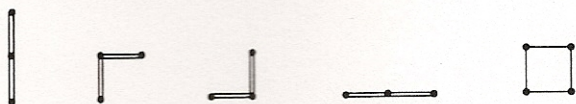


- l. (Unordered) pairs of lattice paths with $n + 1$ steps each, starting at $(0, 0)$, using steps $(1, 0)$ or $(0, 1)$, ending at the same point, and only intersecting at the beginning and end:

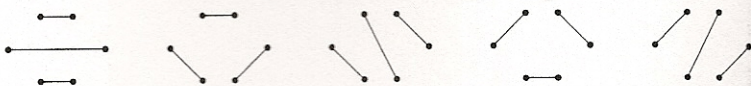


- m. (Unordered) pairs of lattice paths with $n - 1$ steps each, starting at $(0, 0)$, using steps $(1, 0)$ or $(0, 1)$, ending at the same point, such that one path never

risers above the other path:



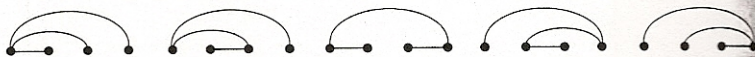
- n. n nonintersecting chords joining $2n$ points on the circumference of a circle:



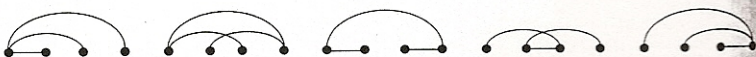
- o. Ways of connecting $2n$ points in the plane lying on a horizontal line by n nonintersecting arcs, each arc connecting two of the points and lying above the points:



- p. Ways of drawing in the plane $n + 1$ points lying on a horizontal line L and n arcs connecting them such that (α) the arcs do not pass below L , (β) the graph thus formed is a tree, (γ) no two arcs intersect in their interiors (i.e., the arcs are noncrossing), and (δ) at every vertex, all the arcs exit in the same direction (left or right):



- q. Ways of drawing in the plane $n + 1$ points lying on a horizontal line L and n arcs connecting them such that (α) the arcs do not pass below L , (β) the graph thus formed is a tree, (γ) no arc (including its endpoints) lies strictly below another arc, and (δ) at every vertex, all the arcs exit in the same direction (left or right):



- r. Sequences of n 1's and n -1's such that every partial sum is nonnegative (with -1 denoted simply as - below):

111--- 11-1-- 11--1- 1-11-- 1-1-1-

- s. Sequences $1 \leq a_1 \leq \dots \leq a_n$ of integers with $a_i \leq i$:

111 112 113 122 123

- t. Sequences $a_1 < a_2 < \dots < a_{n-1}$ of integers satisfying $1 \leq a_i \leq 2i$:

12 13 14 23 24

- u. Sequences $a_1, a_2, \dots, a_n$ of integers such that $a_1 = 0$ and $0 \leq a_{i+1} \leq a_i + 1$:

000 001 010 011 012

- v. Sequences $a_1, a_2, \dots, a_{n-1}$ of integers such that $a_i \leq 1$ and all partial sums are nonnegative:

0, 0 0, 1 1, -1 1, 0 1, 1

- w. Sequences $a_1, a_2, \dots, a_n$ of integers such that $a_i \geq -1$, all partial sums are nonnegative, and $a_1 + a_2 + \dots + a_n = 0$:

0, 0, 0 0, 1, -1 1, 0, -1 1, -1, 0 2, -1, -1

- x. Sequences $a_1, a_2, \dots, a_n$ of integers such that $0 \leq a_i \leq n - i$, and such that if $i < j$, $a_i > 0$, $a_j > 0$, and $a_{i+1} = a_{i+2} = \dots = a_{j-1} = 0$, then $j - i > a_i - a_j$:

000 010 100 200 110

- y. Sequences $a_1, a_2, \dots, a_n$ of integers such that $i \leq a_i \leq n$ and such that if $i \leq j \leq a_i$, then $a_j \leq a_i$:

123 133 223 323 333

- z. Sequences $a_1, a_2, \dots, a_n$ of integers such that $1 \leq a_i \leq i$ and such that if $a_i = j$, then $a_{i-r} \leq j - r$ for $1 \leq r \leq j - 1$:

111 112 113 121 123

- aa. Equivalence classes B of words in the alphabet $[n - 1]$ such that any three consecutive letters of any word in B are distinct, under the equivalence relation $uijv \sim ujiv$ for any words u, v and any $i, j \in [n - 1]$ satisfying $|i - j| \geq 2$:

$\{\emptyset\}$ $\{1\}$ $\{2\}$ $\{12\}$ $\{21\}$

(For $n = 4$ a representative of each class is given by $\emptyset, 1, 2, 3, 12, 21, 13, 23, 32, 123, 132, 213, 321, 2132$.)

- bb. Partitions $\lambda = (\lambda_1, \dots, \lambda_{n-1})$ with $\lambda_1 \leq n - 1$ (so the diagram of λ is contained in an $(n - 1) \times (n - 1)$ square), such that if $\lambda' = (\lambda'_1, \lambda'_2, \dots)$ denotes the conjugate partition to λ then $\lambda'_i \geq \lambda_i$ whenever $\lambda_i \geq i$:

(0, 0) (1, 0) (1, 1) (2, 1) (2, 2)

- cc. Permutations $a_1 a_2 \dots a_{2n}$ of the multiset $\{1^2, 2^2, \dots, n^2\}$ such that: (i) the first occurrences of $1, 2, \dots, n$ appear in increasing order, and (ii) there is no subsequence of the form $\alpha\beta\alpha\beta$:

112233 112332 122331 123321 122133

- dd. Permutations $a_1 a_2 \dots a_{2n}$ of the set $[2n]$ such that: (i) $1, 3, \dots, 2n - 1$ appear in increasing order, (ii) $2, 4, \dots, 2n$ appear in increasing order, and (iii) $2i - 1$ appears before $2i$, $1 \leq i \leq n$:

123456 123546 132456 132546 135246

- ee. Permutations $a_1 a_2 \cdots a_n$ of $[n]$ with longest decreasing subsequence of length at most two (i.e., there does not exist $i < j < k$, $a_i > a_j > a_k$), called 321-avoiding permutations:

123 213 132 312 231

- ff. Permutations $a_1 a_2 \cdots a_n$ of $[n]$ for which there does not exist $i < j < k$ and $a_j < a_k < a_i$ (called 312-avoiding permutations):

123 132 213 231 321

- gg. Permutations w of $[2n]$ with n cycles of length two, such that the product $(1, 2, \dots, 2n) \cdot w$ has $n + 1$ cycles:

$$(1, 2, 3, 4, 5, 6)(1, 2)(3, 4)(5, 6) = (1)(2, 4, 6)(3)(5)$$

$$(1, 2, 3, 4, 5, 6)(1, 2)(3, 6)(4, 5) = (1)(2, 6)(3, 5)(4)$$

$$(1, 2, 3, 4, 5, 6)(1, 4)(2, 3)(5, 6) = (1, 3)(2)(4, 6)(5)$$

$$(1, 2, 3, 4, 5, 6)(1, 6)(2, 3)(4, 5) = (1, 3, 5)(2)(4)(6)$$

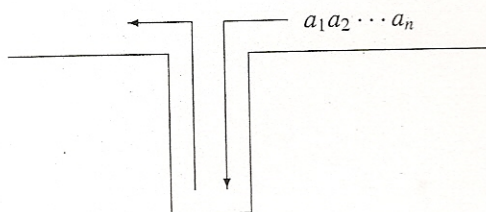
$$(1, 2, 3, 4, 5, 6)(1, 6)(2, 5)(3, 4) = (1, 5)(2, 4)(3)(6)$$

- hh. Pairs (u, v) of permutations of $[n]$ such that u and v have a total of $n + 1$ cycles, and $uv = (1, 2, \dots, n)$:

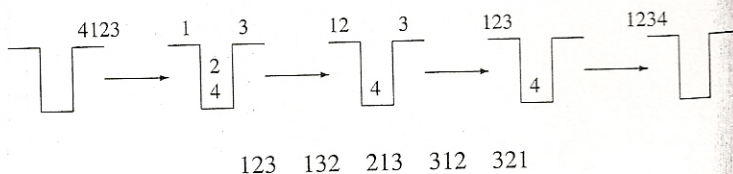
$$(1)(2)(3) \cdot (1, 2, 3) \quad (1, 2, 3) \cdot (1)(2)(3) \quad (1, 2)(3) \cdot (1, 3)(2)$$

$$(1, 3)(2) \cdot (1)(2, 3) \quad (1)(2, 3) \cdot (1, 2)(3)$$

- ii. Permutations $a_1 a_2 \cdots a_n$ of $[n]$ that can be put in increasing order on a single stack, defined recursively as follows: If $\emptyset$ is the empty sequence, then let $S(\emptyset) = \emptyset$. If $w = unv$ is a sequence of distinct integers with largest term n , then $S(w) = S(u)S(v)n$. A *stack-sortable* permutation w is one for which $S(w) = w$:



For example,



- jj. Permutations $a_1 a_2 \cdots a_n$ of $[n]$ that can be put in increasing order on two parallel queues. Now the picture is



123 132 213 231 312

- kk. Fixed-point-free involutions w of $[2n]$ such that if $i < j < k < l$ and $w(i) = k$, then $w(j) \neq l$ (in other words, 3412-avoiding fixed-point-free involutions):

(12)(34)(56) (14)(23)(56) (12)(36)(45) (16)(23)(45) (16)(25)(34)

- ll. Cycles of length $2n + 1$ in $\mathfrak{S}_{2n+1}$ with descent set $\{n\}$:

2371456 2571346 3471256 3671245 5671234

- mm. Baxter permutations (as defined in Exercise 6.55) of $[2n]$ or of $[2n + 1]$ that are reverse alternating (as defined at the end of Section 3.16) and whose inverses are reverse alternating:

132546 153426 354612 561324 563412

1325476 1327564 1534276 1735462 1756342

- nn. Permutations w of $[n]$ such that if w has ℓ inversions then for all pairs of sequences $(a_1, a_2, \dots, a_\ell), (b_1, b_2, \dots, b_\ell) \in [n - 1]^\ell$ satisfying

$$w = s_{a_1} s_{a_2} \cdots s_{a_\ell} = s_{b_1} s_{b_2} \cdots s_{b_\ell},$$

where s_j is the adjacent transposition $(j, j + 1)$, we have that the ℓ -element multisets $\{a_1, a_2, \dots, a_\ell\}$ and $\{b_1, b_2, \dots, b_\ell\}$ are equal (thus, for example, $w = 321$ is not counted, since $w = s_1 s_2 s_1 = s_2 s_1 s_2$, and the multisets $\{1, 2, 1\}$ and $\{2, 1, 2\}$ are not equal):

123 132 213 231 312

- oo. Permutations w of $[n]$ with the following property: Suppose that w has ℓ inversions, and let

$$R(w) = \{(a_1, \dots, a_\ell) \in [n - 1]^\ell : w = s_{a_1} s_{a_2} \cdots s_{a_\ell}\},$$

where s_j is as in (nn). Then

$$\sum_{(a_1, \dots, a_\ell) \in R(w)} a_1 a_2 \cdots a_\ell = \ell!.$$

$$R(123) = \{\emptyset\}, \quad R(213) = \{(1)\}, \quad R(231) = \{(1, 2)\}$$

$$R(312) = \{(2, 1)\}, \quad R(321) = \{(1, 2, 1), (2, 1, 2)\}$$

- pp. Noncrossing partitions of $[n]$, i.e., partitions $\pi = \{B_1, \dots, B_k\} \in \Pi_n$ such that if $a < b < c < d$ and $a, c \in B_i$ and $b, d \in B_j$, then $i = j$:

123 12-3 13-2 23-1 1-2-3

- qq. Partitions $\{B_1, \dots, B_k\}$ of $[n]$ such that if the numbers $1, 2, \dots, n$ are arranged in order around a circle, then the convex hulls of the blocks $B_1, \dots, B_k$ are pairwise disjoint:



- rr. Noncrossing Murasaki diagrams with n vertical lines:



- ss. Noncrossing partitions of some set $[k]$ with $n + 1$ blocks, such that any two elements of the same block differ by at least three:

1-2-3-4 14-2-3-5 15-2-3-4 25-1-3-4 16-25-3-4

- tt. Noncrossing partitions of $[2n + 1]$ into $n + 1$ blocks, such that no block contains two consecutive integers:

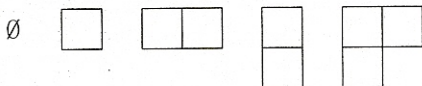
137-46-2-5 1357-2-4-6 157-24-3-6
17-246-3-5 17-26-35-4

- uu. *Nonnesting partitions* of $[n]$, i.e., partitions of $[n]$ such that if a, e appear in a block B and b, d appear in a *different* block B' where $a < b < d < e$, then there is a $c \in B$ satisfying $b < c < d$:

123 12-3 13-2 23-1 1-2-3

(The unique partition of $[4]$ that isn't nonnesting is 14-23.)

- vv. Young diagrams that fit in the shape $(n - 1, n - 2, \dots, 1)$:



- ww. Standard Young tableaux of shape (n, n) (or equivalently, of shape $(n, n - 1)$):

123 124 125 134 135
456 356 346 256 246

or

123 124 125 134 135
45 35 34 25 24

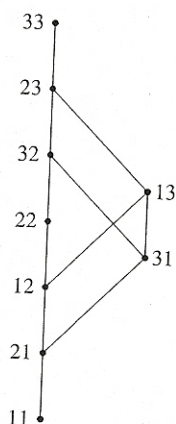


Figure 6-5. A poset with $C_4 = 14$ order ideals.

- xx. Pairs (P, Q) of standard Young tableaux of the same shape, each with n squares and at most two rows:

$$(123, 123) \quad \begin{pmatrix} 12 & 12 \\ 3 & 3 \end{pmatrix} \quad \begin{pmatrix} 12 & 13 \\ 3 & 2 \end{pmatrix} \quad \begin{pmatrix} 13 & 12 \\ 2 & 3 \end{pmatrix} \quad \begin{pmatrix} 13 & 13 \\ 2 & 2 \end{pmatrix}$$

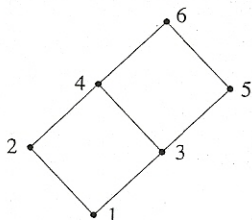
- yy. Column-strict plane partitions of shape $(n-1, n-2, \dots, 1)$, such that each entry in the i -th row is equal to $n-i$ or $n-i+1$:

$$\begin{array}{ccccc} 33 & 33 & 32 & 32 & 22 \\ 2 & 1 & 2 & 1 & 1 \end{array}$$

- zz. Convex subsets S of the poset $\mathbb{Z} \times \mathbb{Z}$, up to translation by a diagonal vector (m, m) , such that if $(i, j) \in S$ then $0 < i - j < n$:

$$\emptyset \quad \{(1, 0)\} \quad \{(2, 0)\} \quad \{(1, 0), (2, 0)\} \quad \{(2, 0), (2, 1)\}$$

- aaa. Linear extensions of the poset $2 \times n$:



123456

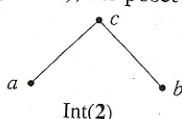
123546

132456

132546

135246

- bbb. Order ideals of $\text{Int}(n-1)$, the poset of intervals of the chain $n-1$:


 $\emptyset, a, b, ab, abc$

- ccc. Order ideals of the poset A_n obtained from the poset $(n-1) \times (n-1)$ by adding the relations $(i, j) < (j, i)$ if $i > j$ (see Figure 6-5 for the Hasse

diagram of A_4 :

$$\emptyset \quad \{11\} \quad \{11, 21\} \quad \{11, 21, 12\} \quad \{11, 21, 12, 22\}$$

- ddd. Nonisomorphic n -element posets with no induced subposet isomorphic to $2 + 2$ or $3 + 1$:



- eee. Nonisomorphic $(n + 1)$ -element posets that are a union of two chains and that are not a (nontrivial) ordinal sum, rooted at a minimal element:



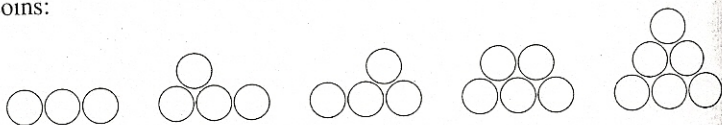
- fff. Relations R on $[n]$ that are reflexive ($i Ri$), symmetric ($i Rj \Rightarrow j Ri$), and such that if $1 \leq i < j < k \leq n$ and $i Rk$, then $i Rj$ and $j Rk$ (in the example below we write ij for the pair (i, j) , and we omit the pairs ii):

$$\emptyset \quad \{12, 21\} \quad \{23, 32\} \quad \{12, 21, 23, 32\} \quad \{12, 21, 13, 31, 23, 32\}$$

- ggg. Joining some of the vertices of a convex $(n - 1)$ -gon by disjoint line segments, and circling a subset of the remaining vertices:



- hhh. Ways to stack coins in the plane, the bottom row consisting of n consecutive coins:



- iii. n -tuples $(a_1, a_2, \dots, a_n)$ of integers $a_i \geq 2$ such that in the sequence $1a_1a_2 \dots a_n1$, each a_i divides the sum of its two neighbors:

$$14321 \quad 13521 \quad 13231 \quad 12531 \quad 12341$$

- jjj. n -element multisets on $\mathbb{Z}/(n + 1)\mathbb{Z}$ whose elements sum to 0:

$$000 \quad 013 \quad 022 \quad 112 \quad 233$$

- kkk. n -element subsets S of $\mathbb{N} \times \mathbb{N}$ such that if $(i, j) \in S$ then $i \geq j$ and there is a lattice path from $(0, 0)$ to (i, j) with steps $(0, 1)$, $(1, 0)$, and $(1, 1)$ that lies entirely inside S :

$$\{(0, 0), (1, 0), (2, 0)\} \quad \{(0, 0), (1, 0), (1, 1)\} \quad \{(0, 0), (1, 0), (2, 1)\} \\ \{(0, 0), (1, 1), (2, 1)\} \quad \{(0, 0), (1, 1), (2, 2)\}$$

- lll. Regions into which the cone $x_1 \geq x_2 \geq \dots \geq x_n$ in $\mathbb{R}^n$ is divided by the hyperplanes $x_i - x_j = 1$, for $1 \leq i < j \leq n$ (the diagram below shows the

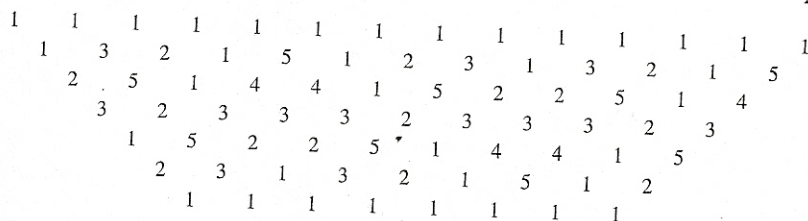
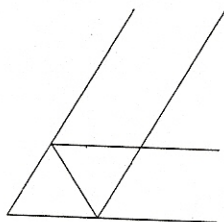


Figure 6-6. The frieze pattern corresponding to the sequence $(1, 3, 2, 1, 5, 1, 2, 3)$.

situation for $n = 3$, intersected with the hyperplane $x_1 + x_2 + x_3 = 0$):



mmm. Positive integer sequences $a_1, a_2, \dots, a_{n+2}$ for which there exists an integer array (necessarily with $n + 1$ rows)

$$\begin{array}{cccccccccccc}
 1 & 1 & 1 & \dots & 1 & 1 & 1 & \dots & 1 & 1 \\
 a_1 & a_2 & a_3 & \dots & a_{n+2} & a_1 & a_2 & \dots & a_{n-1} & \\
 b_1 & b_2 & b_3 & \dots & b_{n+2} & b_1 & \dots & b_{n-2} & & \\
 & & & \vdots & & & & & & \\
 r_1 & r_2 & r_3 & \dots & r_{n+2} & r_1 & & & & \\
 1 & 1 & 1 & \dots & 1 & & & & &
 \end{array} \quad (6.54)$$

such that any four neighboring entries in the configuration s_t^r satisfy $st = ru + 1$ (an example of such an array for $(a_1, \dots, a_8) = (1, 3, 2, 1, 5, 1, 2, 3)$ (necessarily unique) is given by Figure 6-6):

$$\begin{array}{ccccccc}
 12213 & 22131 & 21312 & 13122 & 31221 & & \\
 & & & & & &
 \end{array}$$

nnn. n -tuples $(a_1, \dots, a_n)$ of positive integers such that the tridiagonal matrix

$$\begin{bmatrix}
 a_1 & 1 & 0 & 0 & \dots & 0 & 0 \\
 1 & a_2 & 1 & 0 & \dots & 0 & 0 \\
 0 & 1 & a_3 & 1 & \dots & 0 & 0 \\
 \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
 0 & 0 & 0 & 0 & \dots & a_{n-1} & 1 \\
 0 & 0 & 0 & 0 & \dots & 1 & a_n
 \end{bmatrix}$$

is positive definite with determinant one:

$$\begin{array}{ccccccc}
 131 & 122 & 221 & 213 & 312 & &
 \end{array}$$

