

用智慧探讨人生真义,
用毅力安排人生时间。

Counting principles

1. If $A \cap B = \emptyset$, then $|A \cup B| = |A| + |B|$.

How about $A \cap B \neq \emptyset$?

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

2. How to enumerate $\left| \bigcup_{i=1}^k A_i \right|$?

It is well-known that $|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| -$

$$|A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3|.$$

(*) 你(好)能推导出一般式吗? ($k \geq 1$)

3. The number of choices

Let A be a set and $|A| = n$. Let B be a set and $|B| = m$.

- (1) The number of choosing one element from A is $|A| = n$.
(ways)

- (2) The number of choosing two elements is $\frac{n(n-1)}{2}$. (?)
(ways)

- (3) The number of choosing one element from A and one element from B independently is $m \cdot n$. ($|A \times B| = m \cdot n$)
(ways)

$$A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}.$$

(*) If we have k sets $A_1, A_2, \dots, A_k$, then the number of choosing one element from each set independently is

$$\text{equal to } |A_1 \times A_2 \times \dots \times A_k| \stackrel{=}{=} \prod_{i=1}^k |A_i|. \quad (\stackrel{=}{=} \text{Notation})$$

(4) The number of choosing t elements ($t \geq 1$) from A is denoted
(ways)

by $\binom{n}{t}$, read "n chooses t".

(5) The number of choosing t elements in consecutive order from
(ways)

A is equal to $n \cdot (n-1) \cdot \dots \cdot (n-t+1) = \frac{n!}{(n-t)!}$.

(6) Since we don't consider "order" in "n chooses t", $\binom{n}{t} = \frac{n!}{t!(n-t)!}$.

If we denote $n \cdot (n-1) \cdot \dots \cdot (n-t+1) = n^{\underline{t}}$, then $\binom{n}{t} = \frac{n^{\underline{t}}}{t!}$.

$$(*) \quad \binom{0}{3} = 0, \quad \binom{3}{0} = 1, \quad \binom{n}{t} = \binom{n}{n-t} \quad \text{where } n \geq t.$$

Basic properties

Proposition Let n and t be positive integers. This implies that

$$\binom{n}{t} = \binom{n-1}{t} + \binom{n-1}{t-1}.$$

Proof. (Method 1) By $\binom{n}{t} = \frac{n!}{t!(n-t)!}$.

(Method 2) Combinatorial proof. (New) (Good idea!)

(Method 2)

 x^k 在 $(1+x)^n$ 的展開中係數為 $\binom{n}{k}$.

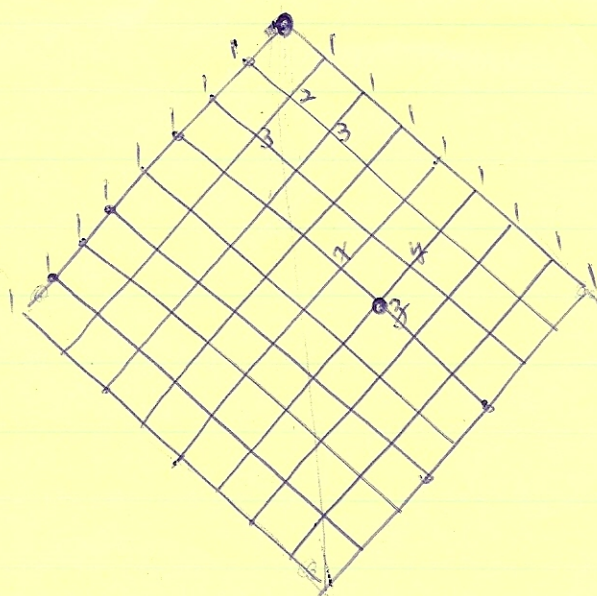
$$(1+x)^{n-1} \cdot 1 + (1+x)^{n-1} \cdot x = (1+x)^n$$

Compare the coefficients of (x^k) :

$$\text{Left: } \binom{n-1}{k} + \binom{n-1}{k-1}.$$

$$\text{Right: } \binom{n}{k}.$$

(Method 3)



$$x+y=z$$

$$\text{right} \leftarrow x = \binom{7}{4} = \binom{7}{3} \rightarrow \text{left}$$

$$\text{right} \leftarrow y = \binom{7}{5} = \binom{7}{2} \rightarrow \text{left}$$

$$\text{right} \leftarrow z = \binom{8}{5} = \binom{8}{3} \rightarrow \text{left}$$

$$\Rightarrow \binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$