

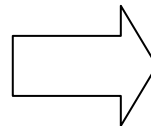
<Step1> 分四個type討論

<一>

0	1	2	3	4
1	0			
2				
3				
4				

<二>

0	1	2	3	4
1	2			
2				
3				
4				



l_5 = number of standard latin square

求出 $l_5 = 56$

<三>

0	1	2	3	4
1	3			
2				
3				
4				

<四>

0	1	2	3	4
1	4			
2				
3				
4				

<Step2> $L_5 = l_5 * 5! * (5-1)! = 161280$

0	1	2	3	4
1	0			
2				
3				
4				

0	1	2	3	4
1	0	3	4	2
2				
3				
4				

0	1	2	3	4
1	0	4	2	3
2				
3				
4				

0	1	2	3	4
1	0	3	4	2
2	3	4		
3	4			
4	2			

0	1	2	3	4
1	0	3	4	2
2	4			3
3	2			
4	3			

0	1	2	3	4
1	0	4	2	3
2	3	4		2
3	4			
4	2			

0	1	2	3	4
1	0	4	2	3
2	4	3		
3	2		4	
4	3			

0	1	2	3	4
1	0	3	4	2
2	3	4	0	1
3	4	1	2	0
4	2	0	1	3

0	1	2	3	4
1	0	3	4	2
2	4	0	1	3
3	2	4	0	1
4	3	1	2	0

0	1	2	3	4
1	0	4	2	3
2	3	0	4	1
3	4	1	0	2
4	2	3	1	0

0	1	2	3	4
1	0	4	2	3
2	4	3	1	0
3	2	0	4	1
4	3	1	0	2

0	1	2	3	4
1	0	3	4	2
2	3	4	1	0
3	4	0	2	1
4	2	1	0	3

0	1	2	3	4
1	0	3	4	2
2	4	1	0	3
3	2	4	1	0
4	3	0	2	1

0	1	2	3	4
1	0	4	2	3
2	3	1	4	0
3	4	0	1	2
4	2	3	0	1

0	1	2	3	4
1	0	4	2	3
2	4	3	0	1
3	2	1	4	0
4	3	0	1	2

<1>

<2>

<3>

<4>

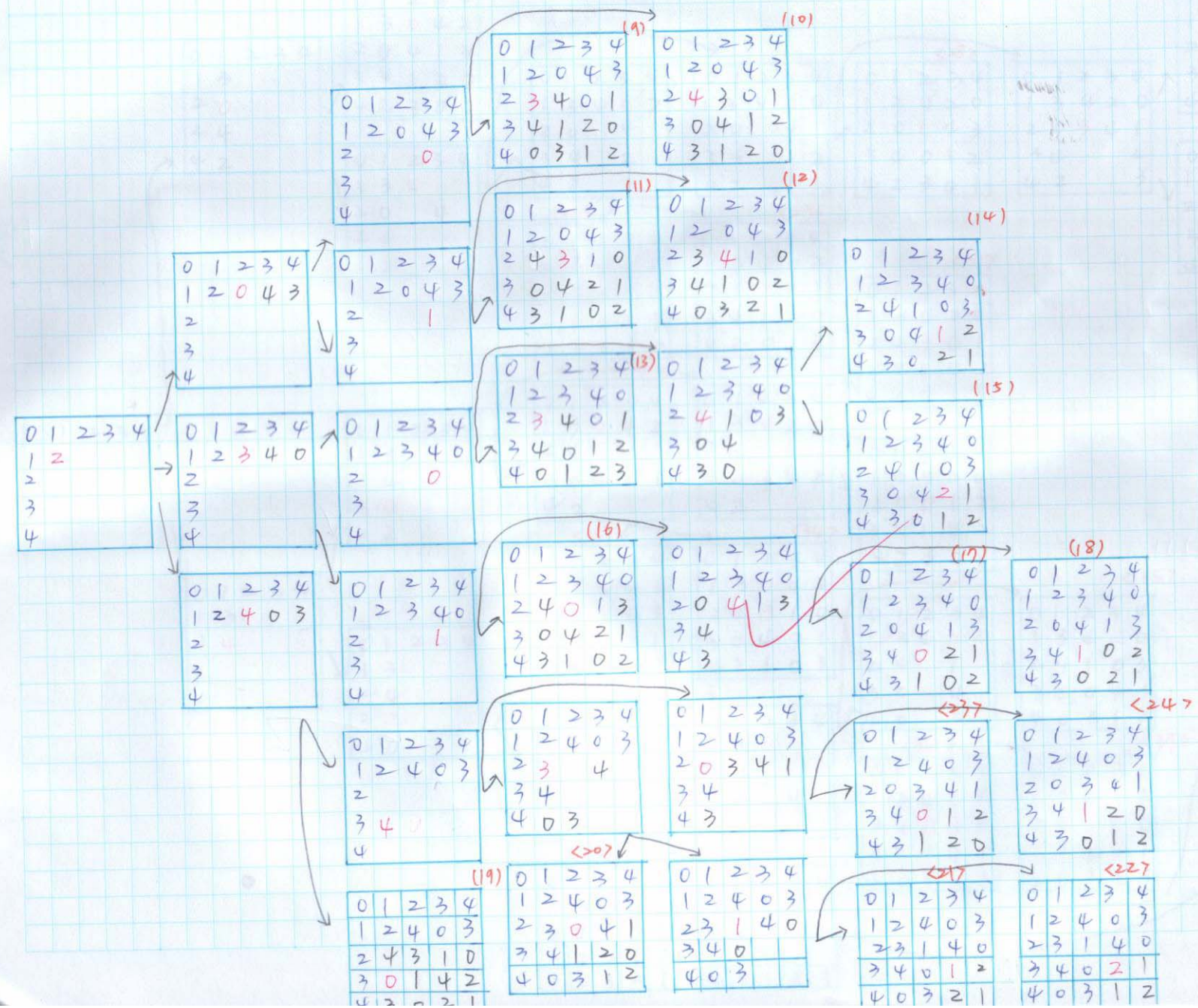
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<6>

<7>

<8>

(=)



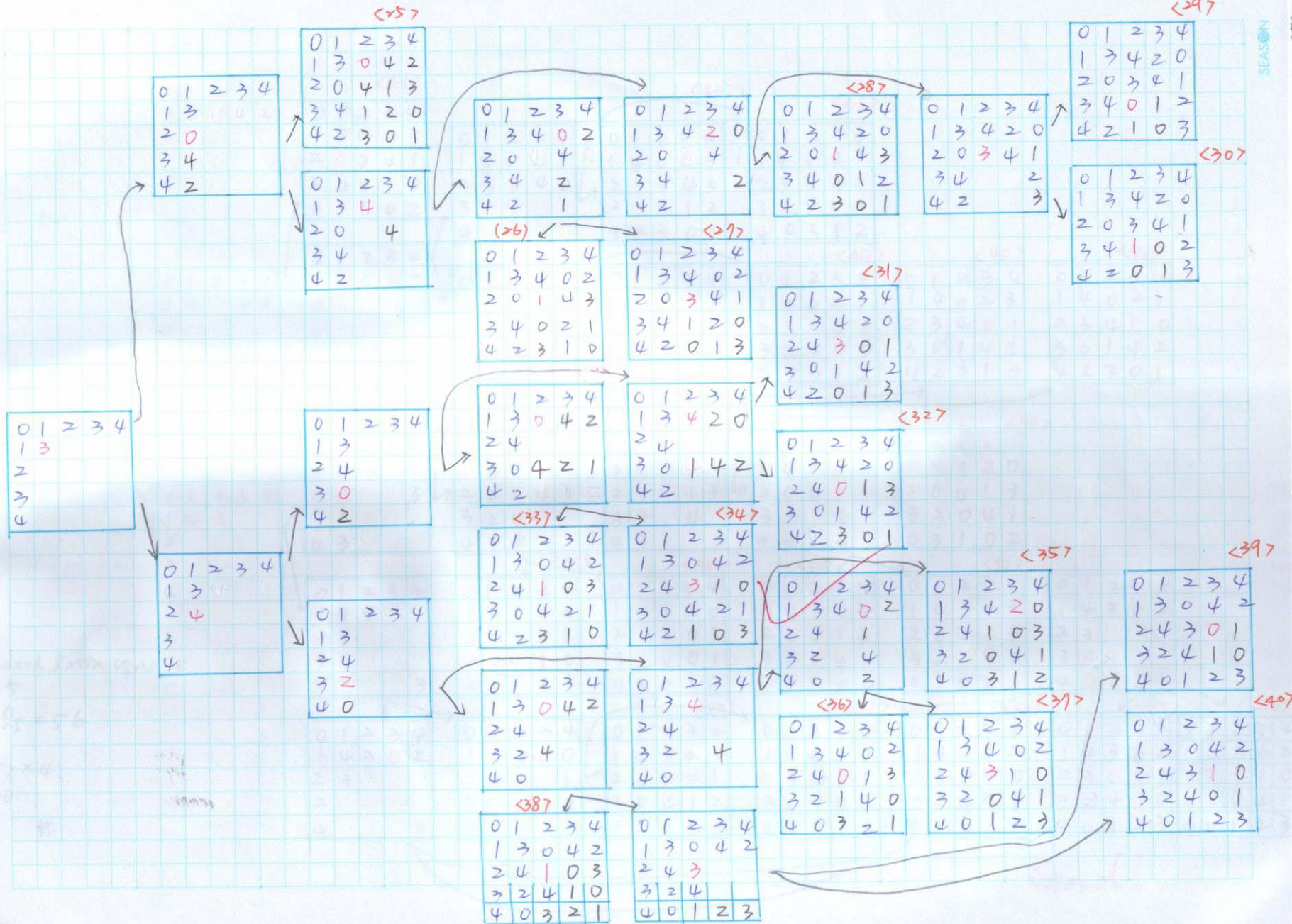
x)

~

SEASON

NO.

(=)



(12)

0	1	2	3	4
1	4			
2				
3				
4				

0	1	2	3	4
1	4	3		
2				
3				
4				

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1	4	0	2	3
2	0	3	4	1
3	2	4	1	0
4	3	1	0	2

0	1	2	3	4
1	4	0	2	3
2	3	1	4	0
3	4			
4	3			

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3	0	4	1	2
4	2	3	0	1

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2	3	1	4	0
3	2	4	0	1
4	0	3	1	2

0	1	2	3	4
1	4	0	2	3
2	3			
3				
4				

0	1	2	3	4
1	4	0	2	3
2	3	4		
3	1	4		
4	3			

0	1	2	3	4
1	4	0	2	3
2	3	4		
3	0	1	4	2
4	2	3		

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4	0	3	1	2

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3	0	1	4	2
4	2	3	1	0

<46>

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2	3	4	1	0
3	0	1	4	2
4	2	3	0	1

0	1	2	3	4
1	4	3		
2	0		3	
3	2			
4	3			

0	1	2	3	4
1	4	3		
2	0	1	4	3
3	2	4		
4	3	0		

0	1	2	3	4
1	4	3		
2	0	4	1	3
3	2	4		
4	3			

0	1	2	3	4
1	4	3	0	2
2	0	4	1	3
3	2	4		
4	3	2		

0	1	2	3	4
1	4	3	2	0
2	0	4	1	3
3	2	0	4	1
4	3	1	0	2

<49>

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1	4	3	0	2
2	0	4	1	3
3	2	0	4	1
4	3	1	0	2

0	1	2	3	4
1	4	3		
2	3			
3				
4				

<47>

0	1	2	3	4
1	4	3	0	2
2	0	1	4	3
3	2	4	1	0
4	3	0	2	1

<48>

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1	4	3	2	0
2	0	1	4	3
3	2	4	0	1
4	3	0	1	2

<50>

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1	4	3	0	2
2	0	4	1	3
3	2	0	4	1
4	3	1	2	0

<51>

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1	4	3	0	2
2	0	4	1	3
3	2	1	4	0
4	3	0	2	1

0	1	2	3	4
1	4	3	0	2
2	3			
3	2			
4	0	1	2	3

0	1	2	3	4
1	4	3	0	2
2	3			
3				
4				

0	1	2	3	4
1	4	3	2	0
2	3			
3	0			
4	2			

<52>

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1	4	3	2	0
2	3	0	4	1
3	0	4	1	2
4	2	1	0	3

<53>

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1	4	3	2	0
2	3	4	0	1
3	0	1	4	2
4	2	0	1	3

<54>

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1	4	3	0	2
2	3	1	4	0
3	0	4	2	1
4	2	0	1	3

<55>

0	1	2	3	4
1	4	3	0	2
2	3	0	4	1
3	2	4	1	0
4	0	1	2	3

<56>

0	1	2	3	4
1	4	3	0	2
2	3	0	4	1
3	2	0	4	1
4	0	1	2	3

of standard latin square

conclusion: $L_5 = 56$

$$\therefore L_5 = 56 \times 5! \times 4! = 16! = 80$$

#.

Good!

Combinatorial Designs HW(I) 1(b)

由陳哲皓同學提供

```
#include<iostream.h>
```

```
void main(void)
```

```
{  
    /*  
    我們現在爲了計算方便,所以就先算 standard latin square of order 6  
    讓第一 row 爲 123456,第一 column 爲 123456  
    因爲每一 row 第一個數字已經被固定了,所以每一個 row 都有 120 種選擇  
    */  
    int total = 0,g=0;  
    int testColumn(int [][][120][6], int, int, int, int, int);  
    int testColumn2(int [][][120][6], int, int, int, int, int, int, int, int);  
    const size = 6;  
    int row[size-1][120][size]={0}; /* 因爲 2~6 row 都有 120 種選擇,讓 row[i][j][]代表  
                                     (i+2) row 的第 j 個選擇 i=0~4 j=0~119*/  
  
    int initial=0,a1=0,a2=0,a3=0,a4=0,a5=0,a6=0;  
    int b1=0,b2=0,b3=0,b4=0,b5=0,b6=0,t=0,k=0;  
    //下面是開始建造 row[i][j][]  
    for(a1=2;a1<=6;a1++) //決定第一個數字  
    {  
        row[a1-2][initial][0]=a1;  
        for(a2=1;a2<=6;a2++) //決定第二個數字  
        {  
            if(a2!=a1)  
                row[a1-2][initial][1]=a2;  
            else  
                continue;  
  
            for(a3=1;a3<=6;a3++) //決定第三個數字  
            {  
                if(a3!=a1 && a3!=a2)  
                    row[a1-2][initial][2]=a3;  
                else  
                    continue;  
  
                for(a4=1;a4<=6;a4++) //決定第四個數字  
                {  
                    if(a4!=a1 && a4!=a2 && a4!=a3)
```



```

        for(b4=0;b4<120;b4++)
            for(b5=0;b5<120;b5++)
                for(b6=0;b6<120;b6++)
                {
                    total = total + testColumn(row, b2, b3, b4, b5, b6);//如果是 latin square 則加一
                    cout << total << "      " << g++ << endl;

                }
/*以下是簡單的測試程式
int kkk[size-1][120][size]={0};
kkk[0][0][0]= 2;
kkk[0][0][1]= 3;
kkk[0][0][2]= 4;
kkk[0][0][3]= 5;
kkk[0][0][4]= 6;
kkk[0][0][5]= 1;

kkk[1][0][0]= 3;
kkk[1][0][1]= 4;
kkk[1][0][2]= 5;
kkk[1][0][3]= 6;
kkk[1][0][4]= 1;
kkk[1][0][5]= 2;

kkk[2][0][0]= 4;
kkk[2][0][1]= 5;
kkk[2][0][2]= 6;
kkk[2][0][3]= 1;
kkk[2][0][4]= 2;
kkk[2][0][5]= 3;

kkk[3][0][0]= 5;
kkk[3][0][1]= 6;
kkk[3][0][2]= 1;
kkk[3][0][3]= 2;
kkk[3][0][4]= 3;
kkk[3][0][5]= 4;

kkk[4][0][0]= 6;
kkk[4][0][1]= 1;
kkk[4][0][2]= 2;
kkk[4][0][3]= 3;

```

```

    kkk[4][0][4]= 4;
    kkk[4][0][5]= 5;
    cout << testColumn(kkk,0,0,0,0,0) << endl;*/
}
//testColumn1 and testColumn2 是用來測試 column 是否有重覆的數字出現
int testColumn2(int row[][120][6], int b2, int b3, int b4, int b5, int b6, int col,int tr)
{
    int column[6] = {0};
    column[0] = col;
    column[1] = row[0][b2][col-1];
    column[2] = row[1][b3][col-1];
    column[3] = row[2][b4][col-1];
    column[4] = row[3][b5][col-1];
    column[5] = row[4][b6][col-1];

    for(int r=0;r<5;r++)
    {
        for(int w=r+1;w<6;w++)
        {
            if(column[r] == column[w])
            {
                tr = 0;
                break;
            }
        }
    }
    return tr;
}

int testColumn(int row[][120][6], int b2, int b3, int b4, int b5, int b6)
{
    int tr = 1;
    for(int s=2;s<7;s++)
    {
        if( testColumn2(row, b2, b3, b4, b5, b6, s, tr) == 0)
        {
            tr = 0;
            break;
        }
    }
    return tr;
}

```

2. List two orthogonal latin squares of order 8.

吳政軒

Sol : Let

$$L^{(1)} = a_{ij} \text{ and } a_{ij} = i+j \quad L^{(2)} = b_{ij} \text{ and } b_{ij} = i+2j$$

A :

0	1	2	3	4	5	6	7
1	0	4	5	2	3	7	6
2	4	0	6	1	7	3	5
3	5	6	0	7	1	2	4
4	2	1	7	0	6	5	3
5	3	7	1	6	0	4	2
6	7	3	2	5	4	0	1
7	6	5	4	3	2	1	0

B :

0	6	1	2	7	3	4	5
1	7	0	4	6	5	2	3
2	3	4	0	5	6	1	7
3	2	5	6	4	0	7	1
4	5	2	1	3	7	0	6
5	4	3	7	2	1	6	0
6	0	7	3	1	2	5	4
7	1	6	5	0	4	3	2

$\therefore Z_2$ is a field and x^3+x+1 is irreducible over Z_2

$$\therefore \frac{Z_2[x]}{\langle x^3+x+1 \rangle} \text{ is a field and } \frac{Z_2[x]}{\langle x^3+x+1 \rangle} = 8$$

Denote

$$0 + \langle x^3+x+1 \rangle \rightarrow 0$$

$$x^2 + \langle x^3+x+1 \rangle \rightarrow 1$$

$$x + \langle x^3+x+1 \rangle \rightarrow 2$$

$$1 + \langle x^3+x+1 \rangle \rightarrow 3$$

$$x^2+x + \langle x^3+x+1 \rangle \rightarrow 4$$

$$x^2+1 + \langle x^3+x+1 \rangle \rightarrow 5$$

$$x+1 + \langle x^3+x+1 \rangle \rightarrow 6$$

$$x^2+x+1 + \langle x^3+x+1 \rangle \rightarrow 7$$

$$\text{If } (a_{ij}, b_{ij}) = (a_{i'j'}, b_{i'j'})$$

$$\Rightarrow i+j = i'+j', \quad i+2j = i'+2j'$$

$$\Rightarrow j-j' = 2(j-j') \quad \text{Since } \frac{Z_2[x]}{\langle x^3+x+1 \rangle} \text{ is a field}$$

$$\Rightarrow j-j' = 0$$

$$\Rightarrow j = j' \quad \text{and} \quad i = i' \quad \text{hence} \quad L^{(1)} \perp L^{(2)}$$

Combinatorial Designs H.W.(I) 連敏筠

3.(a)(1) Suppose $A_1, A_2, \dots, A_r$ are r MOLS of order n .

$\because$ If $A \perp B$, then $\alpha(A) \perp \beta(B)$ where α, β are permutations of entries.

$\therefore$ We can relabel the entries, such that each has first row $0, 1, \dots, n-1$.

Consider the r entries in the $(1, 0)$ position. None of these can be 0, since 0 already occurred in the first row. So $r \leq n-1$.

(2) Let $F = GF(n)$, since n be a prime power.

Define: For each $k \in F^*$ $L^{(k)} = [l^{(k)}_{i,j}] = [i + kj]$.

Claim: $L^{(k)}$ is a latin square, $\forall k \in F^*$.

i.e. claim: $l^{(k)}_{i,j} \neq l^{(k)}_{i,j'}, \forall j \neq j'$ and $l^{(k)}_{i,j} \neq l^{(k)}_{i',j}, \forall i \neq i'$

$$[l^{(k)}_{i,j}] = i + kj \quad [l^{(k)}_{i,j'}] = i + kj' \quad \text{If}$$

$$l^{(k)}_{i,j} = l^{(k)}_{i,j'} \Rightarrow i + kj = i + kj' \Rightarrow k(j - j') = 0 \Rightarrow j = j'. (\text{矛盾})$$

盾) Since F is a field.

Similarly to $l^{(k)}_{i,j} \neq l^{(k)}_{i',j}$.

Claim: $\forall k, h \in F^*, L^{(k)} \perp L^{(h)}$.

Suppose not.

$$\exists i \neq i', j \neq j' \text{ s.t. } l^{(k)}_{i,j} = l^{(k)}_{i',j'} \Rightarrow l^{(h)}_{i,j} = l^{(h)}_{i',j'}$$

$$\begin{cases} i + kj = i' + kj' \\ i + hj = i' + hj' \end{cases} \Rightarrow (k - h)j = (k - h)j' \quad (k \neq h \text{ and integral domain})$$

$$\Rightarrow j = j' \Rightarrow i = i' (\text{矛盾})$$

由(1)知 $M(n) \leq n-1$, and (2) construct $n-1$ MOLS.

Therefore, $M(n) = n-1$ when n is a prime power.

(b)

Let $n = p_1^{n_1} p_2^{n_2} \cdots p_r^{n_r}$ where p_i 's are prime, ($p_1 < p_2 < \cdots < p_r$)

Since $n \not\equiv 2 \pmod{4} \Rightarrow$ if $p_1 = 2$ then $n_1 \geq 2$.

By (a) $\forall p_i^{n_i} : \text{prime power. } M(p_i^{n_i}) = p_i^{n_i} - 1$

$\therefore A_1 \perp A_2$ and $B_1 \perp B_2 \Rightarrow A_1 \times B_1 \perp A_2 \times B_2$ and $M(p_i^{n_i}) = p_i^{n_i} - 1 \geq 2$.

$\therefore$ We can choose two orthogonal latin square from each size of $p_i^{n_i}$.

$A_1 \perp B_1, A_2 \perp B_2, \cdots A_r \perp B_r$, where A_i, B_i are of size $p_i^{n_i} \times p_i^{n_i}$

$\Rightarrow A_1 \times A_2 \times \cdots \times A_r \perp B_1 \times B_2 \times \cdots \times B_r$

$\Rightarrow M(n) \geq 2$ when $n \not\equiv 2 \pmod{4}$.

(c)

Let $A_1', \cdots, A_{n-2}'$ are $n-2$ MOLS of order n . We can permutation entries s.t each has first row $0, 1, \dots, n-1$. Let $A_1, \cdots, A_{n-2}$ are $n-2$ mutually orthogonal standard latin squares of order n .

Let $A = [a_{i,j}]$ where $a_{0,j} = j$ and $a_{i,j} = \{0, 1, \dots, n-1\} \setminus (\bigcup_{k=1}^{n-2} \{a_{i,j}^{(k)}\} \cup i)$ for $1 \leq i \leq n-1$

(1) Claim: $a_{i,j}$ 是唯一的.

Proof.

$\forall (A_i, A_j), (i, i)$ pairs 已經出現在第一個row.

$\therefore \forall i \neq 0$ if $a_{i,j}^{(r)} = k$ then $a_{i,j}^{(l)} \neq k \forall l \in \{0, 1, \dots, n-1\} \setminus \{r\}$

$\therefore |\bigcup_{k=1}^{n-2} a_{i,j}^{(k)}| = n-2 \Rightarrow a_{i,j}$ 被唯一決定.

(2) Claim: A is a latin square.

Proof. Suppose not. $\exists a_{i,j} = a_{i,l} = r$ where $j, l \neq r$.

$\therefore$ each row 一定要有 r .

$$\therefore a_{i,b}^{(k)} = r \text{ for } b \in \{0, 1, \dots, n-1\} \setminus \{r, i, l\} \forall k$$

$\forall s \neq t, a_{i,j}^{(s)} \neq a_{i,j}^{(t)} \therefore r$ 只有 $n-3$ 個選擇 (矛盾)

$$\therefore a_{i,j} \neq a_{i,l}.$$

Simialrly to $a_{i,j} \neq a_{l,j}$.

(3) Claim: $A_i \perp A \quad \forall i = 1, 2, \dots, n-2$

Proof. Fix $l \in A_i$. Let $U = \{(x, y) \mid a_{xy}^i = l\}$.

(1) Count 所有數字在 $A_1, A_2, \dots, A_{i-1}, A_{i+1}, \dots, A_{n-2}$ 中 U 的位置出現的次數

(看每個 A_i 再去作總和)

$$\begin{cases} l \text{ 出現 } n + (n-3) \text{ 次} = 2n-3 \text{ 次} \\ j \text{ 出現 } (n-3) \text{ 次 } \quad \forall j \in [0, \dots, n-1] \setminus \{l\} \end{cases}$$

(2) Count 所有數字在 $A_1, A_2, \dots, A_{i-1}, A_{i+1}, \dots, A_{n-2}, A$ 中 U 的位置出現的次數

(對每個位置經過 $A_1, \dots, A_{n-2}, A$ 再做總和)

$$\begin{cases} l \text{ 出現 } (n-1) + (n-1) \text{ 次} = 2n-2 \text{ 次} \\ j \text{ 出現 } (n-2) \text{ 次 } \quad \forall j \in [0, \dots, n-1] \setminus \{l\} \end{cases}$$

由(1),(2)知道在 A 中每個數字在 U 的位置恰出現一次

$$\therefore A_i \perp A \quad \forall i = 1, 2, \dots, n-2.$$

Combinatorial designs

Hui-Fen, Tseng

December 3, 2007

5. Find infinite many latin squares(with distinct orders) which have no orthogonal mate.

sol:

By induction, If L has an orthogonal mate then L has a transversal. Hence, we make latin square with distinct orders which have no transversal, we guarantee those latin squares which have no orthogonal mate.

When m is odd. We make a latin square.

$$\begin{array}{|c|c|} \hline \text{LS}(m) \\ \text{on } [0, m-1] \\ \hline \text{LS}(m) \\ \text{on } [m, 2m-1] \\ \hline \end{array} \begin{array}{|c|c|} \hline \text{LS}(m) \\ [m, 2m-1] \\ \hline \text{LS}(m) \\ [0, m-1] \\ \hline \end{array} = \begin{array}{|c|c|} \hline A & B \\ \hline C & D \\ \hline \end{array} \quad 2m \times 2m. \quad \forall m \in \mathbb{N}$$

which LS(m) on $[0, m-1]$:= a latin square of order m and its' entry belongs to $[0, m-1]$

if we find a transversal $\{1, 2, \dots, m-1\}$ in a latin square A.

By def of transversal, we can't find $m, \dots, 2m-1$ in B and C. Thus, we find entry $m, \dots, 2m-1$ in D. But it's impossible, there're only $0, \dots, m-1$ entries. Hence, we can't find a transversal $\{0, 1, 2, \dots, 2m-1\}$ in this latin square, then those latin squares won't have orthogonal mate. We have done.

Combinatorial Design Homework (I)

裴若宇

6. Find four latin squares L_1, L_2, L_3 and L_4 such that $L_1 \perp L_2, L_2 \perp L_3, L_3 \perp L_4, L_1$ is not orthogonal to L_3 and L_4 and L_2 is not orthogonal to L_4 .

[Sol.]

We define five latin squares : $A_1 = [a_{i,j}^{(1)}] = i + j, A_2 = [a_{i,j}^{(2)}] = i + 2j$ for $i, j \in \mathbb{Z}_3$,

$B_1 = [b_{i,j}^{(1)}] = i + j, B_2 = [b_{i,j}^{(2)}] = i + x \cdot j, B_3 = [b_{i,j}^{(3)}] = i + (x+1) \cdot j$ for $i, j, x, x+1$ in

$\langle F, +, \cdot \rangle$ where $F = \mathbb{Z}_3[x] / \langle x^2 + x + 1 \rangle$

A_1 :

0	1	2
1	2	0
2	0	1

A_2 :

0	2	1
1	0	2
2	1	0

B_1 :

0	1	2	3
1	0	3	2
2	3	0	1
3	2	1	0

B_2 :

0	2	3	1
1	3	2	0
2	0	1	3
3	1	0	2

B_3 :

0	3	1	2
1	2	0	3
2	1	3	0
3	0	2	1

By homework 3 (a), we know that A_1, A_2 are mutually orthogonal latin squares of order 3, B_1, B_2, B_3 are mutually orthogonal latin squares of order 4.

Now, define $L_1 = A_1 \times B_1, L_2 = A_2 \times B_2, L_3 = A_1 \times B_3, L_4 = A_2 \times B_1$,

By the lemma : 「 If $A_1 \perp A_2$ and $B_1 \perp B_2$, then $(A_1 \times B_1) \perp (A_2 \times B_2)$ 」 ,

$L_1 \perp L_2, L_2 \perp L_3, L_3 \perp L_4, L_1$ is not orthogonal to L_3 and L_4 and L_2 is not orthogonal to L_4 .

Combinatorial Designs H.W.(I) 施智懷

7. Method 1.(Hall's Theorem)

$S = \{S_1, \dots, S_n\}$ has an SDR if and only if for each $1 \leq k \leq n$,

$$\left| \bigcup_{j=1}^k S_{i,j} \right| \geq k .$$

→ This is obvious.(by definition of SDR)

← By induction on n

(i) $n=1$ is true

(ii) Assume that the assertion holds for each $m < n$

(iii) Let $S = \{S_1, \dots, S_n\}$

For any $k : 1 \leq k \leq n$, $\left| \bigcup_{j=1}^k S_{i,j} \right| \geq k .$

Consider: $S' = \{S_1, \dots, S_{n-1}\}$

Case(1) for any h sets in S' the union contains at least $h+1$ elements.

Let $x \in S_n$ and $S'' = \{S_1 - \{x\}, S_2 - \{x\}, \dots, S_{n-1} - \{x\}\}$

Clearly, S'' has SDR → S has SDR.

Case(2) exist h sets in S' such that its union contains exactly h elements.

W.L.O.G , let $S^* = S_1 \cup S_2 \cup \dots \cup S_h$ then $|S^*| = h$

Consider: $\{S_{h+1} \setminus S^*, \dots, S_n \setminus S^*\} = S^{**}$

$$\left| \bigcup_{i=h+1}^n (S_i \setminus S^*) \right| = \left| \bigcup_{i=1}^n (S_i \setminus S^*) \right| = \left| \left(\bigcup_{i=1}^n S_i \right) \setminus S^* \right| = \left| \left(\bigcup_{i=1}^n S_i \right) \right| - |S^*| \geq n - h$$

By induction hypothesis $\rightarrow \{S_1, \dots, S_h\}, S^*$ has SDR

$\rightarrow S$ has SDR

Method 2.

An X, Y – bipartite G has a matching that saturates X if and only if $|N(S)| \geq |S|$ for all $S \subseteq X$.

Proof:

$\rightarrow$ This is obvious since the $|S|$ vertices matched to S must belong to $N(S)$.

$\leftarrow$ Suppose not

Claim: If G does not have a matching that saturate X , then

there is a set $S \subseteq X$ such that $|N(S)| < |S|$.

proof:

Let M be a maximum matching of G , $|M| \leq |X|$

$\rightarrow \exists u \in X$ s.t. u is unsaturated by M

let S : the set of vertices in X that is reachable from u by M -alternating paths.

let T : the set of vertices in Y that is reachable from u

by M-alternating paths.

Fact 1. $|T| = |S| - 1$

Let $G^* = G[M], T^* = N_{G^*}[S]$

if $\exists v \in T \setminus T^* \rightarrow \exists M - \text{augmenting } u - v \text{ path}$

$\rightarrow$ M is not a maximum matching $\rightarrow \leftarrow$

$$\therefore |T| = |T^*| = |S - u| = |S| - 1$$

Fact 2. $T = N(S)$

Trivial $T \subseteq N(S)$

If $\exists v \in N(S) \setminus T$.

Note: $\forall w \in N(v) \cap S$, w is saturated by M and $N(v) \cap S \neq \emptyset$

$\rightarrow \exists M - \text{augmenting } u - v \text{ path}$

$\rightarrow$ M is not a maximum matching $\rightarrow \leftarrow$

$\therefore$ By Fact 1,2, done!

8. Find a latin square of order n (as small as possible) such that the square contains subsquares of order 2,3,4 and 5 respectively.

(sol)

建構方式說明如下：

- 首先,由於 order 越小越好,考慮將 LS(2)分別內嵌於 LS(4)和 LS(5) (By Ryser's Theorem, it can be done), 然後將 LS(3)和 LS(4)作部份重疊(如下圖所示). 從 LS(2)開始填數字(0,1),接著把 LS(2)擴大成 LS(5) (0,1,2,3,4)和 LS(4) (0,1,5,6),最後填 LS(3) (0,2,3).

3	1	0	2	4				
0	2	1	4	3				
1	0	4	3	2				
2	4	3	0	1	5	6		
4	3	2	1	0	6	5		
			5	6	0	1	●	●
			6	5	1	0	2	3
					●	3	0	2
					●	2	3	0

- 先計算每個數字各出現多少次

(0: 9 次 1: 7 次 2: 8 次 3: 8 次 4: 5 次 5: 4 次 6: 4 次

共 45 個,剩 36 格尚未填數字)

3. Theorem(Ryser):

A completed $r \times s$ partial latin rectangle R can be embedded in a latin square of order n if and only if
for all $0 \leq i \leq n-1$, $R(i) \geq r+s-n$. (Ryser's Condition)

由 Ryser's Condition 可知, n 越小 $\Leftrightarrow R(i)$ 越大.

因此考慮讓每個元素出現的次數多一點.

由於 $0 \sim 6$ 中, 出現次數最少的是 5 和 6 (皆為四次),

所以先考慮 $n = 14$ ($9+9-n \geq 4 \Rightarrow n \leq 14$), 然後在不超過可填空格數的前提之下, 讓一些出現次數較少的數字多出現幾次, 即讓 n 變小 (如下圖所示).

	(45)	(45)	(41)	(50)	(53)	(51)
0: 9						
1: 7						→ 8
2: 8						
3: 8						
4: 5						→ 6 → 7 → 8
5: 4						→ 5 → 6 → 7 → 8
6: 4						→ 5 → 6 → 7 → 8
	$\begin{smallmatrix} 7 \\ 5 \\ 13 \end{smallmatrix} : 4$	$\begin{smallmatrix} 7 \\ 5 \\ 12 \end{smallmatrix} : 5$	$\begin{smallmatrix} 7 \\ 5 \\ 11 \end{smallmatrix} : 6$	$\begin{smallmatrix} 7 \\ 5 \\ 10 \end{smallmatrix} : 7$	$\begin{smallmatrix} 7 \\ 5 \\ 9 \end{smallmatrix} : 8$	
	$17 \times 4 = 28$	$6 \times 5 = 30$	$5 \times 6 = 30$	$4 \times 7 = 28$	$3 \times 8 = 24$	
共 45 個	共 13 個	共 11 個	共 8 個	共 8 個	共 8 個	
$n=9$	$n=14$	$n=13$	$n=12$	$n=11$	$n=10$ (impossible)	

4. 由 Ryser's Theorem 可知要內嵌 LS(5) 的話, $n \geq 10$.

由步驟 3 可知 $n = 10 \Leftrightarrow R(i) \geq 8$.

觀察數字 4 可知, 如果 $n = 10$, 則數字 4 最少還要再多出現 $8 - 5 = 3$ 次, 但是數字 4 只能填在圖一的●中, 所以數字 4 最多只能再出現兩次.

$\Rightarrow n \neq 10$.

5. Consider $n = 11$, 可將所需的數字給填進去(如下圖所示).

3	1	0	2	4	8	7	9	10
0	2	1	4	3	9	10	5	6
1	0	4	3	2	10	9	6	7
2	4	3	0	1	5	6	7	8
4	3	2	1	0	6	5	8	9
8	9	7	5	6	0	1	10	4
7	8	10	6	5	1	0	2	3
10	6	5	8	9	7	3	0	2
9	5	8	7	10	4	2	3	0

6. 先將每列缺少的數字寫在旁邊, 然後填進去.

再將每行缺少的數字寫在旁邊, 然後填進去.

7. 則下圖即為所求($n = 11$).

3	1	0	2	4	8	7	9	10	6	5	→	5	6
0	2	1	4	3	9	10	5	6	8	7	→	7	8
1	0	4	3	2	10	9	6	7	5	8	→	5	8
2	4	3	0	1	5	6	7	8	10	9	→	9	10
4	3	2	1	0	6	5	8	9	7	10	→	7	10
8	9	7	5	6	0	1	10	4	2	3	→	2	3
7	8	10	6	5	1	0	2	3	9	4	→	4	9
10	6	5	8	9	7	3	0	2	4	1	→	1	4
9	5	8	7	10	4	2	3	0	1	6	→	1	6
5	10	6	9	7	2	8	4	1	3	0			
6	7	9	10	8	3	4	1	5	0	2			
↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓			
5	7	6	9	7	2	4	1	1	0	0			
6	10	9	10	8	3	8	4	5	3	2			

Note：步驟六只適用於每一列及每一行所需填入的數字個數較少時,若所需填入的數字個數較多時,可依照 Ryser's Theorem 的證明方式去填數字.

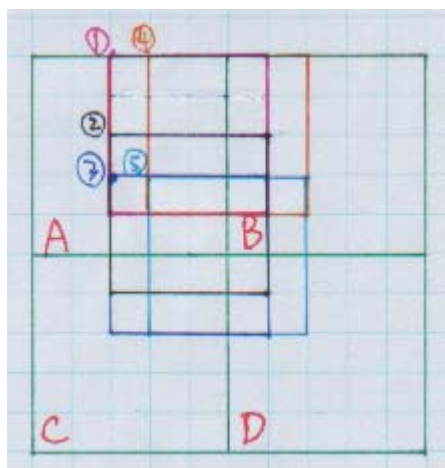
P.S：

Claim：不存在符合題意的 LS(10).

Suppose not.

首先考慮 LS(10)內嵌 LS(5).

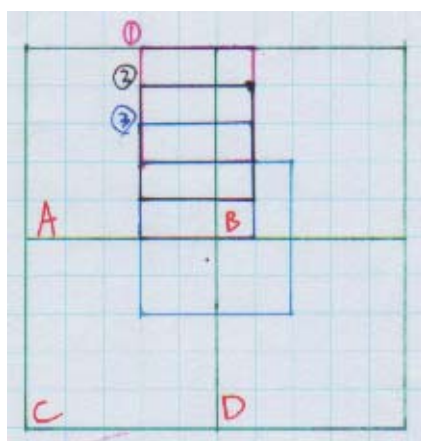
W.L.O.G,假設為下圖的型式,則 A.B.C.D 皆為 LS(5).



By Ryser's Theorem, $LS(4)$ 不可能出現在 A.B.C or D.

因此 $LS(4)$ 出現的位置只可能為上圖中的五種情況.

不過這五種情況只有正中央有可能(如下圖所示).



接著考慮 $LS(3)$ 出現的位置.

同樣地, by Ryser's Theorem, $LS(3)$ 不可能出現在 A.B.C or D,

而且 $LS(3)$ 也不可能出現在 $LS(4)$ 內部.

所以 $LS(3)$ 出現的位置只可能為上圖中的三種情況,但這三種情況皆不可能.

因此不存在符合題意的 $LS(10)$. $\square$

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9. Prove that if A is a critical set of a latin square of order n , then A contains at least $n+1$ fill entries.

Proof.

Suppose not $\Rightarrow$ If we let the critical set of order n is C_n , then $|C_n| \leq n$.

Since we know those critical set have to cover at least $n-1$ rows and $n-1$ columns, and use at least $n-1$ distinct elements in $\{0, 1, \dots, n-1\}$, so $|C_n| \geq n-1$.

Case1

If $|C_n| = n-1$, we know that those $(n-1)$ elements are

0				
	1		k	
		2		
			$\ddots$	
				$n-2$

distinct and in different rows and columns. W.L.O.G, we let this partial square $L = [l_{ij}]$, where $l_{ii} = i-1$,

$\forall i = 1, 2, \dots, n-1$. Now choose any position not in (i, i) , $i = 1, 2, \dots, n-1$. put an element $k \neq i-1$ and $k \neq j-1$.

By Lars Anderson's Theorem, we can complete this latin square. So any set C_n , where $|C_n| = n-1$, is not a critical set.

Case2

If $|C_n| = n$, same with case 1, $L = [l_{ij}]$ where $l_{ii} = i-1$,

$\forall i = 1, 2, \dots, n-1$.

(i) Let the n -th element m , be putted in (n, n) , where $0 \leq m \leq n-2$. We use the α , β and γ to change the partial square to $l_{11} = l_{22} = 0$ and $l_{ii} = i-2$, where $3 \leq i \leq n$.

Then put any element k in (i, j) , where $i \neq j$, $3 \leq i, j \leq n$, $k \neq i-2$ and $k \neq j-2$. ($n \geq 5$)

Then by Lars Anderson's Theorem, we can complete this Latin square.

If $m = n-1$, then use the same condition, we can complete this square.

0				
	1			
		2		
			$\ddots$	
				$n-2$
				m

↓

0				
0				
	1		k	
		2		
			$\ddots$	
				$n-2$

0				
	1		k	
		2		
			$\ddots$	
				$n-1$

$$(1) \quad \begin{array}{|c|c|c|} \hline 0 & & \\ \hline 0 & & \\ \hline & 1 & n-2 \\ & 2 & \\ & & \ddots \\ & & n-3 \\ \hline & & \\ \hline \end{array}$$

(2)

0 0 0		
	1 2 $\ddots$ $n-4$	

(3)		
1		
2		
...		
$n-4$		
	$n-3$	$n-2$
	$n-2$	$n-3$

By Lars Anderson's Theorem, we can complete this Latin square.

(3) There are two elements appear tow times, then we can use the α 、 β and γ to change the partial latin square to $l_{ii} = i - 1$, where $i = 1, 2, \dots, n - 3$ and

$$l_{n-2,n-2} = l_{n-1,n-1} = n-3, l_{n-2,n-1} = l_{n-1,n-2} = n-2$$

By Lars Anderson's Theorem, we can complete this Latin square.

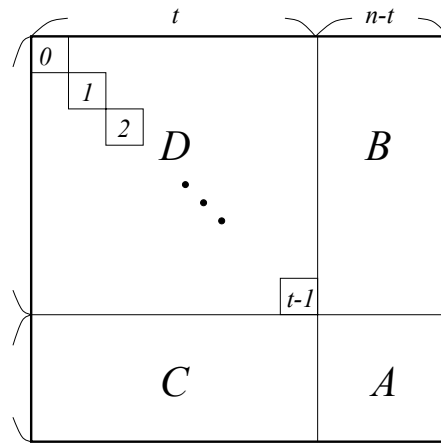
Combinatorial Designs

Chiu Yu-Chieh

10. Prove that there exists a partial transversal in a latin square of order n (given) which contains at least $\lfloor \frac{2}{3}n \rfloor$ distinct entries.

Proof. It's true for latin squares of order 1 and 2.

For $n \geq 3$, use the proof of Woolbright. After permutating the latin square, we can have a maximal transversal in a block D like below:



A just has elements from 0 to $t - 1$, or we can find a larger transversal. We can find $(n - t)$ t 's in B since A has not. Let $(n - t)$ t 's go to the diagonal of D first, and then go down to find $(n - t)$ columns in C . These $(n - t)$ columns just have elements from 0 to t , or we can find a larger transversal by deleting one diagonal elements of D and adding one element in B and C respectively.

Similarly, we can find $(n - t)$ $t + 1$'s in C , and we can find $(n - t)$ rows corresponding to $(n - t)$ $t + 1$'s.

Now, we have three blocks of order $(n - t)$. (2 in C , 1 in A)

$\therefore$ The rows of the latin square is n ,

$\therefore n \geq 3(n - t)$

$$\Rightarrow \frac{n}{3} \geq n - t \Rightarrow t \geq \frac{2}{3}n \rightarrow t \geq \lfloor \frac{2}{3}n \rfloor$$