

Introduction to Graph Ramsey Theory

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Preface

The past three decades have seen enormous development in graph theory, and one striking feature is the invention of many modern methods which involve ideas from various branches of mathematics such as probability, algebra, geometry, and analysis. Graph Ramsey theory is an important area that serves not only as an abundant source but also as a testing ground of these methods. Given the inconvenience to access sporadic results in an extensive literature, we set out to describe the material in this elementary book which aims to provide an introduction to graph Ramsey theory, with focus on methods. The mathematical prerequisites for this book are minimal: we only assumed familiarity with elementary calculus, linear algebra, and graph theory. To make this book as self-contained as possible, we attempted to develop the theory from scratch except the use of a few theorems in number theory (yet without proofs), for instance, some constructions heavily rely on the structural properties of finite fields, so we laid down the background beforehand.

Of course, no book could give a complete treatment of the subject indicated in their titles; simplicity is our first criterion for selecting topics. By so doing, we hope that, on the one hand, readers can easily extract the basic idea underlying each approach and quickly master the methods; on the other hand, teachers can completely finish one topic in a single lesson. (We are sorry for not being able to incorporate many deep results into this introductory book.) The selected topics are independent; beginners may skip the chapters, sections, and proofs marked with asterisks.

Despite substantial advance in graph Ramsey theory, most outstanding problems are far from being solved and beyond the power of traditional methods. The new insights generated by tackling these problems will most likely lead to new tools and techniques. It is due to the abundance and hardness of problems that graph Ramsey theory is full of vitality and hence deserves much more research efforts. We believe that this book, intended for beginning graduate students, can serve as an entrance to this beautiful theory. To facilitate better understanding of the material, this book contains some standard exercises.

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Chapter 1

Existence and Upper Bound

1.1 Terminology and Notation

“Ramsey theory” refers to the study of partitions of large structures. Typical results state that a certain substructure must occur in some classes of the partition. That is to say, complete disorder is impossible. The smallest size of the large structure is called *Ramsey number*.

Ramsey theory can be viewed as a generalization of the following naive principle, called Pigeonhole Principle:

Let k and n be positive integers. Assume that a set consisting of N objects is partitioned into n classes, then some class receives more than k objects if N is large enough.

Here the set of size N is the large structure. The smallest N is $kn + 1$. For $N \geq kn + 1$, the average size of the substructures is more than k , so there exists a class contains at least $k + 1$ objects. The argument uses the *averaging technique* that is an important tool in Ramsey theory.

A cornerstone method in mathematics is *mathematical induction*, which deduces a general statement for infinite many parameters from finite many cases. In contrast to induction, Ramsey theory does the job to obtain a general statement for all large parameters by excluding finite many exceptions.

Graph Ramsey theory is a fascinating branch of combinatorics. There are many difficult open problems in this area. Our lectures are devoted to the study of the behavior of various graph Ramsey functions. We use

Modern Graph Theory by Bollobás (1998) for terminology and notation which are not defined here. We assume that the readers have read some standard textbooks in graph theory, a few of which are listed at the end of this chapter.

Let $G = (V, E)$ be a graph with vertex set $V = V(G)$ and $E = E(G)$. The number of vertices of V is called the *order* of G , which is the size of V . The size of $E(G)$ is particularly called the *size* of G , denoted by $e(G)$. We always assume that G is simple, that is to say, it has neither multiple edges nor loops. For a vertex v of G , call the number of edges incident with v as its *degree*, denoted by $\deg(v)$, or $d(v)$, or even simply d_v . Note the degree of v is the number of its neighbors, which are vertices adjacent to v , since the graph is simple. If $W \subseteq V$, and $F \subseteq E$, set

$$E(W) = \{e \in E(G) : \text{both vertices of } e \text{ are in } W\}$$

and

$$V(F) = \{v \in V(G) : v \text{ is a vertex of an edge in } F\}.$$

Sometimes we may need subscript such as $E_G(W)$ to avoid confusion. A graph H is called a subgraph of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. The subgraph of G with vertex set W and edge set $E(W)$ is called the subgraph *induced* by W ; and that with vertex set $V(F)$ and edge set F is called the subgraph induced by F . A subgraph with vertex set V and any edge set is called a *spanning* subgraph.

If $W \subseteq V(G)$ and $E(W) = \emptyset$, we say that W is an *independent* set of G . The maximum cardinality of all independent sets of G is called *independence number* of G , denoted by $\alpha(G)$, which is one of the important parameters of graph.

In dealing with the problems of independence numbers, subgraphs induced by subsets of vertices are important.

Proposition 1.1 *Let $G = (V, E)$ be a graph and let H be a subgraph of G induced by a subset of V . Then $\alpha(G) \geq \alpha(H)$.*

As usual, we write a complete graph on n vertices as K_n and write a complete k -partite graph with classes of sizes $n_1, \dots, n_k$ as $K(n_1, \dots, n_k)$, and $K_k(n)$ simply if $n_i = n$ for $1 \leq i \leq k$. The complete bipartite graph $K(t, s)$ is also denoted by $K_{t,s}$. The graph $K_{1,s}$ is called a *star*. The path and cycle of length n are denoted by P_n and C_n , respectively. If for any pair of distinct vertices u and v of G there is a path in G with end-vertices u and v , then G is said to be *connected*. A connected graph with no cycle is called a *tree*. The *edge density* of graph G of order n is defined to be $|E(G)|/n$, which is half of the average degree.

Let G and H be graphs with disjoint vertex sets. We use $G \cup H$ to denote the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$, and it is called a *union* of G and H . We denote $\cup_{i=1}^k G$ by kG . A *forest* is a union of trees. We use $G + H$ to denote the graph with vertex set $V(G) \cup V(H)$ and edge set consisting of all edges of G and H as well as all edges joining G and H . Two graphs G and H are called *complementary* if they have the same vertex set V and their edges are a partition of complete graph on V . Denote by $\overline{G}$ for the complementary graph of G .

If we color the vertex set of G with some colors, a subset of $V(G)$ consisting of all vertices with the same color is called a *color class*. If in a coloring of $V(G)$ any color class induces an independent set, then we say that this coloring is a *proper vertex coloring*. Equivalently, any pair of adjacent vertices have distinct colors in a proper vertex coloring. The minimum number of the colors used over all proper vertex colorings of G is called *chromatic number* of G and denoted by $\chi(G)$, which is the minimum k such that $V(G)$ can be partitioned into k independent sets.

Let $G_1, \dots, G_k$ be graphs. The *Ramsey Number* $r(G_1, \dots, G_k)$ is defined to be the smallest positive integer N such that in any edge coloring of complete graph K_N with colors $\{1, \dots, k\}$, there exists some i , such that G_i is contained in the subgraph spanned by all edges in color i . This spanned subgraph may have isolated vertices which are those (if any) not incident to any edge in color i . Note that in definition of Ramsey numbers, there are no restraints on the way of coloring of edges. We write k -coloring Ramsey number $r(G, \dots, G)$ as $r_k(G)$. In particular, if all G_i are complete, we write $r(G_1, \dots, G_k)$ as $r(n_1, \dots, n_k)$, where n_i is the order of G_i , which is called *classical Ramsey number*.

Graham, Rothschild and Spencer in their book *Ramsey Theory* presented an exciting development of Ramsey theory. The number of research papers before 1970s is not substantial. The Combinatorial Conference at Balatonfüred, Hungary 1973, in honor of Paul Erdős for his 60th birthday, was a milestone in Ramsey theory history. There were more than two dozen talks devoted to what is now called *Ramsey theory*. Many papers have been published after this conference. About 40 papers which are classified in *generalized Ramsey theory* and indexed as 05C55 are reviewed in *Mathematical Reviews* of the American Mathematical Society each recent year. But most of them are concerned the related topics.

The case $k = 1$ is trivial and $r_1(G)$ is the order of G . The most studied case is $k = 2$ in which we often refer the two colors for edges as red and blue. For two colors, the Ramsey number $r(G, H)$ can be alternatively defined as the smallest positive integer N such that for any graph F of order N , either F contains G as a subgraph or $\overline{F}$ contains H as a subgraph, where $\overline{F}$ is the complementary graph of F . We write $r(G)$ for $r(G, G)$ and call

it the *diagonal* Ramsey number of G . So diagonal classic Ramsey number $r(n)$ is $r(n, n) = r(K_n, K_n)$. If either G or H is not complete, then $r(G, H)$ is called a *generalized Ramsey number*.

It is a simple fact that $r(K_1, G) = 1$ and $r(K_2, G) = |V(G)|$ for any graph G . To see the latter, let $n = |V(G)| \geq 2$, and color the all edges of K_{n-1} blue, then there is neither red K_2 nor blue G , so $r(K_2, G) \geq n$. For the inverse inequality, consider any coloring of edges of K_n , if there are red edges, we have a red K_2 , otherwise, all edges of K_n are blue, thus we have a blue K_n hence a blue G , so $r(K_2, G) \leq n$.

If G_1 is subgraph of G and H_1 is a subgraph of H , then it is easy to see that $r(G_1, H_1) \leq r(G, H)$. Call a graph *empty graph* if its edge set is empty. An empty graph of order m is the complementary graph of K_m . If G is not an empty graph, then $r(\overline{K}_m, G) = m$.

Proposition 1.2 *For any graphs G and H , $r(G, H) = r(H, G)$.*

Proof. Denote by N for $r(G, H)$. There is a coloring of edges of K_{N-1} with red and blue such that there is neither red G nor blue H . For each edge, switch its color to the other. In the new coloring, there is neither red H nor blue G , so $r(H, G) \geq N = r(G, H)$. Similarly $r(G, H) \geq r(H, G)$. $\square$

Generally, in multi-color Ramsey number $r(G_1, \dots, G_k)$, G_i and G_j are exchangeable, namely

$$r(\dots, G_i, \dots, G_j, \dots) = r(\dots, G_j, \dots, G_i, \dots).$$

It is trivial to see

$$r_k(\dots, G_{i-1}, K_2, G_{i+1}, \dots) = r_{k-1}(\dots, G_{i-1}, G_{i+1}, \dots).$$

In the two-coloring of the edge set of a complete graph with colors red and blue, we often denote the subgraph spanned by all red edges including all isolated vertices by $\langle R \rangle$, or simply R . Similarly, $\langle B \rangle$ or B is the subgraph spanned by all blue edges including isolated vertices. In a specific two-coloring of the edge set of G , for any vertex u , we thus can say that it has red degree $\deg_R(u)$, blue degree $\deg_B(u)$, red neighborhood $N_R(u)$ and blue neighborhood $N_B(u)$, respectively.

In order to verify that $r(G, H) \geq N + 1$ for some N , one must have an edge coloring of K_N in red and blue, such that there is neither a monochromatic red G nor a monochromatic blue H . Since the two monochromatic graphs, which contain all vertices, can be referred as F and $\overline{F}$, respectively, so verifying $r(G, H) \geq N + 1$ is the same to find a graph F of order N such that F does not contains G and $\overline{F}$ does not contain H . Such graph F of order $N = r(G, H) - 1$ is called a *Ramsey graph* for $r(G, H)$.

Note that in fact any graph F can yield a lower bound, good or bad, for a Ramsey number. For example, if $\omega(F) = m$, and $\alpha(F) = n$, then we have $r(m+1, n+1) \geq |V(F)| + 1$. However, it is very difficult to find a good lower bound for most Ramsey numbers.

1.2 Existence and Upper Bounds

In 1930 Frank Ramsey, in a fundamental paper on mathematical logic, gave a result whose special case can be stated in graph language as follows.

Theorem 1.1 (Ramsey, 1930) *Let $k \geq 2$ and let $n_1, \dots, n_k$ be positive integers. The Ramsey number $r(n_1, \dots, n_k)$ exists, and if $n_i \geq 2$ for all $i \leq k$, then*

$$r(n_1, \dots, n_k) \leq \sum_{i=1}^k r(\dots, n_{i-1}, n_i - 1, n_{i+1}, \dots) - k + 2.$$

Proof. The proof is by induction on $n_1 + \dots + n_k$. As we shall prove the inequalities, it will follow that $r(n_1, \dots, n_k)$ exists. Since $r(n_1, \dots, n_k) = 1$ if $n_i = 1$ for some $i \leq k$ and

$$r_k(\dots, n_{i-1}, 2, n_{i+1}, \dots) = r_{k-1}(\dots, n_{i-1}, n_{i+1}, \dots),$$

and $r(2, n) = r(n, 2) = n$, we may assume that

$$N_i = r(\dots, n_{i-1}, n_i - 1, n_{i+1}, \dots)$$

exists for any $i \leq k$, and $n_i \geq 2$. Let

$$N = \sum_{i=1}^k N_i - k + 2$$

and consider a coloring of the edges of K_N with colors $\{1, 2, \dots, k\}$. We have to show that in this coloring there is some i the subgraph spanned by all edge in color i contains K_{n_i} . To this end, let v be vertex of this colored K_N , and let $\deg_i(v)$ be the degree in color i . Clearly there is some i so that $\deg_i(v) \geq N_i$ since otherwise $\deg_i(v) \leq N_i - 1$ then

$$N - 1 = \sum_{i=1}^k \deg_i(v) \leq \sum_{i=1}^k (N_i - 1) = N - 2$$

leading to a contradiction. By symmetry we may assume that $i = 1$, thus vertex v has N_1 neighbors connected by edges in the first color. Consider the

complete graph K_{N_1} spanned by these N_1 vertices. If K_{N_1} has a complete K_{n_1-1} monochromatic in color 1, we are done since which forms a red K_{n_1} with vertex v . Otherwise, by the definition N_1 , the graph K_{N_1} contains a monochromatic K_{n_j} for some $j \geq 2$. $\square$

For a graph G of order at least two, denote by G' for a graph obtained by deleting one vertex from G . With a similar proof, we have a generalization of Theorem 1.1.

Theorem 1.2 *Let $k \geq 2$ be an integer and let $G_1, \dots, G_k$ be graphs. Then the Ramsey number $r(G_1, \dots, G_k)$ exists, and if each G_i has at least two vertices, then*

$$r(G_1, \dots, G_k) \leq \sum_{i=1}^k r(\dots, G_{i-1}, G'_i, G_{i+1}, \dots) - k + 2.$$

The popularization of Ramsey's Theorem began with its rediscovery in a classic paper of Paul Erdős and George Szekeres in 1935.

Theorem 1.3 (Erdős-Szekeres, 1935) *For any integers $m \geq 2$ and $n \geq 2$, Ramsey number $r(m, n)$ exists, and it satisfies*

$$r(m, n) \leq r(m-1, n) + r(m, n-1)$$

and

$$r(m, n) \leq \binom{m+n-2}{m-1}.$$

Proof. The first inequality is a special case of Theorem 1.1. Then by the first inequality we have

$$\begin{aligned} r(m, n) &\leq r(m-1, n) + r(m, n-1) \\ &\leq \binom{m+n-3}{m-2} + \binom{m+n-3}{m-1} \\ &= \binom{m+n-2}{m-1}. \end{aligned}$$

$\square$

The first inequality in the following theorem is a special case of Theorem 1.1. We shall prove the theorem with emphasis on the condition for the strict inequality, which is needed for exact values of small Ramsey numbers.

Theorem 1.4 *Let G and H be graphs with at least two vertices. Then*

$$r(G, H) \leq r(G', H) + r(G, H').$$

If both $r(G', H)$ and $r(G, H')$ are even, then

$$r(G, H) \leq r(G', H) + r(G, H') - 1.$$

Proof. Let $N = r(G', H) + r(G, H')$. In any red-blue coloring of edges of K_N , a vertex v has either red degree at least $r(G', H)$ or blue degree at least $r(G, H')$. Suppose $\deg_R(v) \geq r(G', H)$, then $N_R(v)$ must contain a red G' or a blue H . Together with v , this either yields a red G or a blue H . The case $\deg_B(v) \geq r(G, H')$ is similar, thus the first inequality follows. For the second inequality, suppose to the contrary, $r(G, H) = N = r(G', H) + r(G, H')$, which is even, then there is a coloring of edges of K_{N-1} , such that there is neither red G nor blue H . For any vertex v , we have $\deg_R(v) \leq r(G', H) - 1$ and $\deg_B(v) \leq r(G, H') - 1$. However

$$\deg_R(v) + \deg_B(v) = N - 2 = (r(G', H) - 1) + (r(G, H') - 1),$$

hence

$$\deg_R(v) = r(G', H) - 1 \quad \text{and} \quad \deg_B(v) = r(G, H') - 1$$

for any vertex v . Consider the number of edges of R , which is

$$e(R) = \frac{1}{2}(N-1)(r(G', H) - 1).$$

But the right hand side is not an integer since both $N - 1$ and $r(G', H) - 1$ are odd, yielding a contradiction. $\square$

1.3 A Two-Step Recursive Upper Bound

The following is a two-step recursive upper bound, due to Li, Rousseau and Zang (2004), in which G'' is a graph obtained by deleting two vertices from G .

Theorem 1.5 *Let G and H be graphs of order at least three, and let $A = r(G'', H)$ and $B = r(G, H'')$. Then*

$$r(G, H) \leq A + B + 2 + 2\sqrt{(A^2 + AB + B^2)/3}.$$

In particular, $r(G, G) \leq 4r(G, G'') + 2$.

Proof. Let $N = r(G, H) - 1$. Then there is a coloring of the edges of K_N in colors red and blue in which there is neither a red G nor a blue H . Let (E_r, E_b) denote the corresponding partition of the edge set of K_N . Denote by V the vertex set of the colored complete graph K_N , and let $F_r = (V, E_r)$ and $F_b = (V, E_b)$ denote the red and blue subgraphs, respectively. The number of monochromatic triangles in the two-colored K_N is

$$M = \frac{1}{3} \left(\sum_{uv \in E_r} |N_r(u) \cap N_r(v)| + \sum_{uv \in E_b} |N_b(u) \cap N_b(v)| \right),$$

where $N_r(u)$ and $N_b(u)$ are neighborhoods of u in F_r and F_b , respectively. Since an edge $uv \in E_r$ and a red copy of G'' in $N_r(u) \cap N_r(v)$ yield a red $K_2 + G''$ hence a red G , we have $|N_r(u) \cap N_r(v)| \leq r(G'', H) - 1 = A - 1$. Similarly, $|N_b(u) \cap N_b(v)| \leq r(G, H'') - 1 = B - 1$ for an edge $uv \in E_b$. Therefore we have the following upper bound for M :

$$M \leq \frac{(A - 1)|E_r| + (B - 1)|E_b|}{3}.$$

As observed by Goodman (1959), since the number of non-monochromatic triangles is $\frac{1}{2} \sum_{v \in V} d_r(v)d_b(v)$, so

$$M = \binom{N}{3} - \frac{1}{2} \sum_{v \in V} d_r(v)(N - 1 - d_r(v)),$$

where $d_r(v) = |N_r(v)|$. Suppose $d_1, d_2, \dots, d_N$ are any real numbers satisfying $\sum_{i=1}^N d_i = s$. Then Cauchy's inequality,

$$\left(\sum_{i=1}^N x_i y_i \right)^2 \leq \left(\sum_{i=1}^N x_i^2 \right) \left(\sum_{i=1}^N y_i^2 \right)$$

by taking $y_i = 1$, gives $N \sum_{i=1}^N d_i^2 \geq (\sum_{i=1}^N d_i)^2 = s^2$, with equality if and only if $d_1 = d_2 = \dots = d_N = s/N$. Hence

$$\sum_{i=1}^N d_i(N - 1 - d_i) = (N - 1)s - \sum_{i=1}^N d_i^2 \leq (N - 1)s - \frac{s^2}{N}.$$

Since $\sum_{v \in V} d_r(v) = 2|E_r|$, application of the above inequality gives the following lower bound for M :

$$\begin{aligned} M &\geq \binom{N}{3} - |E_r| \left(N - 1 - \frac{2|E_r|}{N} \right) \\ &= \frac{N(N - 1)(N - 5)}{24} + \frac{2}{N} \left(|E_r| - \frac{1}{2} \binom{N}{2} \right)^2. \end{aligned}$$

This inequality is strict unless $\frac{2|E_r|}{N}$ is an integer and $d_r(v) = \frac{2|E_r|}{N}$ for each $v \in V$. Set

$$|E_r| = \frac{1}{2} \binom{N}{2} + x \quad \text{and} \quad |E_b| = \frac{1}{2} \binom{N}{2} - x,$$

where $|x| \leq \frac{1}{2} \binom{N}{2}$. Then the two bounds for M give

$$\begin{aligned} \frac{N(N-1)(N-5)}{24} + \frac{2x^2}{N} &\leq \frac{(A-1)|E_r| + (B-1)|E_b|}{3} \\ &= \frac{N(N-1)(A+B-2)}{12} + \frac{(A-B)x}{3}. \end{aligned}$$

Equivalently,

$$\frac{N(N-1)[N - (2A + 2B + 1)]}{24} \leq \frac{(A-B)x}{3} - \frac{2x^2}{N}.$$

Maximizing the right-hand side on variable $x \in (-\infty, \infty)$ without considering the constraint on x , the maximum value is taken at $x = \frac{(A-B)N}{12}$ and we have

$$(N-1)[N - (2A + 2B + 1)] \leq \frac{(A-B)^2}{3},$$

which gives $N-1 \leq A+B+2\sqrt{(A^2+AB+B^2)/3}$ as required. $\square$

The following is a result often used in estimating small Ramsey numbers.

Corollary 1.1 (Walker, 1968) *Let $n \geq 3$ be an integer, then*

$$r(n, n) \leq 4r(n-2, n) + 2.$$

1.4 Hypergraph and Infinite Version ★

We shall conclude this chapter with hypergraph and infinite versions of Ramsey theorem.

A *hypergraph* $\mathcal{G}$ on vertex set V is a pair $(V, \mathcal{E})$, where the edge set $\mathcal{E}$ is a family of subsets of V . Let $V^{(r)}$ be the family of all r -subsets of V . If $\mathcal{E} \subseteq V^{(r)}$, then $\mathcal{G}$ is called r -uniform. Thus a 2-uniform hypergraph is just a graph. When $\mathcal{E} = V^{(r)}$, we say that $\mathcal{G}$ is complete, denoted by $K_n^{(r)}$, where $n = |V|$. If all elements in $V^{(r)}$ are colored in k colors, a subset $X \subseteq V$ is said to be monochromatic when any element in $X^{(r)}$, not element

of X itself, has the same color. That is to say, the subset X induces a monochromatic complete sub-hypergraph of $\mathcal{G}$.

Define $r_k^{(r)}(n_1, \dots, n_k)$ to be the minimum integer N such that every coloring of $V^{(r)}$ with $|V| = N$ in colors $\{1, \dots, k\}$, then for some i , there exists $X \subseteq V$ with $|X| = n_i$ that induces a monochromatic $K_{n_i}^{(r)}$ in color i . Write $r_k^{(r)}(n)$ for $r_k^{(r)}(n, \dots, n)$, $r^{(r)}(m, n)$ for $r_2^{(r)}(m, n)$ and $r_k(n_1, \dots, n_k)$ for $r_k^{(2)}(n_1, \dots, n_k)$. Note that $r^{(1)}(m, n) = m + n - 1$ and for $r \geq 2$,

$$r^{(r)}(m, n) = \begin{cases} \max\{m, n\} & \text{if } r = \min\{m, n\}, \\ \min\{m, n\} & \text{if } r > \min\{m, n\}. \end{cases}$$

Theorem 1.6 *Let $r, m, n \geq 2$ be integers. Then $r^{(r)}(m, n)$ exists that satisfies*

$$r^{(r)}(m, n) \leq r^{(r-1)}\left(r^{(r)}(m-1, n), r^{(r)}(m, n-1)\right) + 1.$$

Proof. From the fact that $r^{(1)}(m, n) = m + n - 1$ and the upper bound that $r^{(2)}(s, t) \leq r^{(2)}(s-1, t) + r^{(2)}(s, t-1)$ in the section 2, we see that the assertion holds for $r = 2$. For $r > 2$, we assume that $r^{(r-1)}(s, t)$ exists.

Let V be a set with $N = r^{(r-1)}\left(r^{(r)}(m-1, n), r^{(r)}(m, n-1)\right) + 1$ vertices. For a given coloring $c : V^{(r)} \rightarrow \{\text{red}, \text{blue}\}$, a vertex $v \in V$ and $X = V \setminus \{v\}$, define a coloring

$$\bar{c} : X^{(r-1)} \rightarrow \{\text{red}, \text{blue}\}, \quad \bar{c}(S) = c(S \cup \{v\})$$

for $S \in X^{(r-1)}$ as $S \cup \{v\} \in X^{(r)}$. From the definition of $r^{(r-1)}(s, t)$ and

$$|X| = r^{(r-1)}\left(r^{(r)}(m-1, n), r^{(r)}(m, n-1)\right),$$

we see that either X contains a subset Y with $|Y| = r^{(r)}(m-1, n)$ such that $Y^{(r-1)}$ is completely red in $\bar{c}$, or X contains a subset Z with $|Z| = r^{(r)}(n, m-1)$ such that $Z^{(r-1)}$ is completely blue in $\bar{c}$. Without loss of generality, we assume that the former is the case. We shall show that Y contains either a subset A with $|A| = m$ such that $A^{(r)}$ is completely red in c or subset B with $|B| = n$ such that $B^{(r)}$ is completely blue in c .

We now have $Y \subseteq X = V \setminus \{v\}$ with $|Y| = r^{(r)}(m-1, n)$. Consider the restriction of the coloring c of $V^{(r)}$ on $Y^{(r)}$. From the definition, we know that either Y contains a subset B with $|B| = n$ such that $B^{(r)}$ is completely blue in c , we thus are done, or Y contains a subset A_0 with $|A_0| = m-1$ such that $A_0^{(r)}$ is completely red in c . Set $A = A_0 \cup \{v\}$, then $|A| = m$ since $v \notin A$. For any $T \in A^{(r)}$, if $v \notin T$, then $T \in A_0^{(r)}$ and hence $c(T)$ is red. Otherwise, $S = T \setminus \{v\} \in A_0^{(r-1)} \subseteq Y^{(r-1)}$, so $\bar{c}(S)$ is red since $Y^{(r-1)}$

is completely red in $\bar{c}$. The definition of $\bar{c}$ implies that $c(T)$ is red since $c(T) = \bar{c}(S)$, thus $A^{(r)}$ is completely red in c . $\square$

The following is an infinite version of Ramsey theorem.

Theorem 1.7 *Let V be an infinite set and let r and k be fixed positive integers. If all elements in $V^{(r)}$ are colored in k colors, then V has an infinite subset X such that all elements of $X^{(r)}$ have the same color.*

Proof. The assertion is trivial for $r = 1$. We thus assume that $r \geq 2$ and the assertion holds for smaller values of r .

Consider any coloring $c : V^{(r)} \rightarrow \{1, 2, \dots, k\}$. Let $V_0 = V$ and pick any $v_1 \in V_0$, and then set $U_1 = V_0 \setminus \{v_1\}$. Define a coloring

$$c_1 : U_1^{(r-1)} \rightarrow \{1, 2, \dots, k\}, \quad c_1(S) = c(S \cup \{v_1\})$$

for any $S \in U_1^{(r-1)}$. By the induction hypothesis there exists an infinite subset $V_1 \subseteq U_1$ such that all elements of $V_1^{(r-1)}$ have the same color in c_1 , say k_1 . Hence any $T \in (V_1 \cup \{v_1\})^{(r)}$ with $v_1 \in T$ has color k_1 in c since $T \setminus \{v_1\} \in V_1^{(r-1)}$ has color k_1 in c_1 and $c(T) = c_1(T \setminus \{v_1\})$. Pick any $v_2 \in V_1$ and set $U_2 = V_1 \setminus \{v_2\}$. Define a coloring

$$c_2 : U_2^{(r-1)} \rightarrow \{1, 2, \dots, k\}, \quad c_2(S) = c(S \cup \{v_2\})$$

for any $S \in U_2^{(r-1)}$. Then there exists an infinite subset $V_2 \subseteq U_2$ such that all elements of $V_2^{(r-1)}$ have the same color in c_2 , say k_2 . Hence any $T \in (V_2 \cup \{v_2\})^{(r)}$ with $v_2 \in T$ has color k_2 in c . Continuing the process we have a sequence of infinite nested sets: $V_0 \supseteq V_1 \supseteq V_2 \supseteq \dots$, and an infinite sequence of distinct vertices $v_1, v_2, v_3, \dots$ with $v_i \in V_{i-1}$ and $v_i \notin V_i$, and a sequence of colors $k_1, k_2, \dots$ such that any $T \in (V_i \cup \{v_i\})^{(r)}$ with $v_i \in T$ has color k_i in c .

Since the infinite sequence of $k_1, k_2, \dots$ takes only k different values, there exists a subsequence $k_{n_1}, k_{n_2}, \dots$, taking a constant, say k' . Set $X = \{v_{n_1}, v_{n_2}, \dots\}$, which is an infinite subset of V . For any $T \in X^{(r)}$ with $T = \{v_{m_1}, \dots, v_{m_r}\}$, $m_i \in \{n_1, n_2, \dots\}$ and $m_1 < m_2 < \dots < m_r$, we have $v_{m_2}, \dots, v_{m_r} \in V_{n_1}$, hence $T \in (V_{n_1} \cup \{v_{n_1}\})^{(r)}$ with $v_{n_1} \in T$ thus $c(T) = k_{m_1} = k'$. Therefore all elements of $X^{(r)}$ have color k' in coloring c , completing the proof. $\square$

The hypergraph version of Ramsey theory can be deduced from its infinite version.

Lemma 1.1 *Let $V_1, V_2, \dots$ be an infinite sequence of pairwise disjoint sets with $0 < |V_i| < \infty$, and let G be the graph on $V = \cup_{i \geq 1} V_i$. Suppose that each vertex in V_i has at least one neighbor in V_{i-1} . Then G contains an infinite path $v_1 v_2 \dots$ with $v_i \in V_i$ for all $i \geq 1$.*

Proof. For any vertex $v_n \in V_n$, let $f(v_n)$ be one of neighbors of v_n . There is a path of form as $v_n v_{n-1} \dots v_1$ with $v_i \in V_i$ and $v_{i-1} = f(v_i)$. Let $\mathcal{P}$ be the set of all such paths, then $\mathcal{P}$ contains infinite paths of finite lengths. Since V_1 is a finite set, there is a vertex $v_1 \in V_1$ that is the end of infinite paths. Let $\mathcal{P}_1 \subseteq \mathcal{P}$ be the set of all such paths. Among all vertices of V_2 that in some path of $\mathcal{P}_1$, there is v_2 that is contained in infinite many paths of $\mathcal{P}_1$ since V_2 is finite and $\mathcal{P}_1$ is infinite. Let $\mathcal{P}_2 \subseteq \mathcal{P}_1$ be the set of paths that contains vertices v_1 and v_2 , then $\mathcal{P}_2$ contains infinite many paths. Continue the process, we find $v_1, v_2, \dots$ with $v_n \in V_n$ for all $n \geq 1$ such that they form a path of G . $\square$

The least integer N in the following theorem is $r_k^{(r)}(n)$, which has been introduced earlier, where r is the uniformity of the hypergraph and k is number of colors. As usual, we write $[N]$ for the set $\{1, 2, \dots, N\}$ and $\mathcal{N}$ for $\{1, 2, \dots\}$.

Theorem 1.8 *Let r and k be given positive integers. Then for each $n \geq 1$ there is an integer $N \geq r$ such that if all elements of $[N]^{(r)}$ are colored in k colors, then there is $S \subseteq [N]$ with $|S| = n$ such that all elements of $S^{(r)}$ are in the same color.*

Proof. Note that if the assertion does not hold for some n , then it does not hold for any large values of n . So we may assume that for some fixed n and for any $N \geq r$, there is a “bad” coloring $c : [N]^{(r)} \rightarrow [k]$ such that if $S \subseteq [N]$ with $|S| = n$, the $S^{(r)}$ is not monochromatic in c . Denote by V_N for the set of all such bad colorings on $[N]^{(r)}$, then $0 < |V_N| < \infty$. For $N > r$ and $c \in V_N$, the restriction of c on $[N-1]^{(r)}$ is an element of V_{N-1} since it is bad. From Lemma 1.1, there is an infinite sequence $c_r, c_{r+1}, \dots$ of bad colorings with $c_N \in V_N$ that is the restriction of c_{N+1} on $[N]^{(r)}$. For all $N \geq m \geq r$, any coloring c_N agrees on $[m]^{(r)}$. So if $X \in \mathcal{N}^{(r)}$, then $c_N(X)$ coincides for all $N > \max X$. We define the this common value $c_N(X)$ as $c(X)$ hence define a coloring $c : \mathcal{N}^{(r)} \rightarrow [k]$. Then c is a bad coloring on $\mathcal{N}^{(r)}$. In fact, for any $Y \subseteq \mathcal{N}$ with $|Y| = n$, then $Y \subseteq [N]$ for some large $N \geq r$. So all elements of $Y^{(r)}$ cannot have the same color since c coincides with the bad coloring c_N on $[N]^{(r)}$. We thus reach a contradiction and finish the proof. $\square$

Exercises

1.1 Prove Theorem 1.2.

1.2 Prove

$$r(K_m + \overline{K}_n, K_p + \overline{K}_q) \leq \binom{m+p-1}{m} n + \binom{m+p-1}{p} q.$$

(A. Thomason, 1982.)

1.3 Let G be a graph of order of N . Prove that the number of triangles contained in G and $\overline{G}$ is

$$\binom{N}{3} - \frac{1}{2} \sum_{v \in V(G)} d(v)(N-1-d(v)).$$

(R. Goodman, 1959.)

1.4 Let $k \geq 2$. Prove that

$$r(n_1 + 1, \dots, n_k + 1) \leq \frac{(n_1 + \dots + n_k)!}{n_1! \dots n_k!}.$$

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Chapter 2

Small Ramsey Numbers

2.1 Ramsey Folklore

What are exact values of $r(m, n)$? To answer this is more challenging than to prove the existence of them. As mentioned in Chapter 1, if $N = r(G, H)$, there exists a graph F of order $N - 1$ that contains no subgraph G and $\overline{F}$ contains no H , F is then called *Ramsey graph* or *extremal graph* for $r(G, H)$. To illustrate the idea, we shall find the first four nontrivial classical Ramsey numbers, which are folklore in graph Ramsey theory.

Theorem 2.1 *Exact values of four Ramsey numbers are as follows.*

$$r(3, 3) = 6, \quad r(3, 4) = 9, \quad r(3, 5) = 14, \quad r(4, 4) = 18.$$

Proof. By the upper bound in Theorem 1.3 in Chapter 1, we have $r(3, 3) \leq \binom{4}{2} = 6$. More direct proof is easy. Let F be a graph of order 6 and $v \in V(F)$. Then v has at least three neighbors in F or in $\overline{F}$, say, three neighbors $S = \{v_1, v_2, v_3\}$ in F . If any two vertices in S are adjacent, then these two vertices and v form a triangle. Otherwise, S induces an triangle in $\overline{F}$. Thus $r(3, 3) \leq 6$. To show $r(3, 3) > 5$, just note $\overline{C_5} = C_5$, which contains no K_3 . So C_5 is an extremal graph of $r(3, 3)$.

Since both $r(3, 3) = 6$ and $r(2, 4) = 4$ are even, by Theorem 1.4 in Chapter 1, we have $r(3, 4) \leq 9$. Also,

$$r(3, 5) \leq r(2, 5) + r(3, 4) \leq 14. \quad (2.1)$$

To obtain a lower bound of $r(3, 5)$, let us consider the graph G on vertex set $Z_{13} = \{0, 1, 2, \dots, 12\}$ (the integers modulo 13). Denote by $A = \{1, 5, 8, 12\} = \{\pm 1, \pm 5\}$, which are non-zero cubic (mod 13) of Z_{13} .

Connect vertices i and j by an edge if and only if $|i - j| \in A$. It is easy to check that G is triangle-free and $\alpha(G) = 4$. Thus $r(3, 5) \geq 14$ hence $r(3, 5) = 14$. Furthermore the equality holds in (2.1) so $r(3, 4) = 9$.

From $r(3, 4) = 9$, we have $r(4, 4) \leq 2r(3, 4) = 18$. To get a lower bound, let us consider the graph with vertex set $Z_{17} = \{0, 1, \dots, 16\}$, in which i and j are adjacent if and only if $i - j \in \{\pm 1, \pm 2, \pm 4, \pm 8\}$, the set of non-zero quadratic (mod 17). This graph shows that $r(4, 4) \geq 18$. We will see the lower bound again by Paley graph later. $\square$

Greenwood and Gleason (1955) computed the four exact values as in above theorem. They also found the right values of $r(3, 3, 3) = 17$, whose right upper bounds can be obtained by Theorem 1.1 in Chapter 1. The lower bound of $r(3, 3, 3)$ will be in the next section, which is the only known exact value among all nontrivial classical Ramsey number of three or more colors. Graver and Yackel (1968) found that of $r(3, 6) = 18$ and $r(3, 7) = 23$. No any other classic Ramsey number is found without aid of computers. Using the computers, Grinstead and Roberts (1982) found that $r(3, 8)$ is 28 or 29 addition to the exact value of $r(3, 9) = 36$; McKay and Zhang (1992) finally determined $r(3, 8) = 28$. MaKay and Radziszowski (1995) computed $r(4, 5) = 25$.

For fixed integer $k \geq 2$ and graphs G_i with small orders, the exact value of $r(G_1, \dots, G_k)$ is called *small Ramsey numbers*. Radziszowski's dynamic survey (1994) offers up-to-date information on small Ramsey numbers. All known non-trivial classical Ramsey numbers $r(m, n)$ and some bounds at present are listed in the following table.

$m \backslash n$	3	4	5	6	7	8	9	10
3	6	9	14	18	23	28	36	40/43
4		18	25	35/41	49/61	56/84	69/115	92/149
5			43/49	58/87	80/143	101/216	121/316	141-442
6				102/165	111/298	127/495	169/780	178/1171
7					205/540	216/1031	232/1713	-/2826
8						282/1870	317/3583	-/6090
9							565/6588	580/12677
10								798/23556

Table 1 Some values and bounds of $r(m, n)$.

The paucity of known exact values $r(m, n)$ indicates the difficulty in this area. Also this paucity stimulates our curiosity to find more classic Ramsey numbers. The following story came from *Ten lectures on the probabilistic method* by Spencer (1994).

Erdős asks us to imagine an alien force, vastly more powerful than us, landing the earth and demanding the value of $r(5,5)$ or they will destroy our planet. In this case, he claims, we should marshall all our computers and all our mathematicians and attempt to find the value. But suppose, instead, that they ask for $r(6,6)$. In that case, we should attempt to destroy the aliens."

Perhaps as the improvement of computers and algorithms, the value of $r(5,5)$ will be obtained in the near future. But for a bit larger n , the exact value of $r(n,n)$ is still far away from tractable. However, more "exact" results are known on generalized Ramsey numbers. We will discuss some of them later in this chapter.

2.2 Finite Field and $r_3(3)$

In this section, we introduce a construction by a finite field. About finite fields, one of basic facts is that there exists a q element field $F(q)$ if and only if q is a prime power. A finite field $F(q)$ is often called Galois field so denoted by $GF(q)$. For the simplicity, we use the notation $F(q)$ or F_q . When $q = p$ is a prime, the addition and multiplication in $F(p)$ are more pleasing since they are arithmetic mod p . A bit later, we will see the operations in $F(p^m)$ with $m \geq 2$ are different (only in appearance essentially). Set $F^*(q) = F(q) \setminus \{0\}$, then $F^*(q)$ is closed for multiplication. In fact it is a multiplicative group. However, If $q = p^m$ is a power of prime p and if $x \in F(q)$, then $px = 0$, where px is the sum of x of p times. Any element of $F^*(q)$ satisfies that the equation $x^{q-1} = 1 \pmod{q}$, hence any element of $F(q)$ satisfies the equation $x^q = x \pmod{q}$.

Let us look a more specific case. The following Fermat theorem is well known in elementary number theory.

Theorem 2.2 (Fermat Theorem) *If prime p and positive integer a satisfy $(a, p) = 1$, then $a^{p-1} = 1 \pmod{p}$.*

Set $Z_p = \{0, 1, \dots, p-1\}$ and $Z_p^* = \{1, 2, \dots, p-1\}$. So the Fermat theorem ensures that any $x \in Z_p^*$ satisfies $x^{p-1} = 1 \pmod{p}$. Here Z_p is a field of order p , and it is the unique field of order p up to isomorphism.

Using finite field, Greenwood and Gleason in 1955 constructed a graph giving the right lower bound of $r_3(3)$, in which more essential properties of finite field $F(p^m)$ with $m \geq 2$ are involved. This is the only graph producing the right lower bound for $r_k(n)$ with $k \geq 3$ and $n \geq 3$. Let us

begin with some basic properties involved in the construction. Of course, we shall constrain ourselves with what are needed here.

Let F be a (finite or infinite) field. A polynomial

$$f(x) = a_0x^m + a_1x^{m-1} + \cdots + a_m$$

is called a polynomial on F if any coefficient $a_i \in F$. Denote by $F[x]$ for the set of all polynomials on F . Then $F[x]$ is a ring on ordinary addition and multiplication of polynomials. The polynomial $f(x)$ on a field F is called *primitive* if it is irreducible and $a_0 = 1$. We now define an equivalence relation $\equiv$ on $F[x]$ for with $g_1(x) \equiv g_2(x)$ if and only if $f(x)|(g_1(x) - g_2(x))$, that is to say, the remainders of $g_1(x)$ and $g_2(x)$ are the same when they are divided by $f(x)$. Let

$$g_i(x) = f(x)h_i(x) + r_i(x),$$

where $h_i(x), r_i(x) \in F[x]$ with the degree of $r_i(x)$ is less m . Denote by $\langle g(x) \rangle$ for the equivalence class that contains $g(x)$, then $\langle g_i(x) \rangle = \langle r_i(x) \rangle$, and $g_1(x) \equiv g_2(x)$ if and only if $r_1(x) = r_2(x)$. Let $F[x]/(f(x))$ be the set of all equivalence classes. Operating addition and multiplication on $F[x]/(f(x))$ in an obvious way, we have a field called *quotient field*.

By identifying any element $a \in F$ as the element $\langle a \rangle \in F[x]/(f(x))$, we see that $F[x]/(f(x))$ is an extension field of F , and any polynomial $g(x)$ on F is a polynomial on $F[x]/(f(x))$. In the extension field, the polynomial $f(x)$ is reducible. In fact, the element $\langle x \rangle$ is a root of $f(x)$ in $F[x]/(f(x))$. Denote the root by β , then $F[x]/(f(x))$ is a linear space with basis $\{1, \beta, \dots, \beta^{m-1}\}$, where m is the degree of $f(x)$.

To make this construction clearer, let us look the complex field in an algebraically constructive way. The polynomial $f(x) = x^2 + 1$ is primitive on real number field R . Denote the root of $f(x) = x^2 + 1$ by i , then $i^2 = -1$ and any complex number can be expressed as $a + bi$ with $a, b \in R$, where $a + bi$ is a signification of the element of $\langle a + bx \rangle$ of the quotient field $R[x]/(x^2 + 1)$. Here $\{1, i\}$ forms a basis of complex field over real field R .

Recall that for a prime p , Z_p is a field in which the addition and multiplication are operated on $(\text{mod } p)$. From the previous discussion, we know that a finite field $GF(p^m)$ of order p^m can be isomorphic with $Z_p[x]/(f(x))$, where $Z_p[x]$ is the polynomial ring on Z_p and $f(x)$ is a primitive polynomial on Z_p of degree m . Note that the number of polynomials with degree less than m is p^m , which is exactly the order of $Z_p[x]/(f(x))$ by the mentioned reason.

Let β be a root of $f(x)$, then the elements of $GF(p^m)$ can be expressed as

$$0, \beta, \beta^2, \dots, \beta^{q-1} = 1,$$

where $q = p^m$. The above expression is only convenient for multiplication as $\beta^i \beta^j = \beta^{i+j}$. However, the operations are on $(\text{mod } f(x))$. For example, if $x^k = h(x)f(x) + r(x)$, where $h(x), r(x) \in Z_p[x]$ with the degree of $r(x)$ is at most $m-1$, then $\beta^k = r(\beta)$. The uniqueness is the advantage of the expression $r(\beta)$ for β^k .

We now consider $F = GF(2^4)$, which is isomorphic to $Z_2[x]/(x^4 + x + 1)$ since $x^4 + x + 1$ is a primitive polynomial in $Z_2[x]$. Let β be a root of $x^4 + x + 1$, then the elements of F may be identified as

$$\{0, \beta, \beta^2, \dots, \beta^{15} = 1\},$$

in which $1, \beta, \beta^2, \beta^3$ form a basis of F as a linear space on Z_2 .

$0 = 0$	$\beta^8 = 1 + \beta^2$
$\beta = \beta$	$\beta^9 = \beta + \beta^3$
$\beta^2 = \beta^2$	$\beta^{10} = 1 + \beta + \beta^2$
$\beta^3 = \beta^3$	$\beta^{11} = \beta + \beta^2 + \beta^3$
$\beta^4 = 1 + \beta$	$\beta^{12} = 1 + \beta + \beta^2 + \beta^3$
$\beta^5 = \beta + \beta^2$	$\beta^{13} = 1 + \beta^2 + \beta^3$
$\beta^6 = \beta^2 + \beta^3$	$\beta^{14} = 1 + \beta^3$
$\beta^7 = 1 + \beta + \beta^3$	$\beta^{15} = 1$

Table 2 The elements in a field $GF(16)$

Note that $-f(\beta) = f(\beta)$ since $-f(x) = f(x)$ in $Z_2[x]$. Let A_0 be the set of all cubic residues of $F^* = F \setminus \{0\}$, then

$$\begin{aligned} A_0 &= \{\beta^3, \beta^6, \beta^9, \beta^{12}, \beta^{15}\} \\ &= \{\beta^3, \beta^2 + \beta^3, \beta + \beta^3, 1 + \beta + \beta^2 + \beta^3, 1\}. \end{aligned}$$

It is easy to see that A_0 is a subgroup of multiplicative group F^* . Set

$$A_1 = \beta A_0 = \{1 + \beta, 1 + \beta + \beta^3, 1 + \beta + \beta^2, 1 + \beta^2 + \beta^3, \beta\}$$

and

$$A_2 = \beta^2 A_0 = \{\beta + \beta^2, 1 + \beta^2, \beta + \beta^2 + \beta^3, 1 + \beta^3, \beta^2\}.$$

The sets A_0, A_1, A_2 are cosets of A_0 in multiplicative group F^* . A set $S \subseteq F^*$ is called *sum-free* if for any $s, t \in S$, $s + t \notin S$. Observe that A_0 is sum-free from the above table hence A_2 and A_3 are sum-free.

We now construct a complete graph on vertex set formed by elements of F . Color the edge connecting vertices u and v with color i if $u - v \in A_i$. Since $u - v = u + v$ in $F = GF(2^4)$, the order of difference is immaterial. Also, $u - v \neq 0$ for distinct u and v , so all edges are colored in three colors.

Lemma 2.1 *There is no monochromatic triangle in the above construction.*

Proof. For any three distinct vertices u, v, w , assume that the edges uv and uw are colored in i , then $v - u \in A_i$, $w - u \in A_i$. Thus

$$w - v = (w - u) + (u - v) \notin A_i$$

since A_i is sum-free. $\square$

Theorem 2.3 $r_3(3) = 17$.

Proof. The graph constructed above shows that $r_3(3) \geq 17$. The right upper bound has been mentioned in Chapter 1 by

$$r(3, 3, 3) \leq 3r(3, 3) - 3 + 2 = 17.$$

A direct proof is easy. For any edge coloring of K_{17} , a vertex v is adjacent to at least 6 vertices in one color among 16 other vertices. If there is an edge among these 6 vertices is colored i , then we have a monochromatic triangle in color i , otherwise all these edges must be in other two colors, which yields also a monochromatic triangle since $r(3, 3) = 6$. $\square$

This construction also gives $r(3, 6) \geq 17$ if we blend two of the colors, which is close to the right value $r(3, 6) = 18$. The above construction can be generalized as follows.

Let $F = GF(q)$ be a Galois field of order q . If the multiplicative group $F^* = F \setminus \{0\}$ has a sum-free subgroup S that contains -1 , then $r_k(3) > q$, where $k = (q - 1)/|S|$.

The subgroup S consisting of all k th power residues contains -1 , where k is odd. Finding such S that is sum-free with many elements as possible seems to be reasonable.

2.3 Paley Graphs

We have seen some recursive bounds in last chapter. However they are not effective on estimating larger classic Ramsey numbers. For the lower bounds, the situation is even worse.

Property 2.1 *Let m and n_i be positive integers. Then*

$$r(m, n_1 + n_2 - 1) \geq r(m, n_1) + r(m, n_2) - 1.$$

Proof. The proof is trivial. Let G_i be a graph of order $r(m, n_i) - 1$ that contain no K_m and $\alpha(G_i) \leq n_i - 1$. Then the graph $G = G_1 \cup G_2$ contains no K_m and $\alpha(G) \leq n_1 + n_2 - 2$, so the required inequality follows. $\square$

The following result is also trivial.

Property 2.2 *Let m_i and n_i be positive integers. Then*

$$\begin{aligned} & r(m_1 m_2 + 1, n_1 n_2 + 1) \\ & \geq [r(m_1 + 1, n_1 + 1) - 1][r(m_2 + 1, n_2 + 1) - 1] + 1. \end{aligned}$$

Proof. Let G be a graph of order $N_1 = r(m_1 + 1, n_1 + 1) - 1$ with $\omega(G) \leq m_1$ and $\alpha(G) \leq n_1$. Let H be a graph of order $N_2 = r(m_2 + 1, n_2 + 1) - 1$ with $\omega(H) \leq m_2$ and $\alpha(H) \leq n_2$. One can “blow up” the graph G by H as follows. For any vertex v of G , replace v by a copy of H , denoted it by H_v . For any pair of vertices x and y from distinct copies H_u and H_v , x and y are adjacent if and only if u and v are adjacent. Inside the same copy H_v , the adjacency of H is preserved. Clearly, the clique number of new graph is at most $m_1 m_2$ and its independent number is at most $n_1 n_2$. So the claimed inequality follows. $\square$

On the other hand, we have been successful to obtain the right lower bound of $r_3(3)$ by using finite field. Let us return to finite fields for help.

An element $a \in F(q)$ is called a *quadratic residue* mod q if there exists $x \in F(q)$ such that $a = x^2 \pmod{q}$. A property of $F(p)$ concerning quadratic residue is the following theorem.

Theorem 2.4 (Euler) *Let p be an odd prime. Any $a \in F^*(p)$ is a quadratic residue mod p if and only if $a^{(p-1)/2} = 1 \pmod{p}$; and it is a quadratic non-residue mod p if and only if $a^{(p-1)/2} = -1 \pmod{p}$. Furthermore, half of elements in $F^*(p)$ are quadratic residue, and the other half are quadratic non-residue.*

Proof. Let $p = 2k + 1$ and let $g(x)$ be a polynomial as

$$g(x) = x^{2k-2} + x^{2k-4}a + \cdots + x^2a^{k-2} + a^{k-1},$$

where $a \in F^*(p)$. Then $x^{2k} - a^k = (x^2 - a)g(x)$. Thus

$$\begin{aligned} x^p - x &= (x^{p-1} - a^{(p-1)/2})x + (a^{(p-1)/2} - 1)x \\ &= (x^2 - a)xg(x) + (a^{(p-1)/2} - 1)x. \end{aligned}$$

If a is a quadratic residue mod p , then $a^{(p-1)/2} = 1 \pmod{p}$. On the other hand, if $a^{(p-1)/2} = -1 \pmod{p}$, then $x^2 - a$ can divide $x^p - x$, which

has p roots in $F(p)$, namely, all elements of $F(p)$, thus $g(x)$ has (at most hence exactly) $2k - 2$ roots and $x^2 - a$ has two roots in $F(p)$, so a is a quadratic residue mod p .

From the Fermat's theorem, we have

$$(a^{(p-1)/2} + 1)(a^{(p-1)/2} - 1) = 0 \pmod{p}.$$

By what just proved, we see that a is a quadratic non-residue mod p if and only if $a^{(p-1)/2} + 1 = 0 \pmod{p}$.

Consider the equation

$$x^{p-1} - 1 = 0 \pmod{p}, \quad (2.2)$$

which is equivalent to

$$x^{(p-1)/2} + 1 = 0 \pmod{p} \quad \text{or} \quad x^{(p-1)/2} - 1 = 0 \pmod{p}. \quad (2.3)$$

Each element of $F^*(p)$ is a solution of equation (2.2), so it is a solution of one of equation system (2.3). But each of the equation system (2.3) has at most $(p-1)/2$ solutions so it has exactly $(p-1)/2$ solutions, completing the proof for the second part of the theorem. $\square$

Corollary 2.1 *Let p be an odd prime. For any $a, b \in F^*(p)$, then the product ab is a quadratic residue mod p if both a and b are quadratic residue or both are non-residue; and ab is a quadratic non-residue mod p if one of a and b is quadratic residue and other is a non-residue.*

$\square$

It seems to be necessary to discuss the density of primes. We know that there are infinite many primes. As usual, let $f(n) \sim g(n)$ signify that $f(n)/g(n) \rightarrow 1$ as $n \rightarrow \infty$, and let $\pi(n)$ be the number of primes p with $p \leq n$, then $\pi(n) \sim n/\log n$, called *the prime number theorem*. For any pair of positive integers a and b with $0 < b \leq a$ and $(a, b) = 1$, let us write

$$Z_a(b) = \{ta + b : t \geq 0\}.$$

Dirichlet's theorem tells us that $Z_a(b)$ contains infinite many primes. Denote by $\pi(n; a, b)$ for the number of primes $p \in Z_a(b)$ with $p \leq n$. The following result is usually called Siegel-Walfisz Theorem, in which $\phi(a)$ is the number of integers $b \in \{1, 2, \dots, a\}$ with $(a, b) = 1$.

Theorem 2.5 *Let a and b be positive integers with $0 < b \leq a$ and $(a, b) = 1$. Then*

$$\pi(n; a, b) \sim \frac{n}{\phi(a) \log n}$$

as $n \rightarrow \infty$. Equivalently, the n th prime in $Z_a(b)$ is asymptotically equal to $\phi(a)n \log n$. Consequently, if p' and p'' are two consecutive primes in the set $Z_a(b)$, then $p' \sim p''$ as $p' \rightarrow \infty$.

By the facts that $\pi(n) = \pi(n; 1, 1)$ and $\phi(1) = 1$, we see that the prime number theorem is a special case of the above theorem. In the following corollary, we write p_n for the n th prime.

Corollary 2.2 *As $n \rightarrow \infty$, $\pi(n) \sim n/\log n$, which is equivalent to $p_n \sim n \log n$. Consequently, $p_n \sim p_{n+1}$.*

Another way to describe the density is the estimate the difference $p_{n+1} - p_n$. It has been shown that for some constant c with $0 < c < 1$, $p_{n+1} - p_n < p_n^c$ for all large n . At present, the best known value of c is less $11/20$.

The first application of Theorem 2.5 is the case $a = 4$. Since $\phi(4) = 2$, so asymptotically, there are half primes $p \leq n$ in form $p \equiv 1 \pmod{4}$ and half that $p \leq n$ in form $p \equiv 3 \pmod{4}$. The common asymptotical value is $n/(2 \log n)$.

The *Paley graph* G_p is defined as follows. Let $p \equiv 1 \pmod{4}$ be a prime and let $F(p)$ be a p element field. The vertex set of G_p is $F(p)$, and its edge set is formed by connecting two vertices x and y with an edge if and only if $x - y$ is a quadratic residue mod p . Note that if $p = 4n + 1$ for some integer $n \geq 1$ hence $x - y$ is a quadratic residue mod p if and only if $y - x$ is. Thus x and y are adjacent if and only if $(x - y)^{(p-1)/2} = 1 \pmod{p}$, and they are nonadjacent if and only if $(x - y)^{(p-1)/2} = -1 \pmod{p}$.

As an example, it is easy to verify that G_5 is C_5 , which is the Ramsey graph for $r(3, 3)$.

In this chapter, we only define the Paley graph for prime p so that the proof is simplified. Also, their parameters such as clique numbers are easier to find by computers. In fact, the Paley graph can be defined for any prime power.

A graph G of order n is said to be a *strongly regular graph* with parameters n, k, λ, μ , denoted by $srg(n, k, \lambda, \mu)$ in short, if it is k -regular, and any pair of vertices have λ common neighbors if they are adjacent, or μ common neighbors if they are not adjacent. It is easy to see that the complement of an srg is also an srg .

Proposition 2.1 *Let G be an $srg(n, k, \lambda, \mu)$, then its complement is also an $srg(n, k_1, \lambda_1, \mu_1)$ with*

$$k_1 = n - k - 1,$$

$$\begin{aligned}\lambda_1 &= n - 2k + \mu - 2, \\ \mu_1 &= n - 2k + \lambda.\end{aligned}$$

The union mK_k is an $srg(mk, k-1, k-2, 0)$, where $\mu = 0$. On the other hand, let G be an $srg(n, k, \lambda, \mu)$. If $\mu = 0$, then G is a union of complete graphs with the same order. A relation among the parameters is as follows.

Proposition 2.2 *Let G be an $srg(n, k, \lambda, \mu)$, then*

$$k(k - \lambda - 1) = \mu(n - k - 1).$$

proof. Consider a vertex x and partition the other vertices into $N(x)$ of neighbors of x and $M(x)$ of non-neighbors of x . By the definition, $N(x)$ contains k vertices, and $M(x)$ contains $n - k - 1$ vertices. Each vertex of $N(x)$ is adjacent to λ vertices in $N(x)$, and hence $k - \lambda - 1$ vertices in $M(x)$. Each vertex in $M(x)$ is adjacent to μ vertices in $N(x)$. Counting the edges between $N(x)$ and $M(x)$ in two ways, the required equality follows.

Theorem 2.6 *Let $p = 4n + 1$ be a prime. Then the Paley graph G_p is an $srg(4n + 1, 2n, n - 1, n)$. Furthermore, it is self-complementary.*

Proof. Let $V(G_p) = \{0, 1, \dots, p - 1\}$. Fix a vertex i , for any $j \in V$, $j \neq i$, among all differences $j - i$, exact $(p - 1)/2$ of them are quadratic residue mod p by Theorem 2.4, so the degree of i is $2n = (p - 1)/2$ hence G_p is $2n$ -regular. Fix $a \in F^*(p)$ that is a quadratic non-residue mod p , define a map ϕ as

$$\phi : V(G_p) \rightarrow V(G_p), \phi(x) = ax.$$

From Corollary 2.1, it is easy to check that $x - y$ is a quadratic residue mod p if and only if $\phi(x) - \phi(y)$ is a quadratic non-residue mod p , so it is an isomorphism between G_p and $\overline{G_p}$. Hence G_p is self-complementary.

Consider any edge ij of G_p . It is easy to know that the map

$$\psi : V(G_p) \rightarrow V(G_p), \psi(x) = (x - i)/(j - i)$$

is an automorphism of G_p with $\psi(i) = 0$ and $\psi(j) = 1$. Hence the number of common neighbors of i and j is as the same as that of 0 and 1. Similarly, the number of common neighbors of any pair of non-adjacent vertices is the same. So G_p is a strongly regular graph with order $4n + 1$ and $k = 2n$. Set the other parameters as λ and μ . Thus by Proposition 2.1, we know that $\overline{G_p}$ is an

$$srg(4n + 1, 2n, \mu - 1, \lambda + 1).$$

Since G_p is self-complementary, we have $\mu = \lambda + 1$. Then by Proposition 2.2, we get $\lambda = n - 1$ and $\mu = n$, hence G_p is an $srg(4n + 1, 2n, n - 1, n)$. $\square$

Denote by $B_n = K_2 + \overline{K_n}$ which contains n triangles sharing a common edges. We call B_n as an n -book. Look back the Paley graph G_p with $p = 4n + 1$, it contains no B_n and $\overline{G_p}$ contains no B_n since G_p is self-complementary. This fact yields that $r(B_n, B_n) \geq 4n + 2$. In fact the equality holds for all n with $4n + 1$ is a prime.

Theorem 2.7 (Rousseau-Sheehan, 1978) *Let m and n be positive integers that satisfies $2(m + n) + 1 > (n - m)^2/3$. Then*

$$r(B_m, B_n) \leq 2(m + n + 1),$$

and if $4n + 1$ is a prime, then $r(B_n, B_n) = 4n + 2$.

Proof. From the facts that $\overline{K_m}$ can be obtained from B_m by deleting two vertices and $r(\overline{K_m}, B_n) = m$, and Theorem 1.5 in Chapter 1, we have

$$r(B_m, B_n) \leq m + n + 2 + 2\sqrt{(m^2 + mn + n^2)/3}.$$

To get the upper bound as desired, note that the Ramsey number is an integer, we need only to verify that

$$2\sqrt{(m^2 + mn + n^2)/3} < m + n + 1,$$

which is equivalent to $2(m + n) + 1 > (n - m)^2/3$, as given. The lower bound for the equality is given by Paley graphs as mentioned. $\square$

2.4 Conjunction of Paley Graphs

Return to the classic Ramsey numbers, the interesting facts are that $G_5 = C_5$ is the Ramsey graph for $r(3, 3)$ and G_{17} is a Ramsey graph for $r(4, 4)$. Unfortunately, no other Paley graphs are found to be the “exact” Ramsey graphs. However, they still give good lower bounds. Using the computers, Shearer (1986) and Mathon (1987) independently calculated some $\alpha(G_p) = \omega(G_p)$ for small prime p listed in Table 3. Let $k = \alpha(G_p) = \omega(G_p)$, then $r(k + 1, k + 1) \geq p + 1$. They found that some form of conjunction of two copies of G_p can give $r(k + 2, k + 2) \geq 2p + 3$. (Mathon’s construction needs to change slightly).

Theorem 2.8 *Let $p \equiv 1 \pmod{4}$ be a prime and let $k = \omega(G_p) = \alpha(G_p)$. Then $r_2(k + 2) \geq 2p + 3$.*

Proof. Set the vertex sets of two copies of G_p to be

$$V = \{0, 1, \dots, p-1\} \quad \text{and} \quad V' = \{0', 1', \dots, (p-1)'\},$$

respectively. Let H_p be the graph on vertex set $V \cup V' \cup \{\lambda, \lambda'\}$. Let λ connect with V , and λ' connect with V' completely. Then add the edges connecting any two vertices x and y' , if and only if x and y are not adjacent in G_p . Writing it in details, the edge set contains (λ, λ') and

$$\begin{array}{ll} (x, \lambda), (x', \lambda'), (x, x'), & x \in V \\ (x, y), (x', y') & (x, y) \in E(G_p) \\ (x, y'), (x', y) & (x, y) \notin E(G_p) \end{array}$$

Note that all “crossing” edges of form (x, x') are in the edge set of H_p .

Claim 1. $\omega(H_p) = \omega(G_p) + 1 = k + 1$. Let S be a maximum clique of H_p . Clearly $|S| \geq 3$. It is easy to see that $|S| \geq k + 1$ since λ and a maximum clique of G_p form a clique of size $k + 1$. Also, if $x \in S$ then $x' \notin S$ since they do not have any common neighbor. We then need to verify that $|S| \leq k + 1$.

Case 1. $\lambda \in S$ or $\lambda' \in S$, say $\lambda \in S$. Then $S \setminus \{\lambda\}$ is a clique of G_p thus $|S| \leq k + 1$.

Case 2. Neither λ nor λ' is in S . Assume

$$S = \{x_1, \dots, x_m, y'_1, \dots, y'_n\},$$

where $x_i, y_j \in V$ and $x_i \neq y_j$ for $i \neq j$. That means

- $x_i - x_j$ and $y_i - y_j$ are quadratic residue mod p ;
- $y'_i - x_j$ are quadratic non-residue mod p , that is to say, $y_i - x_j$ are quadratic residue mod p .

Then for any $a \in V$, the set $\{x_1 - a, \dots, x_m - a, (y_1 - a)', \dots, (y_n - a)'\}$ is also a maximum clique of H_p . Hence we may assume that

$$S = \{0, x_2, \dots, x_m, y'_1, \dots, y'_n\},$$

and

- $x_i, x_i - x_j$ and $y_i - y_j$ are quadratic residue mod p ;
- y_j and $y_j - x_i$ are quadratic non-residue mod p .

Set

$$T = \{1/x_2, \dots, 1/x_m, 1/y_1, \dots, 1/y_n\},$$

where $1/x$ is the inverse of x in field $F(p)$. It then suffices to verify that T is a clique of G_p for Claim 1. For any $i \neq j$,

$$\left(\frac{1}{x_i} - \frac{1}{x_j}\right)^{(p-1)/2} = \frac{(x_j - x_i)^{(p-1)/2}}{(x_i x_j)^{(p-1)/2}} = 1 \pmod{p},$$

so $1/x_i$ and $1/x_j$ are adjacent in G_p since x_i and x_j are adjacent in G_p . Similarly, $1/y_i$ and $1/y_j$ are adjacent in G_p . For any i and j , note $x_i - y_j$ and y_j are quadratic non-residue mod p , then

$$\left(\frac{1}{x_i} - \frac{1}{y_j}\right)^{(p-1)/2} = \frac{(y_j - x_i)^{(p-1)/2}}{(x_i y_j)^{(p-1)/2}} = \frac{-1}{-1} = 1 \pmod{p},$$

so $1/x_i$ and $1/y_j$ are adjacent in G_p , as required.

Claim 2. $\alpha(H_p) = \omega(\overline{H_p}) = k + 1$. Clearly $\alpha(H_p) \geq k + 1$ since a maximum independent set in G_p and λ' form an independence set of size $k + 1$ in H_p . To verify the inverse inequality, let H be the graph obtained from H_p by deleting crossing edges of form (x, x') with $x \in V$ and (λ, λ') . It is easy to see that $\overline{H}$ is isomorphic to H_p by extending the self-isomorphism of G_p and $\overline{G_p}$. Thus

$$\alpha(H_p) \leq \alpha(H) = \omega(\overline{H}) \leq \omega(H_p) = k + 1.$$

From Claim 1 and Claim 2, both H_p and $\overline{H_p}$ do not contain clique of size $k + 2$, so $r_2(k + 2) \geq |V(H_p)| + 1$, completing the proof. $\square$

In the following table, the entries in the first column are values of $k = \omega(G_p) = \alpha(G_p)$; the second column p_1 and the third column p_ℓ are the first and largest such prime p that $\omega(G_p) = k$; the fourth column are number of such primes p ; and the last two columns are the lower bounds obtained by Theorem 2.6 and Theorem 2.8, in which the better one of two lower bounds is listed only, and $r(k) = r_2(k) = r(k, k)$.

$\alpha(G_p)$	p_1	p_ℓ	n	$r(\alpha + 1) \geq p_\ell + 1$	$r(\alpha + 2) \geq 2p_\ell + 3$
2	5	5	1	$r(3) \geq 6$	
3	13	17	2	$r(4) \geq 18$	
4	29	37	2	$r(5) \geq 38$	
5	41	101	6	$r(6) \geq 102$	$r(7) \geq 205$
6	97	109	2		
7	113	281	10	$r(8) \geq 282$	$r(9) \geq 565$
8	173	373	7		
9	229	797	15	$r(10) \geq 798$	$r(11) \geq 1597$
10	557	709	3		$r(12) \geq 1421$
11	433	1277	32		$r(13) \geq 2557$
12	613	1493	13		$r(14) \geq 2989$
13	853	2741	53		$r(15) \geq 5485$
14	1373	2801	17		$r(16) \geq 5605$

Table 3 Independence numbers of small G_p

Note that the best known lower bounds of $r_2(n)$ for $n \geq 6$ at present come all from Table 3.

The lower bound $r(12, 12) \geq 1421$ in last column can be improved as $r(12, 12) \geq 1618$ by using Property 2.1 twice and the fact $r(11, 11) \geq 1597$.

It is time to propose a problem concerning the asymptotic behavior of diagonal Ramsey numbers $r_2(n)$.

Problem 2.1 *Prove or disprove that for any $\epsilon > 0$ fixed, then*

$$r_2(n + 1) \geq (2 - \epsilon)r_2(n)$$

for all sufficiently large n .

Denote by $B_{m,n}$ for the graph $K_m + \overline{K}_n$, in which n isolated vertices are adjacent to each vertex of a K_m . Thus $B_{1,n} = K_{1,n}$ and $B_{2,n} = B_n$. We have shown that $r(B_{2,n}) \leq 4n + 2$. It is easy to see that $r(B_{1,n}) \leq 2r(\overline{K}_n, B_{1,n}) = 2n$. The following is a conjecture of A. Thomason (1979).

Conjecture 2.1 *For all positive integers m and n ,*

$$r(B_{m,n}) \leq 2^m(m + n - 2) + 2.$$

This conjecture is very hard even its special case. For example, it implies that $r(K_m)$ has an upper bound $(m - 1)2^{m-1} + 2$ from the fact that

$$r(K_m) \leq r(B_{m-2,m})$$

since either m vertices induces an independent set or they contain an edge which together with K_{m-2} yielding a K_m .

2.5 Some Linear Formulas

As we have mentioned, more “exact” results have been obtained on generalized cases. Some of them are formulas. But all these formulas are linear.

The first generalized Ramsey number was due to Gerencsér and Gyárfás in 1966, who computed the Ramsey number $r(P_m, P_n)$. However, the general problem was anticipated in a paper of Erdős (1947), in which the right upper bound for $r(P_m, P_n)$ was determined.

Theorem 2.9 *Let $n \geq m \geq 1$ be integers and let P_n be a path of length n . Then*

$$r(P_n, P_m) = n + \lfloor (m+1)/2 \rfloor.$$

Proof. We only write the proof of the diagonal case, the proof for general case is similar. So we shall show

$$r(P_n, P_n) = n + \lfloor (n+1)/2 \rfloor.$$

The assertion is trivial for $n = 1$ and $n = 2$, so we assume $n \geq 3$ and the assertion holds for $n - 1$. The facts that the graph $G = K_n \cup K_\ell$, where $\ell = \lfloor (n-1)/2 \rfloor$ and its complement contain no P_n give $r(P_n, P_n) \geq N$, where $N = n + \lfloor (n+1)/2 \rfloor$. Let us consider an edge coloring of K_N in red and blue. We shall show that there is a monochromatic path of length n . If not, by assumption, we have a monochromatic, say red, path P of length $n - 1$. Consider two disjoint blue paths Q_1 and Q_2 consisting of edges between $V(P)$ and $V(K_N) \setminus V(P)$, such that their end-vertices are in $V(K_N) \setminus V(P)$, and they do not contain end-vertices of P . Furthermore, the sum of their lengths are maximum. We shall prove that the following claim by induction on n .

Claim. $V(P) \cup V(Q_1) \cup V(Q_2) = V(K_N)$.

It is easy to see that $|(V(Q_1) \cup V(Q_2)) \setminus V(P)| \geq 2$ since otherwise there is at most one vertex outside $V(P)$ so $|V(P)| \geq N - 1 \geq n + 1$. Thus

$$|V(P) \cup V(Q_1) \cup V(Q_2)| \geq n + 2.$$

Suppose to the contrary, the claim is false, there is a vertex $x \in V(K_N)$ with $x \notin V(P) \cup V(Q_1) \cup V(Q_2)$. From the structures of Q_1 and Q_2 ,

$$|V(P) \cap V(Q_i)| = \frac{1}{2}(|V(Q_i)| - 1) = |V(Q_i) \setminus V(P)| - 1.$$

Thus we have

$$\begin{aligned}
|V(P) \setminus (V(Q_1) \cup V(Q_2))| &\geq n + 2 - |(V(Q_1) \cup V(Q_2)) \setminus V(P)| \\
&> n + 2 - |V(K_N) \setminus V(P)| \\
&= n + 2 - \lfloor (n + 1)/2 \rfloor \\
&= \lfloor n/2 \rfloor + 2.
\end{aligned}$$

Thus at least $\lfloor n/2 \rfloor + 3$ vertices of P , hence at least $\lfloor n/2 \rfloor + 1$ internal vertices of P are not covered by $Q_1 \cup Q_2$. This implies that there is an internal edge uv of P such that $u, v \notin V(Q_1 \cup Q_2)$.

One of edges xu and xv , say xu , must be blue as otherwise the length of P can be increased by replacing uv by ux and xv . Denote by $u_0, v_0, u_1, v_1, u_2, v_2$ for the end-vertices of P, Q_1, Q_2 , respectively. Then the edge u_1v and vu_2 must be blue by the same argument as above. Now

$$Q'_1 = Q_1 + u_1v + vu_2 + Q_2,$$

and $Q'_2 = \{x\}$ are two paths with the sum of lengths greater than that of Q_1 and Q_2 satisfying the same conditions, yielding a contradiction and proving the claim.

Consider the four edges $u_0u_1, u_0u_2, v_0v_1, v_0v_2$. All of these are blue by the maximality of P . Therefore the cycle

$$C = Q_1 + u_1u_0 + u_0u_2 + Q_2 + v_2v_0 + v_0u_2$$

is monochromatically blue. Since the vertices of C are in $V(P)$ and out of $V(P)$ alternatively, the length of C is

$$2(N - n) = 2\lfloor (n + 1)/2 \rfloor.$$

If n is odd, the length of C is $n + 1$, which gives an path of length n by removing an edge of C . So we assume that n is even and the length of C is n . If there is a blue edge between C and $V(K_N) \setminus V(C)$, we can find such blue path again. Thus we assume that all edges between C and $V(K_N) \setminus V(C)$ are red. Then a red path with length $2\lfloor (n + 1)/2 \rfloor \geq n$ can be found easily. This contradicts to the maximality of P , completing the proof. $\square$

The following is an early result of Chvátal and Harary (1972).

Theorem 2.10 *Let $n \geq 1$ be an integer. Then*

$$r(K_{1,n}) = \begin{cases} 2n - 1 & \text{if } n \text{ is even} \\ 2n & \text{if } n \text{ is odd.} \end{cases}$$

Proof. From the recursive bounds in Section 1, we have

$$r(K_{1,n}) \leq 2r(\overline{K}_n, K_{1,n}) = 2n.$$

For an even n , the above inequality is strict. Let G be the graph $K_{n-1} \cup K_{n-1}$. Then $\Delta(G) = n - 2$ and $\Delta(\overline{G}) = n - 1$, which shows that $r(K_{1,n}) \geq 2n - 1$ hence $r(K_{1,n}) = 2n - 1$.

For an odd n , take disjoint vertex sets V_i for $i = 1, 2$, and $U_i \subseteq V_i$ with $|V_i| = n - 1$ and $|U_i| = (n - 1)/2$. Consider a graph on vertex set $V_1 \cup V_2$ and one more vertex w . The edge set consists of all edges in V_1 and V_2 , a match between U_1 and U_2 of size $(n - 1)/2$, and all edges connecting w and $V_1 \setminus U_1$ and $V_2 \setminus U_2$. This is a regular graph of order $2n - 1$ with degree $n - 1$, which yields that $r(K_{1,n}) \geq 2n$ hence $r(K_{1,n}) = 2n$. $\square$

The above result was generalized by Burr and Roberts (1973).

Theorem 2.11 *Let m and n be positive integers. Then*

$$r(K_{1,m}, K_{1,n}) = \begin{cases} m + n - 1 & \text{if } m \text{ and } n \text{ are both even} \\ m + n & \text{otherwise.} \end{cases}$$

We need some lemmas for the proof of the above result. A *factor* of a graph G is a spanning subgraph of G . If $G_1, \dots, G_k$ are pairwise edge-disjoint factors of G such that $E(G) = \cup_{i=1}^k E(G_i)$, then we say that G is the edge sum of the factors $G_1, \dots, G_k$. An r -regular factor of a graph G is referred to as an r -factor. If there exists a factorization of G such that each factor is a r -factor, then we say that G is r -factorable. A 1-factor of a graph (of even order) is a perfect matching of the graph. A cycle of a graph G containing all vertices is called a *hamiltonian* cycle.

Lemma 2.2 *Let $n \geq 1$ be integer. Then K_{2n} is 1-factorable, and K_{2n+1} can be factored into n hamiltonian cycles.*

Proof. The result is trivial for $n = 1$. Thus we assume that $n \geq 2$. Set $V(K_{2n}) = \{v_0, v_1, \dots, v_{2n-1}\}$. Arrange the vertices $v_1, v_2, \dots, v_{2n-1}$ in a regular $(2n - 1)$ -gon, and place v_0 in the center. Joint every two vertices by a straight line segment. For $i = 1, 2, \dots, 2n - 1$, define the edge set of the factor F_i to consist of the edge $v_0 v_i$ together with all those edges perpendicular to $v_0 v_i$. Then each F_i is a perfect matching and K_{2n} is factored into $F_1, \dots, F_{2n-1}$.

For the second assertion, set $V(K_{2n+1}) = \{v_0, v_1, \dots, v_{2n}\}$. Arrange the vertices $v_1, v_2, \dots, v_{2n}$ in a regular $2n$ -gon, and place v_0 in some other convenient position. Joint every two vertices by a straight line segment

producing a K_{2n+1} . Define the edge set of the factor F_1 to consist of edges v_0v_1, v_0v_{n+1} , all edges parallel to v_1v_2 and all edges parallel to v_2v_{2n} . In general, for $i = 1, 2, \dots, n$, define the edge set of the factor F_i to consist of edges v_0v_i, v_0v_{n+i} , all edges parallel to v_iv_{i+1} and all edges parallel to $v_{i+1}v_{i-1}$, where the subscripts are expressed modulo $2n$. Then each F_i is a hamiltonian cycle and K_{2n+1} is factored into $F_1, \dots, F_n$. $\square$

Proof of Theorem 2.11. From the recursive upper bounds in Section 1, we have

$$r(K_{1,m}, K_{1,n}) \leq r(\overline{K}_m, K_{1,n}) + r(K_{1,m}, \overline{K}_n) = m + n,$$

and if m and n are both even, the inequality is strict. The upper bound required follows.

If each of m and n is odd as is $m + n - 1$. By Lemma 2.2, the edge set of the complete graph K_{m+n-1} can be partitioned into $(m + n - 2)/2$ hamiltonian cycles. Let R and B be the edge sums of $(m-1)/2$ and $(n-1)/2$ of these cycles, respectively. Then R is an $(m-1)$ -regular and B is an $(n-1)$ -regular, which yields that $r(K_{1,m}, K_{1,n}) \geq m + n$.

Assume that one of m and n is odd and one is even. Then $m + n - 1$ is even. By Lemma 2.2, the complete graph K_{m+n-1} is 1-factorable. Let R and B be the edge sums of $m-1$ and $n-1$ of these factors, respectively, so that R is $(m-1)$ -regular and B is $(n-1)$ -regular. Thus $r(K_{1,m}, K_{1,n}) \geq m + n$.

Finally, assume that both m and n are even. Then $m + n - 2$ is even and K_{m+n-2} can be factored into $m + n - 3$ matchings. Let R and B be the edge sums of $m-1$ and $n-2$ of these matchings, respectively, yielding that $r(K_{1,m}, K_{1,n}) \geq m + n - 1$ and completing the proof. $\square$

We have seen that $r(P_m, P_n) = n + \lfloor (m+1)/2 \rfloor$ for $n \geq m$ and $r(K_{1,m}, K_{1,n}) = m + n + \delta$, where $\delta = 0$ or $\delta = 1$. It is believed that $r(T_{1+m}, T_{1+n})$ is bounded between $r(P_m, P_m)$ and $r(K_{1,m}, K_{1,n})$, where T_{1+m} is a tree of m edges. The upper bound can be deduced if a conjecture of Erdős and Sós in 1963 is confirmed, which stated that a graph with N vertices and more than $(m-1)N/2$ edges contains every tree with m edges. The truth of this conjecture for the star $K_{1,m}$ is trivial and the case of a path with m edges is covered by a well-known result of Erdős and Gallai (1959), see Chapter 3 for more details.

The following result is due to Chvátal (1977), which stimulated generalized Ramsey theory greatly.

Theorem 2.12 *Let T_n be a tree on n vertices, then*

$$r(T_n, K_k) = (k-1)(n-1) + 1.$$

Proof. The complete $(k-1)$ -partite graph $K_{k-1}(n-1)$ yields the lower bound $r(K_k, T_n) \geq (k-1)(n-1) + 1$. To get the reverse inequality, we use induction on n . Let G be a graph on $N = (k-1)(n-1) + 1$ vertices. Suppose that $\overline{G}$ contains no K_k , that is to say, $\alpha(G) \leq k-1$. We shall see that G contains T_n . Let $S \subset V(G)$ be a maximum independent set of size $\ell \leq k-1$. Outside S , there are $N - \ell \geq (k-1)(n-2) + 1$ vertices, which must contain T_{n-1} by induction, where T_{n-1} is a tree obtained from T_n by deleting a vertex v of degree one. Let u be the vertex of T_{n-1} adjacent with v in T_n . Since S is maximum, so u has at least one neighborhood in S , yielding T_n as claimed. $\square$

We shall return to the above result in Chapter 13. Let us conclude this section with the following problem.

Problem 2.2 Let T_{1+m} be a tree with m edges. Find a good expression for $r(T_{1+m}, T_{1+n})$.

Exercises

2.1 Prove that $r(C_4, C_4) = 6$.

2.2 Prove that the Ramsey graph for $r(3, 4)$ can be obtained from that for $r(3, 5)$ by deleting a vertex and its neighbors. When we can do so for $r(m, n-1)$ from $r(m, n)$?

2.3 Prove Proposition 2.1.

2.3 Show that

$$r(T, K_{n_1}, \dots, K_{n_k}) = (m-1)(r(K_{n_1}, \dots, K_{n_k}) - 1) + 1,$$

where T is a tree of order m .

2.4 Let $\ell = r(K_n, K_{n-2})$. Show $r(K_n, K_n) \leq r(B_\ell, B_\ell)$ hence Walker's bound $r(K_n, K_n) \leq 4r(K_n, K_{n-2}) + 2$.

2.5 Show the line graph of K_n with $n \geq 4$, denoted by $T(n)$, is an $srg(\binom{n}{2}, 2(n-2), n-2, 4)$. Using $T(6)$ and $T(7)$, show that $r(B_2, B_5) = 16$ and $r(B_4, B_6) = 22$. (Rousseau-Sheehan, 1978.)

2.5 Show the line graph of $K_{n,n}$ with $n \geq 2$, denoted by $L(n)$, is an $srg(n^2, 2(n-2), n-2, 2)$. Using $L(4)$, show that $r(B_3, B_5) = 17$. (Rousseau-Sheehan, 1978.)

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Chapter 3

Bounding Independence Numbers

3.1 Turán Lower Bounds

By the definition of Ramsey numbers, we see that $r(H, K_n)$ is the minimum integer N such that any H -free graph G of order N satisfies that $\alpha(G) \geq n$, where a graph is H -free means the graph contains no H as a subgraph. So it is important to get some bounds on independence numbers, in particular, that of H -free graphs. A well-known lower bound is as follows.

Theorem 3.1 (Turán's Lower Bound) *Let G be a graph of order N with average degree d , then*

$$\alpha(G) \geq \frac{N}{1+d}.$$

This result is commonly called the Turán lower bound since it can be an easy corollary of a famous theorem of Turán, see Chapter 8. However, a slightly stronger form was first explicitly given by Wei and Caro. An elegant probabilistic proof due to Alon and Spencer (1992) will be in Chapter 4.

Theorem 3.2 (Caro and Wei) *Let G be a graph of order N , then*

$$\alpha(G) \geq \sum_{v \in V(G)} \frac{1}{1 + \deg(v)}.$$

The first proof. Let G be a graph on vertex set $\{1, 2, \dots, N\}$ and denote by d_i as the degree of vertex i . We will use induction on N . Let H be the subgraph from G by deleting vertex i and its neighbors. It is clear

$$\alpha(G) \geq 1 + \alpha(H)$$

for any vertex i . (There must be some vertex i such that the equality holds). Assume that d_1 is the minimum degree and vertices $\{2, \dots, k\}$ are all neighbors of the vertex 1, where $k = 1 + d_1$. Let

$$f(G) = \sum_{v \in V(G)} \frac{1}{1 + \deg(v)} = \sum_{i=1}^N \frac{1}{1 + d_i}.$$

By the induction $\alpha(H) \geq f(H)$, thus if we can prove $1 + f(H) \geq f(G)$, then

$$\alpha(G) \geq 1 + \alpha(H) \geq 1 + f(H) \geq f(G),$$

we are done. The assertion $1 + f(H) \geq f(G)$ is

$$1 + \frac{1}{1 + d'_{k+1}} + \dots + \frac{1}{1 + d'_N} \geq \frac{1}{1 + d_1} + \dots + \frac{1}{1 + d_N},$$

where d'_i signifies the degree of vertex i in H . Since $d'_i \leq d_i$, it suffices to verify that

$$1 + \frac{1}{1 + d_{k+1}} + \dots + \frac{1}{1 + d_N} \geq \frac{1}{1 + d_1} + \dots + \frac{1}{1 + d_N}.$$

Equivalently,

$$1 \geq \frac{1}{1 + d_1} + \dots + \frac{1}{1 + d_k}.$$

This is clear since $d_1 \leq d_i$ and $k = 1 + d_1$. □

In the algorithm employed in the above proof, we delete a vertex of minimum degree and its neighborhood in each step. In the following proof, another algorithm will be used, in which only a vertex of maximum degree is deleted until we obtain a graph without any edge.

The second proof. Let $V(G) = \{1, 2, \dots, N\}$. Let G_1 be the subgraph from G by deleting a vertex of maximum degree of the graph. Assume this vertex is 1, say. Let

$$f(G) = \sum_{v \in V(G)} \frac{1}{1 + \deg(v)} = \sum_{i=1}^N \frac{1}{1 + d_i}.$$

It is clear the statement is true for an empty graph. We shall show that $f(G)$ increases when a vertex of maximum degree is deleted unless no edge left. Assume the neighborhood of vertex 1 is $\{2, \dots, k\}$ with $k = 1 + d_1 \geq 2$. We will show $f(G_1) \geq f(G)$. That is,

$$\begin{aligned} & \frac{1}{1 + d'_2} + \dots + \frac{1}{1 + d'_k} + \frac{1}{1 + d_{k+1}} + \dots + \frac{1}{1 + d_N} \\ & \geq \frac{1}{1 + d_1} + \dots + \frac{1}{1 + d_N}, \end{aligned}$$

where d'_i signifies the degree of vertex i in G_1 . Note that $d_i = d'_i + 1$, so we shall prove,

$$\frac{1}{d_2} + \dots + \frac{1}{d_k} \geq \frac{1}{1 + d_1} + \frac{1}{1 + d_2} + \dots + \frac{1}{1 + d_k}.$$

Equivalently,

$$\frac{1}{d_2(1 + d_2)} + \dots + \frac{1}{d_k(1 + d_k)} \geq \frac{1}{1 + d_1},$$

which can be seen easily since $d_i(1 + d_i) \leq d_1(1 + d_1)$.

Suppose that after deletions of ℓ steps we obtain an empty graph G_ℓ , then

$$\alpha(G) \geq \alpha(G_1) \geq \dots \geq \alpha(G_\ell) = f(G_\ell) \geq f(G_{\ell-1}) \geq \dots \geq f(G),$$

completing the proof. $\square$

Note In the procedure of the proofs, we only need to delete the vertex whose degree is no more than (no less than) that of its neighbors.

The Turán's lower bound $\alpha(G) \geq N/(1 + d)$ is sharp on graphs of unions of complete graph with the same order; its discrete form $\alpha(G) \geq \sum 1/(1 + d_v)$ is sharp on union of complete graphs (not necessary the same order). The latter is stronger than the former since $\sum 1/(1 + d_v) > N/(1 + d)$ if G is not regular.

Proof of Theorem 3.1. Note that the function $g(x) = 1/(x + 1)$ is convex since $g''(x) > 0$ for $x \geq 0$. Using Jensen's inequality for a convex function g ,

$$\frac{1}{N} \sum_{i=1}^N g(x_i) \geq g\left(\frac{1}{N} \sum_{i=1}^N x_i\right),$$

then Theorem 3.1 follows from Theorem 3.2.

We now introduce the concept of asymptotic formula. Let $f(x)$ and $g(x)$ be two functions. If $f(x)/g(x) \rightarrow 0$ as $x \rightarrow \infty$, we write $f(x) = o(g(x))$. For example, $\log x = o(x^\delta)$ for any $\delta > 0$, where $\log x$ to stand for the logarithmic function on natural base e . So $f(x) = o(1)$ signify that $f(x) \rightarrow 0$ as $x \rightarrow \infty$, that is, $f(x)$ is a small term tending to zero as $x \rightarrow \infty$. Using ϵ -language, $f(x) = o(1)$ means that for any $\epsilon > 0$, there exists an $X = X(\epsilon) > 0$ such that if $x > X$, then $|f(x)| < \epsilon$. We often ignore the existence of the large X by saying that for large x or simply $x \rightarrow \infty$ instead. When $f(x) = (1 + o(1))g(x)$, we say that $g(x)$ is an asymptotic formula of $f(x)$, denoted by $f(x) \sim g(x)$ sometimes. When we write $f(x) \leq (1 + o(1))g(x)$ or $f(x) \geq (1 - o(1))g(x)$, the term $o(1)$ stands for a positive term tending to zero. But in an asymptotic equality such as $f(x) = (1 + o(1))g(x)$, the small term $o(1)$ is not necessarily positive.

For two functions $f(x)$ and $g(x)$, we write $f(x) = O(g(x))$ if there exists a positive constant c such that $f(x) \leq cg(x)$ for all large x , and write $f(x) = \Omega(g(x))$ if $g(x) = O(f(x))$. We write $f(x) = \Theta(g(x))$ if $f(x) = O(g(x))$ and $f(x) = \Omega(g(x))$. If both $f(x)$ and $g(x)$ are positive, then $f(x) = \Theta(g(x))$ means that there exist positive constants c_1 and c_2 such that for all large x ,

$$c_1 g(x) \leq f(x) \leq c_2 g(x).$$

In this case, we say that $f(x)$ has order $g(x)$. As an asymptotic formula or order of $f(x)$, the function $g(x)$ should be simple.

In order to investigate the asymptotic behavior of a nonlinear Ramsey function, it is often impossible to find its formula. It is even very difficult to find the correct order.

Let us look back the upper bound $r(m, n) \leq \binom{m+n-2}{m-1}$ obtained in Chapter 1. For fixed m and $n \rightarrow \infty$, the upper bound is asymptotically equal to $\frac{n^{m-1}}{(m-1)!}$ and its order of magnitude is n^{m-1} . We of course have a weaker bound $r(m, n) \leq n^{m-1}$. This can be proved by the lower bound of independence number just proved. To get this, we use induction on m starting at $m = 2$. Assume it holds for $m - 1$, then for a graph G of order $N = r(m, n) - 1$ which contain no K_m and $\alpha(G) \leq n - 1$. Let $v \in V(G)$. Easy to see $\deg(v) \leq r(m - 1, n) - 1$ hence the average degree d of G is no more than $r(m - 1, n) - 1$. By Turán's theorem,

$$n - 1 \geq \alpha(G) \geq \frac{N}{1 + d} \geq \frac{r(m, n) - 1}{r(m - 1, n)}.$$

Then

$$r(m, n) \leq (n - 1)r(m - 1, n) + 1 \leq (n - 1)n^{m-2} + 1 \leq n^{m-1}.$$

This bound is weaker than Erős-Szekeres bound in Chapter 1. Why? This is because we did not consider the structures of the graph. As a further step, we shall pay attention to the neighborhoods of the graphs in the next section.

3.2 Triangle-free Graphs

From the argument in last section, it is expected that the upper bound of Ramsey function can be improved if a better lower bound of independence number is employed. Let us gain an overview of the technique employed by Ajtai, Komlós and Szemerédi. In order to produce a larger independent set in a greedy algorithm, we hope to delete less vertices and more edges in each step so that the remaining graph is more sparse. To obtain some criterion, we define $Q(v)$ to be the number of edges that incident with a neighbor of v , and $Q_0(v) = \sum_{u \in N(v)} \deg(u)$. Note that if we delete a vertex v and its neighbors, we delete exactly $Q(v)$ edges. Clearly $Q(v) \leq Q_0(v)$, and in a triangle-free graph $Q(v) = Q_0(v)$ for any vertex v . We will establish a criterion $R(v)$ for choosing v to satisfy $R(v) \geq 0$. Before this, let us have a property of $Q_0(v)$.

Lemma 3.1 *Let $G = (V, E)$ be any graph with average degree d . Then the average value of $Q_0(v)$ over $v \in V$ is at least d^2 .*

Proof. Denote by N for the order of G . Then

$$\begin{aligned} \frac{1}{N} \sum_{v \in V} Q_0(v) &= \frac{1}{N} \sum_{v \in V} \sum_{u \in N(v)} \deg(u) \\ &= \frac{1}{N} \sum_{u \in V} \deg(u) \sum_{v \in N(u)} 1 = \frac{1}{N} \sum_{u \in V} \deg^2(u) \\ &\geq \left(\frac{1}{N} \sum_{u \in V} \deg(u) \right)^2 = d^2. \end{aligned}$$

In the last line we have used the Jensen's inequality for the convex function $f(x) = x^2$. $\square$

Ajtai, Komlós and Szemerédi(1980) defined a vertex v to be a *groupie* if the average degree of its neighbors is at least the average degree of G .

Lemma 3.2 *Every graph has a groupie.*

Proof. For a graph $G = (V, E)$ with average degree d , we need to find a vertex $v \in V$ so that

$$Q_0(v) - d \deg(v) \geq 0.$$

This is clear since the average value of the left hand side is at least zero from Lemma 3.1. $\square$

By deleting groupies recursively, they proved that for any triangle-free graph G of order N and average degree d ,

$$\alpha(G) \geq cN \frac{\log d}{d}, \quad (3.1)$$

where $c = 1/100$, and $\log x = \log_e x$ is the natural logarithmic function as mentioned. Ajtai, Erdős, Komlós and Szemerédi (1980,1981) conjectured in 1981 that the best constant is $1 - o(1)$, where the small term $o(1)$ tends to zero as $d \rightarrow \infty$. By adding the details to the proof for (3.1), Griggs (1983) improved the constant to $1/2.4$. These results considerably improved the Turán inequality when d is large. Now, let $N = r(3, n) - 1$, there is a triangle-free graph G on N vertices that contains no independence number $n - 1$. Since each neighborhood of a triangle-free graph is an independent set, thus $\alpha(G) \geq \Delta(G) \geq d$, where d is the average degree of G . Thus $n - 1 \geq cN \frac{\log(n-1)}{n-1}$, it follows by

$$r(3, n) = N + 1 \leq \left(\frac{1}{c} + o(1) \right) \frac{n^2}{\log n},$$

as $n \rightarrow \infty$. This bound is much stronger than $\binom{n+1}{2} = \frac{1}{2}n(n+1)$, which we have obtained in Chapter 1. A deep result of Kim (1995) proved that

$$r(3, n) \geq \left(\frac{1}{162} - o(1) \right) \frac{n^2}{\log n}.$$

This fact and the argument just mentioned imply that one cannot expect to improve lower bound (3.1) more than a multiplicative constant even just for triangle-free graphs.

The method of Ajtai, Komlós and Szemerédi(1980) is now called “semi-random method” or “nibble method”, in which they selected their objects in many small “nibbles” rather than a big “bite”, and then analyzed how the nibbles change the structure of the remainder. It was the method that Kim made an elaborate use in his random structure to obtain his lower bound of $r(3, n)$.

In order to find a larger independent set, the key step is to determine the vertex, which together with its neighbors, will be deleted. We now

look how Shearer (1983) determined this vertex for triangle-free graphs. Suppose that $f(x)$ is the function for which we want to prove

$$\alpha(G) \geq Nf(d)$$

for a triangle-free graph G . Naturally, we *assume that $f(x)$ is positive, decreasing*, and more importantly, $f(x) \geq c \log x/x$ for some constant $c > 1/100$ when x is sufficiently large. Let $P(v)$ denote $\deg(v) + 1$ and $Q(v)$ denote the number of edges incident with v or one of neighbors of v . Let H denote the graph obtained from G by deleting v and its neighbors. Note we delete exactly $Q(v) = Q_0(v)$ edges since G is triangle-free. Then H has $N - P(v)$ vertices and $Nd/2 - Q(v)$ edges. So its average degree is

$$d_H = \frac{Nd - 2Q(v)}{N - P(v)}.$$

By induction,

$$\alpha(G) \geq 1 + \alpha(H) \geq 1 + [N - P(v)]f(d_H).$$

We do not know which of d and d_H is bigger. That is why the algorithm of Ajtai et. al. deletes a groupie and its neighbors. However we can swap $f(d_H)$ with $f(d)$, if we *further assume that $f(x)$ is convex* so that we can use the fact

$$f(x) \geq f(d) + f'(d)(x - d).$$

Thus we have

$$1 + [N - P(v)]f(d_H) \geq Nf(d) + R(v)$$

where

$$R(v) = 1 - P(v)f(d) + [dP(v) - 2Q(v)]f'(d).$$

The only thing left to prove is $R(v) \geq 0$ for some v . To find such vertex v , let us look the average of $R(v)$ as follows.

$$\begin{aligned} \frac{1}{N} \sum_v R(v) &\geq 1 - (d+1)f(d) + [d(d+1) - 2d^2]f'(d) \\ &= 1 - (d+1)f(d) - d(d-1)f'(d), \end{aligned}$$

where we used the fact that the average of $Q(v)$ is at least d^2 by Lemma 3.1 and the assumption that $f'(x) < 0$. So we need that the function $f(x)$ satisfies the following differential equation

$$x(x-1)f'(x) + (x+1)f(x) = 1.$$

Solving this differential equation, Shearer thus obtained that

$$f(x) = \frac{x \log x - x + 1}{(x-1)^2}.$$

Luckily enough, $f(x)$ is positive, decreasing and convex. Moreover, $f(x) \sim \log x/x$ as $x \rightarrow \infty$.

3.3 Graphs with Sparse Neighborhoods

Since the improvements of Griggs and Shearer are only for triangle-free graphs, so their improvements can be applied only for $r(3, n)$ in Ramsey theory. We should do more. Our function is the solution of the differential equation

$$x(x-m)f'(x) + (x+1)f(x) = 1.$$

We shall adopt such a form of the solution that we can prove its increasing and convex property.

For $m \geq 1$ and $x \geq 0$, set

$$f_m(x) = \int_0^1 \frac{(1-t)^{1/m}}{m + (x-m)t} dt.$$

Our aim is to prove that $\alpha(G) \geq N f_{m+1}(d)$ for any graph G with order N and average degree d , in which each subgraph induced by a neighborhood has a maximum degree at most m . This result appeared in Li, Rousseau and Zang (1996, 2001). In order to prove, we need some analytic properties of the function involved.

Lemma 3.3 *For $m \geq 1$ and $x \geq 0$, the function*

$$f_m(x) = \int_0^1 \frac{(1-t)^{1/m}}{m + (x-m)t} dt$$

satisfies the differential equation

$$x(x-m)f'_m(x) + (x+1)f_m(x) = 1. \quad (3.2)$$

Moreover, $f_m(x)$ is completely monotonic on $[0, \infty)$, that is to say, $(-1)^k f_m^{(k)}(x) > 0$ for all $k \geq 0$ and $x \geq 0$. In particular, $f_m(x)$ is positive, decreasing, and convex.

Proof. By differentiating under the integral and then integrating by parts, we have

$$\begin{aligned}
& x(x-m)f'_m(x) \\
&= -x(x-m) \int_0^1 \frac{(1-t)^{1/m}t}{(m+(x-m)t)^2} dt \\
&= x \int_0^1 (1-t)^{1/m} t \frac{d}{dt} \left(\frac{1}{m+(x-m)t} \right) dt \\
&= -x \int_0^1 \left(1 - \frac{t}{m(1-t)} \right) \frac{(1-t)^{1/m}}{m+(x-m)t} dt \\
&= -xf_m(x) + \int_0^1 \frac{(1-t)^{1/m}}{m} \left[\frac{1}{1-t} - \frac{m}{m+(x-m)t} \right] dt \\
&= -xf_m(x) + 1 - f_m(x).
\end{aligned}$$

Hence (3.2) follows. The complete monotonicity of $f_m(x)$ can be seen by repeated differentiating under the integral. $\square$

Corollary 3.1 *The following two statements hold.*

$$f_m(x) \begin{cases} \leq 1/(x+1) & \text{if } x \leq m; \\ \geq 1/(x+1) & \text{if } x \geq m. \end{cases}$$

For $m \geq 1$, $f_m(x) \geq \frac{\log(x/m)-1}{x}$.

Proof. By the lemma, the function $f_m(x)$ satisfies the differential equation

$$x(x-m)f'_m(x) + (x+1)f_m(x) = 1,$$

and $f'_m(x) \leq 0$. So

$$x(x-m)f'_m(x) \begin{cases} \geq 0 & \text{if } x \leq m; \\ \leq 0 & \text{if } x \geq m. \end{cases}$$

The first statement follows.

To justify the second statement for the case $x > m$, note that

$$\begin{aligned}
f_m(x) &\geq \int_0^1 \frac{(1-t)dt}{m+(x-m)t} \\
&= \frac{x \log(x/m) - (x-m)}{(x-m)^2} \\
&> \frac{\log(x/m) - 1}{x}.
\end{aligned}$$

The last inequality holds since it amounts to

$$(2x - m) \log(x/m) > x - m$$

or

$$(2t - 1) \log t > t - 1$$

for $t > 1$. Set $g(t) = (2t - 1) \log t - t + 1$. Then $g'(t) = 2 \log t - 1/t + 1$ and $g''(t) = 2/t + 1/t^2$. From $g(1) = g'(1) = 0$ and $g''(t) > 0$, it follows that $g(t) > 0$ for $t > 1$. $\square$

By simple integrating, we see

$$f_1(x) = \frac{x \log x - x + 1}{(x - 1)^2},$$

which is the function that Shearer found and applied to triangle-free graph. The following property is not necessary for the proof, but it is helpful for us to understand the function series f_m .

Lemma 3.4 *For fixed $x \geq 0$, $f_m(x)$ is decreasing on m , that is*

$$f_m(x) \geq f_{m+1}(x).$$

Proof. We have

$$f_m(x) = \int_0^1 \frac{(1-t)^{1/m} dt}{m + (x-m)t} = \int_0^1 \frac{t^{1/m} dt}{mt + x(1-t)}.$$

By setting $t = u^m$,

$$f_m(x) = \int_0^1 \frac{mu^m du}{mu^m + x(1-u^m)}.$$

To prove that $f_m(x)$ decreases with increasing m , it suffices to show that if $0 \leq r < s$, then

$$\frac{ru^r}{ru^r + x(1-u^r)} > \frac{su^s}{su^s + x(1-u^s)}, \quad 0 < u < 1.$$

Equivalently, $(s-r)u^{r+s} + ru^r - su^s > 0$ for $0 < u < 1$. Setting $s = r + \delta$, the inequality to be proved becomes

$$\delta u^{r+\delta} + r - (r+\delta)u^\delta > 0, \quad 0 < u < 1.$$

Let $g(u) = \delta u^{r+\delta} + r - (r+\delta)u^\delta$. Note $g(1) = 0$ and

$$g'(u) = \delta(r+\delta)u^{\delta-1}(u^r - 1) < 0, \quad 0 < u < 1,$$

completing the proof. $\square$

In conclusion, we have the following lemma.

Lemma 3.5 *Let $f_m(x)$ be the function defined in Lemma 3.3 for $x \geq 0$ with $m \geq 1$. Then $f_m(0) = 1$ and*

$$\frac{x \log x - x + 1}{(x-1)^2} \geq f_m(x) > \frac{\log(x/m) - 1}{x} \quad (x > 0).$$

In particular, for any fixed $m \geq 1$, as $x \rightarrow \infty$,

$$f_m(x) \sim \frac{\log x}{x}.$$

Now we come to the our main result in this section.

Theorem 3.3 *Let G be a graph with N vertices and average degree d . If any subgraph induced by a neighborhood has the maximum degree at most a , then*

$$\alpha(G) \geq N f_{a+1}(d).$$

Proof. We prove by induction on N , the number of vertices of G . If $N \leq a + 2$, then $d \leq a + 1$. By the corollary, we have $1/(d+1) \geq f_{a+1}(d)$. It follows from Turán's theorem that $\alpha(G) \geq \frac{N}{d+1} \geq N f_{a+1}(d)$. So we suppose $N > a + 2$ hereafter. By the preceding argument, we may also assume $d > a + 1$. We shall let G_v stand for the subgraph of a graph G induced by the neighborhood of v . In case some vertex v of G has degree $N-1$, by virtue of Turán's theorem, we have $\alpha(G_v) \geq \frac{N-1}{a+1}$ as the maximum degree of G_v is at most a . It follows that

$$\alpha(G) \geq \alpha(G_v) \geq \frac{N-1}{a+1} \geq \frac{N}{a+2} = N f_{a+1}(a+1) \geq N f_{a+1}(d).$$

So we suppose henceforth that

the maximum degree of G is at most $N-2$.

For each $v \in V(G)$, let $P(v)$ denote $\deg(v) + 1$ and let $Q(v)$ denote the number of edges of G that are incident with either v or one of its neighbors. Since the average degree of G_v is at most a , G_v contains at most $\frac{a}{2} \deg(v)$ edges. It follows that

$$Q(v) \geq Q_0(v) - \frac{a}{2} \deg(v),$$

where $Q_0(v) = \sum_{u \in N(v)} \deg(u)$. Consequently, the average value of Q satisfies

$$\frac{1}{N} \sum_{v \in V} Q(v) \geq d^2 - \frac{ad}{2},$$

where the fact that the average of $Q_0(v)$ is at least d^2 is from Lemma 3.1.

Set

$$R(v) = 1 + [P(v)d - 2Q(v)]f'_{a+1}(d) - P(v)f_{a+1}(d).$$

Note that the coefficient of $Q(v)$ is positive since $f'_{a+1}(d) < 0$. Then

$$\begin{aligned} \frac{1}{N} \sum_{v \in V} R(v) &\geq 1 + [(d+1)d - 2d^2 + ad]f'_{a+1}(d) - (d+1)f_{a+1}(d) \\ &= 1 - d(d-a-1)f'_{a+1}(d) - (d+1)f_{a+1}(d) \\ &= 0. \end{aligned}$$

Hence there exists a vertex $v_0 \in V(G)$ such that $R(v_0) \geq 0$. Let $R(v_0) = \hat{R}$, $P(v_0) = \hat{P}$ and $Q(v_0) = \hat{Q}$. Then

$$\hat{R} = 1 + (\hat{P}d - 2\hat{Q})f'_{a+1}(d) - \hat{P}f_{a+1}(d) \geq 0.$$

Delete v_0 and its neighbors from G , in view of that the maximum degree of G is at most $N-2$, we obtain a nontrivial graph H with $N - \hat{P}$ vertices and $Nd/2 - \hat{Q}$ edges. Note that any subgraph induced by a neighborhood of H has a maximum degree at most a , so by induction hypothesis,

$$\alpha(H) \geq (N - \hat{P})f_{a+1}\left(\frac{Nd - 2\hat{Q}}{N - \hat{P}}\right).$$

Clearly $\alpha(G) \geq 1 + \alpha(H)$. Since f_{a+1} is convex, $f_{a+1}(x) \geq f_{a+1}(d) + f'_{a+1}(d)(x-d)$ for all $x \geq 0$. Combining these two facts, we obtain

$$\begin{aligned} \alpha(G) &\geq 1 + \alpha(H) \\ &\geq 1 + (N - \hat{P})f_{a+1}\left(\frac{Nd - 2\hat{Q}}{N - \hat{P}}\right) \\ &\geq 1 + (N - \hat{P})\left\{f_{a+1}(d) + f'_{a+1}(d)\left(\frac{\hat{P}d - 2\hat{Q}}{N - \hat{P}}\right)\right\} \\ &= 1 + (N - \hat{P})f_{a+1}(d) + (\hat{P}d - 2\hat{Q})f'_{a+1}(d) \\ &= Nf_{a+1}(d) + \hat{R} \\ &\geq Nf_{a+1}(d). \end{aligned}$$

This completes the proof. $\square$

Remark. Like the Turán lower bound, which has discrete form, the Theorem 3.3 has discrete form as

$$\alpha(G) \geq \sum_{v \in V(G)} f_{a+1}(d_v).$$

See Li, Rousseau and Zang (2002), and Li and Zang (2005).

In application, we also need some generalization of Theorem 3.3. The condition that any neighborhood has maximum degree at most m is equivalent that the graph does not contain $B_{m+1} = K_1 + K_{1,m+1}$, which is a star of $m + 1$ edges.

Lemma 3.6 *Let G be a connected graph with $n \geq 3$ vertices. If for all distinct nonadjacent vertices u and v ,*

$$\deg(u) + \deg(v) \geq k,$$

where $k < n$, then G contains a path of length k .

Proof. Let $P : u_0, u_1, \dots, u_m$ be a longest path of length m in G . Since P is a longest path, each of u_0 and u_m is adjacent only to vertices of P .

If u_0 and u_i are adjacent, $1 \leq i \leq m$, then we assume that u_{i-1} and u_m are non-adjacent; for otherwise the cycle

$$C : u_0 u_1 \dots u_{i-1} u_m u_{m-1} \dots u_i u_0$$

of length $m + 1$ is present in G . If $m + 1 = n$, then $m \geq k$, we are done. So the cycle C cannot contain all vertices of G . There exists a vertex w not on C adjacent with a vertex of C ; however, this implies that G contains a path of length $m + 1$, which is impossible. Similarly, u_0 and u_m are non-adjacent in G . Furthermore, for each vertex of $\{u_1, u_2, \dots, u_m\}$ adjacent to u_0 there is a vertex of $\{u_0, u_1, \dots, u_{m-1}\}$ not adjacent with u_m . Thus $\deg(u_m) \leq m - \deg(u_0)$ so that

$$m \geq \deg(u_0) + \deg(u_m) \geq k.$$

□

Corollary 3.2 *Let G be a connected graph with n vertices and minimum degree at least $\frac{k}{2}$, where $k < n$, then G contains a path of length k .*

□

Lemma 3.7 *Let G be a graph of order N that contains no a path of length $k \geq 1$. Then it has at most $\frac{k-1}{2}N$ edges.*

Proof. We fix k and apply induction on N . The assertion is clearly true if $N \leq k$. Assume now that $N > k$ and the assertion holds for smaller values of N .

If G is disconnected, then the induction hypothesis implies the result. Now, if G is connected, then it has a vertex x of degree at most $(k-1)/2$. Thus the number of edges of G is

$$\deg(x) + e(G-x) \leq \frac{k-1}{2} + \frac{k-1}{2}(n-1) = \frac{k-1}{2}n.$$

□

Lemma 3.8 *A graph G with average degree d contains a subgraph in which every vertex has degree (in the subgraph) at least $d/2$.*

Proof. It is trivial for $d \leq 2$, so assume $d > 2$. If the minimum degree of G is at least $d/2$, we are at home. Otherwise, there are some vertex, say v , whose degree is less than $d/2$, delete it from G . Then the average degree of the remaining graph is at least

$$\frac{2}{n-1} \left(\frac{nd}{2} - \frac{d}{2} \right) = d.$$

Repeat Procedure if necessary till we obtain the graph with minimum degree at least $d/2$. □

Lemma 3.9 *A graph G of order N with $e(G) > (m-1)N$ edges contains every tree with m edges.*

Proof. Now our graph has average degree more than $2(m-1)$ so it contains a subgraph, denoted by H , whose minimum degree is more than $m-1$. Let T_{m+1} be a tree of m edges. Suppose that T_k is contained in H that is the subtree of T_{m+1} . If $k < m+1$, there is a vertex, say v , from which we can obtain a larger subtree of T_{m+1} by joining it with any vertex of H out of T_k . This can always be done since $\deg(v) \geq m$. □

Conjecture 3.1 (Erdős-Sós) *A graph G of order N with $e(G) > \frac{m-1}{2}N$ edges contains every tree with m edges.*

Now we generalize our main result in this section. Note the condition that $N(v)$ has a maximum degree at most m controls the number of edges in $N(v)$ which are counted twice in summation $\sum_{wv \in E(G)} \deg(w)$. Now, the condition that $N(v)$ contains no T_{m+1} can do the same thing.

Theorem 3.4 *Let T_{m+1} be a tree with m edges and let G be a graph with N vertices and average degree d . If each neighborhood of G does not contain T_{m+1} , then*

$$\alpha(G) \geq N f_{2m-1}(d).$$

If T_{m+1} is a star or a path, then $f_{2m-1}(d)$ can be replaced by f_m .

Note that if Erdős-Sós conjecture holds, then for any tree T_{m+1} , the function $f_{2m-1}(d)$ can be replaced by f_m . However, by the property of f_m , the difference between $f_m(x)$ and $f_{2m-1}(x)$ is slight.

We conclude this section with a conjecture of Ajtai, Erdős Komlós and Szemerédi(1981), which says that the independence numbers of K_k -free graphs have the lower bound similar with that of triangle-free graphs.

Conjecture 3.2 *Let $k \geq 3$ be a fixed integer. There exists a constant $c = c(k) > 0$ such that if G is a K_k -free graph with order N and average degree $d > 0$, then*

$$\alpha(G) \geq cN \frac{\log d}{d}.$$

Shearer in 1995 proved that $\alpha(G) \geq cN \frac{\log d}{d \log \log d}$ for the graphs described in above conjecture. To confirm the conjecture, a factor $\log \log d$ in the denominator needs to be taken away.

3.4 $r(k, n)$ with Large n

Now we study the asymptotic upper bound for $r(k, n)$ with $k \geq 2$ fixed and $n \rightarrow \infty$.

Recall the function in last section,

$$f_a(x) = \int_0^1 \frac{(1-t)^{1/a} dt}{a + (x-a)t} \geq \frac{\log(x/a) - 1}{x}, \quad (3.3)$$

and for a graph G of order N and average degree d in which each subgraph induced by a neighborhood has a maximum degree at most a , then $\alpha(G) \geq N f_{a+1}(d)$.

Theorem 3.5 *Let $k \geq 2$ be fixed. Then for all large n ,*

$$r(k, n) \leq (1 + o(1)) \frac{n^{k-1}}{(\log n)^{k-2}}.$$

Proof We prove by induction on k . For $k = 2$, it follows the trivial case $r(2, n) = n$.

For the case $k = 3$. Let G be graph of order $N = r(3, n) - 1$ which contains no triangles and $\alpha(G) < n$. Since G is triangle-free, each subgraph induced by a neighborhood is empty thus its average degree is zero. Let d be average degree of G , since each neighborhood is an independence set,

$n > \alpha(G) \geq d$. By inequality that has been just mentioned with noting that $a = 0$,

$$n - 1 \geq N f_1(d) \geq N f_1(n) \geq N \frac{\log n - 1}{n}.$$

Then $r(3, n) - 1 < \frac{n(n-1)}{\log n - 1}$, follows by $r(3, n) \leq (1 + o(1)) \frac{n^2}{\log n}$.

Suppose the statement holds for $2, 3, \dots, k$. We proceed to the induction step. Let G be a graph of order $N = r(k+1, n) - 1$ such that G contains no K_{k+1} and that $\alpha(G) \leq n - 1$. Then for each vertex v of G , we have

- the degree of v is at most $r(k, n) - 1$, and
 - the maximum degree of G_v is at most $r(k-1, n) - 1$,
- where G_v is the subgraph induced by the neighborhood of v .

Thus, it follows that

$$n > \alpha(G) \geq N f_m(r(k, n) - 1) \geq N f_m(r(k, n)), \quad (3.4)$$

where $m = r(k-1, n)$. (Note that both $r(k, n)$ and m go to ∞ as $n \rightarrow \infty$). Now let ϵ be an arbitrary number with $0 < \epsilon < 1$. Then there exists an $M > 0$ such that $f_m(x) > (1 - \epsilon) \log(x/m)/x$ whenever $x/m > M$. Let us decompose the set of *large* natural numbers into $\{n'\}$ and $\{n''\}$ so that

$$\frac{r(k, n')}{r(k-1, n')} > (n')^{1-\epsilon}$$

and

$$\frac{r(k, n'')}{r(k-1, n'')} \leq (n'')^{1-\epsilon}.$$

Thus $\log\{\frac{r(k, n')}{r(k-1, n')}\} \geq (1 - \epsilon) \log n'$. Without loss of generality, we may suppose that for all n' , $(n')^{1-\epsilon} > M$. So, by (3.4), we have

$$n' > N f_m(r(k, n')) \geq (1 - \epsilon) N \frac{\log(r(k, n')/m)}{r(k, n')} \geq \frac{(1 - \epsilon)^2 N \log n'}{r(k, n')},$$

where $m = r(k-1, n')$. Hence $N \leq \frac{n'}{(1-\epsilon)^2} \frac{r(k, n')}{\log n'}$, and the desired inequality for $r(k+1, n')$ follows from the induction hypothesis on $r(k, n')$. Recall the corollary in the last section, $f_m(x) \geq 1/(1+x)$ if $x \geq m$. So

$$n'' > N f_m(r(k, n'')) \geq N/(1 + r(k, n'')),$$

where $m = r(k-1, n'') < r(k, n'')$. Hence

$$N \leq n''[1 + r(k, n'')] \leq n''[1 + (n'')^{1-\epsilon} r(k-1, n'')].$$

The desired inequality for $r(k+1, n'')$ follows from the induction hypothesis on $r(k-1, n'')$ since $(n'')^{2-\epsilon} < (n''/\log n'')^2$ for large n'' . This completes the proof the theorem. $\square$

From the Chapter 2,

$$r(K_{1,\ell}, K_n) = r(K_1 + \ell K_1, K_n) = \ell(n-1) + 1 \sim \ell n.$$

Starting at this result similarly with the proof of above Theorem we can obtain the following result.

Theorem 3.6 *For any fixed $k, \ell \geq 1$*

$$r(K_k + \ell K_1, K_n) \leq (\ell + o(1)) \frac{n^k}{(\log n)^{k-1}}$$

as $n \rightarrow \infty$. Furthermore

$$r(K_k + n K_1, K_n) \leq (1 + o(1)) \frac{n^{k+1}}{(\log n)^{k-1}}.$$

Bollobás and Erdős asked whether or not $r(K_k + K_{1,\ell}, K_n)$ really grows linearly with ℓ . We cannot solve this problem. However, we will prove

$$r(K_k + n K_1, K_n) \geq c \frac{n^{k+1}}{(\log n)^k}$$

in a later chapter. So the second upper bound in above theorem is very close to the right order of magnitude.

Let us propose a problem which is a special form of a conjecture of Erdős', see Alon and Spencer (1992).

Conjecture 3.3 *Prove the order of $r(k, n)$ is $\frac{n^{k-1}}{(\log n)^{k-2}}$ as $k \geq 2$ fixed and $n \rightarrow \infty$.*

We will show that $r(k, n) \geq c(k)(n/\log n)^{(k+1)/2}$. For $r(3, n)$ Kim showed that $r(3, n) \geq cn^2/\log n$, thus the above conjecture is true for $k = 3$. Further more, the upper bound $n^2/\log n$ of $r(3, n)$ may be very close to the truth. The comparison of the value $r(3, n)$ and $n^2/\log n$ is in the following table.

n	3	4	5	6	7	8	9	10	11	12
$r(3, n)$	6	9	14	18	23	28	36	40/43	46/51	52/59
$n^2/\log n$	8.2	11.5	15.5	20.1	25.2	30.8	36.9	43.4	50.5	57.9
Error(%)	36.5	28.2	11.0	11.6	9.5	9.9	2.4			

Table 1 Values $r(3, n)$ and $n^2/\log n$

Problem 3.1 Prove or disprove that the asymptotic formula of $r(3, n)$ is $n^2 / \log n$.

Exercises

3.1 Show if G is a graph of order N and average degree d , then

$$\alpha(G) \geq (1 - d/2)N.$$

(This is slightly stronger than Turán bound for $d < 1$.)

3.2 It is well known that a linear first-order differential equation $\frac{dy}{dx} = p(x)y + q(x)$ has a unique solution

$$y = e^{\phi(x)} \left(y_0 + \int_{x_0}^x q(t)e^{-\phi(t)} dt \right)$$

satisfying $y_0 = y(x_0)$, where $\phi(x) = \int_{x_0}^x p(t)dt$. Solve the equation in Lemma 3.3 this way, try to show that the function is decreasing and convex. Explain why we shall adopt the form of the solution as an integral on parameter.

3.3 Show that $r(3, n) < n^2 / \log(n/e)$.

3.4 Using Kim's result that $r(3, n) \geq cn^2 / \log n$, show that $r(4, n) \leq (1 + o(1))r(3, n)n / \log n$.

3.5 Complete the proof of Theorem 3.6.

3.6 Prove that there exists some constant $c > 0$ such that for any regular graph G on n vertices with girth at least 6, then its independence number is at least $c(n \log n)^{2/3}$.

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Chapter 4

Basic Probabilistic Method

4.1 Random Graphs

Probabilistic method is a powerful tool in Ramsey theory. In fact, the graph Ramsey theory is the birthplace of random graphs. The main reason is that there are many problems in Ramsey theory that traditional combinatorial methods do not work well. The standard texts on random graphs, we refer the books *Random Graphs* by Bollobás (2001), *Probabilistic Method* by Alon and Spencer (1998), and *Random Graphs* by Janson, Łuczak, and Ruciński. Random graphs began with some sporadic papers of Erdős in the 1940s and 1950s, in which he used random methods to show the existence of graphs with seemingly contradictory properties. A paper “On the Evolution of Random Graphs” of Erdős and Rényi in 1960 was very important for the development of the theory of random graphs. Among all these results, Erdős gave an exponential lower bound for the diagonal Ramsey number $r(n, n)$. Today random methods have been used in many other areas.

Every probability space whose points are graphs gives a notion of a random graph. For a family of graphs $\mathcal{G} = \{G_1, G_2, \dots\}$ with probabilities $\Pr(G_i)$ such that $0 \leq \Pr(G_i) \leq 1$ and $\sum_{i \geq 1} \Pr(G_i) = 1$, we have a probability space of random graphs. Each G_i is called a random graph of $\mathcal{G}$ with probability $\Pr(G_i)$. We shall consider the probability space consists of graphs on a fixed set $V = [n] = \{1, 2, \dots, n\}$, where the vertices in V are *distinguishable*, so edges on V are distinguishable, too. Note that the

complete graph K_n on set $V = [n]$ has

$$\binom{n}{1} + \binom{n}{2}2 + \cdots + \binom{n}{k}2^{\binom{k}{2}} + \cdots + \binom{n}{n}2^{\binom{n}{2}}$$

subgraphs. The general term corresponds the subgraphs that have exactly k vertices, and the last term $2^{\binom{n}{2}}$ corresponds all spanning subgraphs.

Let us label all edges of K_n on vertex set $[n]$ as $e_1, e_2, \dots, e_m$, where $m = \binom{n}{2}$. Note the number of graphs on vertex set $[n]$ is 2^m since the edges are distinguishable. The space $\mathcal{G}(n; p_1, \dots, p_m)$ is defined for $0 \leq p_i \leq 1$ as follows. To get a random element of this space, one selects the i th edge e_i independently, with probability p_i . Putting it another way, the ground set of $\mathcal{G}(n; p_1, \dots, p_m)$ is the set of all 2^m graphs on $V = [n]$, and the probability of a graph H with vertex set $V(H) = [n]$ and edge set $E(H) = \{e_j : j \in S\}$ is

$$(\prod_{j \in S} p_j) (\prod_{j \notin S} (1 - p_j)),$$

where $S \subseteq \{1, \dots, m\}$ is the index set of edges of H . That is to say, each of the edges of H has to be selected and none of the edges of $\overline{H}$ is allowed to be selected. Write $q_j = 1 - p_j$ and $G(p_1, \dots, p_m)$ for a random element in $\mathcal{G}(n; p_1, \dots, p_m)$, then

$$\Pr(G(p_1, \dots, p_m) = H) = (\prod_{j \in S} p_j) (\prod_{j \notin S} q_j).$$

Since the vertices (and edges) are distinguishable, $G(p_1, \dots, p_n) = H$ is not equivalent to that $G(p_1, \dots, p_n)$ is isomorphic to H . To see that $\mathcal{G}(n; p_1, \dots, p_n)$ is true a probability space, let us verify that

$$\begin{aligned} \sum_H \Pr(G(p_1, \dots, p_n) = H) &= \sum_{S \subseteq [m]} (\prod_{j \in S} p_j) (\prod_{j \notin S} q_j) \\ &= \prod_{j=1}^m (p_j + q_j) \\ &= 1. \end{aligned}$$

We shall concentrate on the case $p_1 = p_2 = \dots = p_m = p$, for which the probability space $\mathcal{G}(n; p_1, \dots, p_m)$ is written as $\mathcal{G}(n, p)$.

In $\mathcal{G}(n, p)$ the probability of a graph H on $[n]$ with k edges is $p^k(1 - p)^{m-k}$: each of the k edges of H has to be selected and none of the edges of $\overline{H}$ is allowed to be selected. Write G_p for a random element of $\mathcal{G}(n, p)$, then

$$\Pr(G_p = H) = p^{e(H)} (1 - p)^{m - e(H)}.$$

In the space $\mathcal{G}(n, 0)$, the probability that the empty graph $\overline{K_n}$ appears is one, and the probability that any other graph appears is zero. Similarly, in

the space $\mathcal{G}(n, 1)$, the only graph that appears is K_n . Other than these two extremal cases, $0 < p < 1$, any graph on vertex set $[n]$ can appear with a positive probability. As p increases from 0 to 1, random graph G_p evolves from empty to full.

In their original paper in 1960, Erdős and Rényi let $\mathcal{G}(n, e)$ be the random graph with vertex set $V = [n]$ and precisely e edges. For $0 \leq e \leq m = \binom{n}{2}$ with e fixed, the space $\mathcal{G}(n, e)$ consists of all $\binom{m}{e}$ spanning subgraphs with exactly e edges: which can be turned into a probability space by taking its elements to be equiprobable. Thus, write G_e for a random graph in the space $\mathcal{G}(n, e)$, for a fixed graph H on vertex set $[n]$, then

$$\Pr[G_e = H] = \binom{m}{e}^{-1},$$

where the event $G_e = H$ means that G_e is precisely H , but not isomorphic to H in general.

It is interesting, as expected, that for $e \sim p\binom{n}{2}$ the spaces $\mathcal{G}(n, e)$ and $\mathcal{G}(n, p)$ are close to each other. In most proofs for existence, the calculations are easier in $\mathcal{G}(n, p)$ than in $\mathcal{G}(n, e)$. So we will work on the probability model $\mathcal{G}(n, p)$ exclusively.

Note Another point of view may be convenient in which one colors all edges of the complete graph K_n with probability p , randomly and independently. Thus random graph G_p is viewed as a random coloring of edge set of K_n . The coloring of edge set of K_n is also said a coloring of K_n in short. Recalling the definition of Ramsey numbers, we see why the relation between random method and Ramsey theory is so natural and tight.

In Ramsey theory, we need to consider the probability that a given subgraph appears. Let F be a given graph on k vertices, and let $S \subseteq [n]$ with $|S| = k$. The event $\cup_S A_S$ signify that F appears in G_p as a subgraph, its probability is hard to calculate since the events A_S have a complex interaction. It is often to bound above this probability by the expectation of the number of copies of F in the random graph. To get the expectation, let us look the number of copies of F in K_k first. This is closely related to the automorphism group of F . Recall a permutation (or a bijection) ϕ of $V(F)$ is called an *automorphism* of F when $uv \in E(F)$ if and only if $\phi(u)\phi(v) \in E(F)$. It is straightforward to verify the set of all automorphisms of F forms a group under the operation of composition. This group is called the automorphism group of F , denoted by $\mathcal{A}(F)$. For example, $\mathcal{A}(K_k)$ is the symmetric group S_k of order $k!$, and $\mathcal{A}(C_k)$ is the

dihedral group D_k of order $2k$. The following results are simple consequence of the definitions.

- For any graph F and its complement $\overline{F}$, $\mathcal{A}(F) = \mathcal{A}(\overline{F})$.
- For any graph F on k vertices, the order of $\mathcal{A}(F)$ is a divisor of $k!$, and it is $k!$ if and only if F is isomorphic to K_k or $\overline{K}_k$.

In random graph space, the graphs are labelled. Two graphs F_1 and F_2 are called *identical* if $V(F_1) = V(F_2)$ and $E(F_1) = E(F_2)$. Clearly, identical graphs are isomorphic, but the inverse statement is not generally true. For example, given a labelling of a star $K_{1,k}$ as $V = \{u, v_1, \dots, v_k\}$ with the center u and $E = \{uv_1, \dots, uv_k\}$. Exchanging the labels of any pair vertices v_i and v_j will yield identical graphs.

Theorem 4.1 *Let F be a graph of order k . Then the number of copies of F from a set of k labels such that no two resulting graphs are identical is $k!/|\mathcal{A}(F)|$.*

Proof. Let $\{v_1, v_2, \dots, v_k\}$ be the set of labels. Certainly there are $k!$ labelings of F from this set of labels with the number of resulting labelled graphs that may be identical. Let $F_1, F_2, \dots, F_{k!}$ be the labelled graphs obtained from F , the relation “ F_i is identical to F_j ” is an equivalence relation. For a given labelled graph F_i , each automorphism gives rise to a labelled graph that is identical to F_i , and conversely. Hence each equivalence class so determined contains $|\mathcal{A}(F)|$ elements, thus implying that there are $k!/|\mathcal{A}(F)|$ equivalent classes in all. This proves the theorem. $\square$

In random graph space $\mathcal{G}(n, p)$, we need to consider the number of copies of F in a labelled complete graph.

Corollary 4.1 *Let F be a graph of order k . Then the number of copies of F in a labelled complete graph of order k is $k!/|\mathcal{A}(F)|$.*

Now let $S \subseteq [n]$ with $|S| = k$ and let X_S be the number of copies of F on S . Then $X = \sum_S X_S$ is the number of copies of F in G_p . Then

$$E(X_S) = \frac{k!}{|\mathcal{A}(F)|} p^{e(F)},$$

and

$$E(X) = \binom{n}{k} \frac{k!}{|\mathcal{A}(F)|} p^{e(F)} = \frac{(n)_k}{|\mathcal{A}(F)|} p^{e(F)},$$

where $(n)_k = n(n-1) \cdots (n-k+1)$ is the falling factorial.

Similar formulas hold for the number of *induced* subgraphs. Let Y be the number of induced F in G_p , then

$$E(Y) = \frac{\binom{n}{k}}{|\mathcal{A}(F)|} p^{e(F)} q^{\binom{k}{2} - e(F)}.$$

Let A_S signify the event that the subgraph induced by S in G_p contains F as a subgraph, then

$$\Pr(A_S) \leq \frac{k!}{|\mathcal{A}(F)|} p^{e(F)}.$$

Hence

$$\Pr(F \subset G_p) = \Pr(\cup A_S) \leq \binom{n}{k} \frac{k!}{|\mathcal{A}(F)|} p^{e(F)} = \frac{\binom{n}{k}}{|\mathcal{A}(F)|} p^{e(F)}, \quad (4.1)$$

where the upper bound is exactly $E(X)$. This can be seen by the fact that X takes only nonnegative integral values and

$$\begin{aligned} \Pr(\cup A_S) &= \Pr(X \geq 1) = \sum_{i \geq 1} \Pr(X = i) \\ &\leq \sum_{i \geq 1} i \Pr(X = i) = E(X). \end{aligned}$$

It seems to be necessary to point out that F is not a random element in $\mathcal{G}(n, p)$ and the above discussion is about appearance of F as a subgraph.

It is worth remarking that $p = p(n)$ is often a function. The space $\mathcal{G}(n, p)$ is of great interest for fixed values of p as well; in particular, $\mathcal{G}(n, 1/2)$ could be viewed as the space of order n : it consists of all 2^m graphs on $[n]$, and the probability of any graph is equiprobable. This is just a classical probability space. Thus $G_{n, 1/2}$ is also obtained by picking any of the 2^m graphs on $[n]$ at random with probability 2^{-m} . No matter how p is fixed or a function, we tend to be interested in what happens as $n \rightarrow \infty$.

Now we have obtained a space of random graphs, every graph invariant becomes a random variable; the nature of such a random variable depends crucially on the space. For instant, the number $X_k(G)$ of complete graphs of order k in G is a random variable on our space of random graphs.

To be proficient in the probabilistic method one must have a feeling for asymptotic calculation. For the sake of convenience, we state some simple inequalities that will be used in the calculations. The following precise formula is called *Stirling formula*.

Lemma 4.1 For all $n \geq 1$,

$$n! = \sqrt{2\pi n} \left(\frac{n}{e}\right)^n \exp\left\{\frac{1}{12n + \theta}\right\},$$

where $0 < \theta = \theta_n < 1$. Thus

$$\sqrt{2\pi n} \left(\frac{n}{e}\right)^n < n! < \sqrt{2\pi n} \left(\frac{n}{e}\right)^n e^{1/12n},$$

and

$$n! = (1 + o(1))\sqrt{2\pi n} \left(\frac{n}{e}\right)^n > \left(\frac{n}{e}\right)^n.$$

□

Lemma 4.2 For any positive integers $N \geq n$,

$$\binom{N}{n} \leq \frac{N^n}{n!} \leq \left(\frac{eN}{n}\right)^n.$$

If $N = N(n)$ and $n = o(N)$, then

$$\binom{N}{n}^{1/n} \sim \frac{eN}{n}$$

as $n \rightarrow \infty$.

Proof. The first two inequalities are immediate, which implies an upper bound $\binom{N}{n}^{1/n} \leq \frac{eN}{n}$. We need a lower bound for the asymptotical equality. Then the Stirling's formula gives

$$\begin{aligned} \binom{N}{n} &= \frac{N!}{n!(N-n)!} \\ &\sim \frac{\sqrt{2\pi N}}{\sqrt{2\pi n}\sqrt{2\pi(N-n)}} \left(\frac{N}{N-n}\right)^N \left(\frac{N-n}{n}\right)^n \\ &\sim \frac{1}{\sqrt{2\pi n}} \left(\frac{N}{N-n}\right)^N \left(\frac{N-n}{n}\right)^n, \end{aligned}$$

where we use the condition $n = o(N)$. Also we have

$$\left(\frac{N}{N-n}\right)^N = \left(1 + \frac{n}{N-n}\right)^N \geq (e - o(1))^n,$$

hence

$$\binom{N}{n}^{1/n} \geq (e - o(1)) \frac{N-n}{n} \sim \frac{eN}{n}$$

as desired. $\square$

The following simple fact from calculus is often used.

Lemma 4.3 For any $0 \leq x \leq 1$ and $k \geq 0$,

$$(1 - x)^k \leq e^{-kx}.$$

We have obtained an upper bound of $r(n, n)$ in Chapter 1. This and Stirling's formula give

$$r(n, n) \leq \binom{2n-2}{n-1} = (1 + o(1)) \frac{4^{n-1}}{\sqrt{\pi n}}.$$

Hence

$$\overline{\lim}_{n \rightarrow \infty} r(n, n)^{1/n} \leq 4.$$

We do not have any general lower bound for $r(n, n)$ yet.

4.2 Elementary Examples

In the original proof of the exponent lower bound for $r(n, n)$ in 1947, Erdős did not use the formal probabilistic language. So his paper has been considered as an informal starting point of random graphs. But in two papers published in 1959 and 1961, in which he gave a lower bound $c(n/\log n)^2$ for $r(3, n)$, he even wrote probabilities in the titles.

Theorem 4.2 As $n \rightarrow \infty$,

$$r(n, n) \geq (1 - o(1)) \frac{n}{e\sqrt{2}} 2^{n/2}.$$

Proof. Consider the random graphs in $\mathcal{G}(N, 1/2)$, or color K_N randomly and independently with probability $p = 1/2$.

Let S be a set of n vertices. Let A_S be the event that S is monochromatic. Then

$$\Pr[A_S] = 2 \left(\frac{1}{2} \right)^{\binom{n}{2}} = 2^{1 - \binom{n}{2}},$$

as for S to hold all $\binom{n}{2}$ edges must be colored the same. Consider the event $\bigcup A_S$ over all n -set on $[N]$. It is difficult to find value for $\Pr[\bigcup A_S]$ as A_S may have a complex interaction. We use the simple fact that the

probability of a disjunction is at most the sum of the probability of the events. Thus

$$\Pr[\cup A_S] \leq \sum \Pr[A_S] = \binom{N}{n} 2^{1-\binom{n}{2}}.$$

If this is less than one, then event $B = \bigcap \overline{A_S}$ has positive probability. Therefore B is not the null event. Thus there is a point in the probability space for which B holds. But a point in the probability space is precisely a coloring of the edges of K_N . And the event B is precisely that under this coloring there is no monochromatic K_n . Hence $r(n, n) > N$.

We need to find the maximum possible N such that

$$\binom{N}{n} 2^{1-\binom{n}{2}} < \left(\frac{eN}{n} \right)^n \frac{2}{2^{n(n-1)/2}} = 2 \left(\frac{e\sqrt{2}N}{n2^{n/2}} \right)^n < 1,$$

only need

$$\frac{e\sqrt{2}N}{n2^{n/2}} < 1.$$

For any $\epsilon > 0$, if n is sufficiently large, the possible N that ensures the above inequality would be

$$N = \left\lfloor \left(1 - \frac{\epsilon}{2}\right) \frac{n}{e\sqrt{2}} 2^{n/2} \right\rfloor.$$

Therefore $r(n, n) > N > (1 - \epsilon) \frac{n}{e\sqrt{2}} 2^{n/2}$, that is

$$r(n, n) \geq (1 - o(1)) \frac{n}{e\sqrt{2}} 2^{n/2},$$

completing the proof. $\square$

Theorem 4.3 *Let m, n and N be positive integers. If for some $0 < p < 1$,*

$$\binom{N}{m} p^{\binom{m}{2}} + \binom{N}{n} (1-p)^{\binom{n}{2}} < 1,$$

then $r(m, n) > N$.

Proof. Consider random graphs G_p in $\mathcal{G}(N, p)$. Let S be a set of m vertices and let A_S be the event that S induces a complete graph. Then $\Pr[A_S] = p^{\binom{m}{2}}$, and

$$\Pr[\cup A_S] \leq \sum \Pr[A_S] = \binom{N}{m} p^{\binom{m}{2}}.$$

Let T be a set of n vertices and let B_T be the event that T induces an independent set. Then

$$\Pr[\cup B_S] \leq \sum \Pr[B_S] = \binom{N}{n} p^{\binom{n}{2}}.$$

Thus

$$\Pr[(\cup A_S) \cup (\cup S_T)] < 1.$$

So there exists a graph on N vertices such that there is neither an induced K_m nor an induced $\overline{K_n}$, thus $r(m, n) > N$. $\square$

The above result is ineffective in bounding $r(3, n)$. We now examine the lower bound of $r(4, n)$. We shall give details in calculation for choosing a suitable value of p and N as large as possible for large n . To make the condition in Theorem 4.3 satisfied, we roughly estimate $\binom{N}{n}$ as $(eN/n)^n$, and $(1-p)^{\binom{n}{2}}$ as $e^{-p\binom{n}{2}}$, hence $\binom{N}{n}(1-p)^{\binom{n}{2}}$ as

$$\left(\frac{eN}{n}\right)^n \exp\left\{-p\binom{n}{2}\right\} = \left(\frac{eN}{ne^{p(n-1)/2}}\right)^n.$$

We have known that $r(4, n) \leq (1 + o(1))n^3/(\log n)^2$, thus $e^{p(n-1)/2} = n^{a+o(1)}$ for some constant a , so we take $p = (c_1 \log n)/(n-1)$. Then

$$\binom{N}{4} p^6 \sim \frac{1}{24} N^4 \left(\frac{c_1 \log n}{n}\right)^6 \sim \frac{c_1^6}{24} \left(N \left(\frac{\log n}{n}\right)^{3/2}\right)^4 < 1,$$

so $N \sim c_2(n/\log n)^{3/2}$ for some constant c_2 .

Formally, we let $p = (c_1 \log n)/(n-1)$ and $N = \lfloor c_2(n/\log n)^{3/2} \rfloor$, where c_1 and c_2 are positive constants to be chosen satisfying that $c_1^6 c_2^4 < 24$. Then

$$\binom{N}{4} p^6 < \frac{N^4}{24} p^6 \leq \frac{c_1^6 c_2^4}{24} \left(\frac{n}{n-1}\right)^6 \leq c_3 < 1$$

for large n , where c_3 is a constant. For the second term, we estimate that

$$\binom{N}{n} < \left(\frac{eN}{n}\right)^n \quad \text{and} \quad 1-p < e^{-p},$$

thus $(1-p)^{\binom{n}{2}} < e^{-pn(n-1)/2} = n^{-c_1 n/2}$ and

$$\binom{N}{n} (1-p)^{\binom{n}{2}} < \left(\frac{eN}{n}\right)^n n^{-c_1 n/2} = \left(\frac{eN}{n^{1+c_1/2}}\right)^n.$$

In order to make the above tending to zero, we have to take $c_1 \geq 1$. On the other hand, in order to take c_2 as large as possible with $c_1^6 c_2^4 < 24$, we have to take c_1 as small as possible. So it has to be $c_1 = 1$.

Now, we may hope to optimize the constant c_2 . Since we need only $c_2 < 24^{1/4}$, so $c_2 = 24^{1/4} - \epsilon$ will be ok. Thus we have

$$r(4, n) \geq (24^{1/4} - o(1)) \left(\frac{n}{\log n} \right)^{3/2}.$$

Hereafter we will choose p with some foresight. For general $m \geq 4$, by taking $p = (m-3) \log n / (n-1)$, the similar calculation yields

$$r(m, n) \geq c \left(\frac{n}{\log n} \right)^{(m-1)/2}.$$

Note It is often to replace $\log n / (n-1)$ in the expression of p by $\log n / n$. Strictly speaking, only if $f(n) - g(n) \rightarrow 0$, we have $e^{f(n)} \sim e^{g(n)}$. But the replacement works. When we use $p = \log n / n$ in above calculation for the lower bound of $r(4, n)$,

$$\binom{N}{n} (1-p)^{\binom{n}{2}} < \left(\frac{eN}{n} \right)^n n^{-(n-1)/2} = \left(\frac{eN}{n^{3/2}} \right)^n n^{1/2} \rightarrow 0.$$

Also, when $N = \lfloor f(n) \rfloor$ or $N = \lceil f(n) \rceil$ and $f(n) \rightarrow \infty$, we often ignore the fact that $f(n)$ may not be an integer by simply taking $N = f(n)$ if this does not affect the proof essentially.

We have seen that the property of G_p is sensitive with the value of p . To ensure that G_p contains no K_m (with a positive probability, more precisely, or $\binom{N}{m} p^{\binom{m}{2}}$ is small), it is better to take smaller p . But it is better to take a bigger p to ensure that there is no induced $\overline{K_n}$ (i.e., $\binom{N}{n} (1-p)^{\binom{n}{2}}$ is small). Our task is to balance both sides to obtain a larger N as possible.

We shall improve the obtained lower bounds for $r(n, n)$ and $r(m, n)$ by the proofs so called *deletion method*.

Theorem 4.4 As $n \rightarrow \infty$,

$$r(n, n) \geq (1 - o(1)) \frac{n}{e} 2^{n/2}.$$

Proof. Consider the random graphs in $\mathcal{G}(N, 1/2)$. Let X be the number of clique or independent set of size n . Then

$$X = \sum X_S,$$

the sum over all n -set S , where X_S is the indicator random variable of the event A_S that S is a clique or independent. That is

$$X_S = \begin{cases} 1 & \text{if } S \text{ induces } K_n \text{ or } \overline{K}_n \\ 0 & \text{otherwise.} \end{cases}$$

Therefore

$$E[X_S] = Pr[A_S] = 2 \left(\frac{1}{2} \right)^{\binom{n}{2}}.$$

By linearity of expectation

$$E[X] = \sum E[X_S] = \binom{N}{n} 2^{1-\binom{n}{2}}.$$

There is a point in the probability space for which X does not exceed its expectation. That is, there is a graph with at most

$$\binom{N}{n} 2^{1-\binom{n}{2}}$$

S that induces a K_n or a $\overline{K}_n$. Fix that graph. For each such S select a point $x \in S$ arbitrarily and delete it from the vertex set. The remaining point V^* have neither K_n nor $\overline{K}_n$. Thus

$$r(n, n) \geq |V^*| \geq N - \binom{N}{n} 2^{1-\binom{n}{2}}.$$

The remaining proof is to find N such that $|V^*|$ as large as possible. By taking $N = \frac{n2^{n/2}}{e}$, then

$$\binom{N}{n} 2^{1-\binom{n}{2}} \leq \left(\frac{eN}{n} \right)^n \frac{2}{2^{n(n-1)/2}} = \left(\frac{e\sqrt{2}N}{n2^{n/2}} \right)^n \leq 2^{n/2}$$

which is $o(N)$. Thus

$$r(n, n) \geq |V^*| \geq N - o(N) = (1 - o(1))N = (1 - o(1)) \frac{n}{e} 2^{n/2}$$

completing the proof. $\square$

Theorem 4.5 *For any positive integer m, n and N , and any real number $0 < p < 1$,*

$$r(m, n) > N - \binom{N}{m} p^{\binom{m}{2}} - \binom{N}{n} (1-p)^{\binom{n}{2}}.$$

Proof. Note that in G_p in $\mathcal{G}(N, p)$ the expectation of the number of clique of size m is $\binom{N}{m}p^{\binom{m}{2}}$, and expectation of number of independent sets of size n is $\binom{N}{n}(1-p)^{\binom{n}{2}}$. By the same argument in the proof for Theorem 4.4 gives the required lower bound. $\square$.

Theorem 4.6 *Let integer $m \geq 3$ be fixed. Then for some constant $c = c(m) > 0$,*

$$r(m, n) \geq c \left(\frac{n}{\log n} \right)^{m/2}$$

for all large n .

Proof. Setting $N = a(n/\log n)^{m/2}$ and $p = (m-2)\log n/(n-1)$ such that $a - \frac{(m-2)\binom{m}{2}}{m!}a^m > 0$, this is possible since $a^m = o(a)$ as $a \rightarrow 0^+$. Then

$$N_1 = \binom{N}{m}p^{\binom{m}{2}} \sim \frac{(m-2)\binom{m}{2}a^m}{m!} \left(\frac{n}{\log n} \right)^{m/2},$$

and

$$N_2 = \binom{N}{n}(1-p)^{\binom{n}{2}} < \left(\frac{eN}{n} \right)^n e^{-pn(n-1)/2} = \left(\frac{eN}{n^{m/2}} \right)^n \rightarrow 0.$$

So if $c < a - \frac{(m-2)\binom{m}{2}a^m}{m!}$

$$r(m, n) \geq N - N_1 - N_2 > c \left(\frac{n}{\log n} \right)^{m/2}.$$

To get the constant c optimal, let c_m be the maximum value of $a - (m-2)\binom{m}{2}a^m/m!$, then $r(m, n) \geq (c_m - o(1))(n/\log n)^{m/2}$, where $c_3 = \sqrt{2}/3$ and $c_4 = \frac{3}{4}(3/32)^{1/3}$. $\square$

Similarly,

$$r(C_m, K_n) > N - (m-1)! \binom{N}{m} p^m - \binom{N}{n} (1-p)^{\binom{n}{2}}.$$

In fact, we have the following lower bound for $r(F, K_n)$.

Theorem 4.7 *Let F be a graph with $m \geq 3$ vertices and $e(F) \geq m$ edges. Then*

$$r(F, K_n) \geq c \left(\frac{n}{\log n} \right)^{e(F)/(m-1)},$$

for all $n \geq 1$, where $c = c(F) > 0$ is a constant. In particular,

$$r(C_m, K_n) \geq c \left(\frac{n}{\log n} \right)^{m/(m-1)}.$$

Proof. We just need to find such constant $c > 0$ that the lower bound holds for all large n . So we always assume that n is sufficiently large. Set $s = e(F)$, and

$$N = a \left(\frac{n}{\log n} \right)^{s/(m-1)} \quad \text{and} \quad p = 2 \left(\frac{s}{m-1} - 1 \right) \frac{\log n}{n-1},$$

where a is a positive constant satisfying

$$c = \frac{1}{2} \left[a - 2^s a^m \left(\frac{s}{m-1} - 1 \right)^s \right] > 0.$$

This is possible since $a^m = o(a)$ as $a \rightarrow 0^+$. In a random graph $G_p \in \mathcal{G}(N, p)$, let N_1 and N_2 be the expected numbers of F and induced $\overline{K}_n$, respectively. Then

$$\begin{aligned} N_1 &\leq m! \binom{N}{m} p^s \sim N^m p^s \\ &\sim 2^s a^m \left(\frac{s}{m-1} - 1 \right)^s \left(\frac{n}{\log n} \right)^{s/(m-1)}, \end{aligned}$$

and

$$\begin{aligned} N_2 &= \binom{N}{n} (1-p)^{\binom{n}{2}} < \left(\frac{eN}{n} \right)^n e^{-pn(n-1)/2} \\ &= \left(\frac{eN}{n^{s/(m-1)}} \right)^n \rightarrow 0. \end{aligned}$$

There exists a graph on N vertices from which we can delete at most $N_1 + N_2$ vertices to get a graph containing no F and no induced $\overline{K}_n$. Thus

$$\begin{aligned} r(F, K_n) &> N - N_1 - N_2 \\ &\geq (2c - o(1)) \left(\frac{n}{\log n} \right)^{s/(m-1)} \\ &> c \left(\frac{n}{\log n} \right)^{s/(m-1)} \end{aligned}$$

for all large n , as desired. $\square$

Note that the order of the lower bound in above theorem is higher than linear, together with Chvátal linear formula of $r(T_m, K_n)$ in Chapter 2, we have the following conclusion.

Corollary 4.2 *For a fixed connected graph G , $r(G, K_n)$ can have a linear upper bound if and only if G is tree.*

Let A be a property of graphs. When the probability that graphs holding A appear in $\mathcal{G}(N, p)$ tends to one, we say that *almost all graphs* have the property A , or *almost surely*, all graphs in $\mathcal{G}(N, p)$ have property A . In Theorem 4.2, we in fact proved the following result.

Theorem 4.8 *Let $\epsilon > 0$ and $N = \left\lfloor \frac{(1-\epsilon)n}{e\sqrt{2}} 2^{n/2} \right\rfloor$. Then almost all graphs G in $\mathcal{G}(N, 1/2)$ satisfy that $\omega(G) < n$ and $\alpha(G) < n$.*

That is to say, all most graphs in $\mathcal{G}(N, 1/2)$ yield the lower bound $r(n, n) \geq (1 - o(1)) \frac{n}{e\sqrt{2}} 2^{n/2}$. Can we construct one (in fact, one family of graphs)? Unfortunately, nobody can give a constructive proof of a lower bound like $r(n, n) \geq a^n$ for any constant $a > 1$. This reveals that finite discrete structures are much more complicated than they seem to be. The graphs of large order we can image must have some uniform description, but typical random graphs do not have such description. This is not very unusual cases. Most real numbers that we can write down are rational, which are recycling. However, almost all real numbers are of irrational.

We have a brief remark on “almost all” arguments in the proofs of several theorems in this section. They are in fact *weighted counting arguments*. The probability space we are dealing is finite, which can be defined as a finite set Ω together with nonnegative weights on the elements that sum to 1. An event A is a subset of Ω . The probability $\Pr(A)$ is the sum of the weights of the elements of A . In particular, in the space $\mathcal{G}(N, p)$, the graphs with around $p\binom{N}{2}$ edges have larger weights.

In this chapter we have

$$\sqrt{2} \leq \underline{\lim} r(n, n)^{1/n} \leq \overline{\lim} r(n, n)^{1/n} \leq 4.$$

There is a big gap between the lower bound and the upper bound. It is likely that in fact, $\lim r(n, n)^{1/n} = 2$, but both the lower bound $\sqrt{2}$ and the upper bound 4 has been standing for more than half century without improvement.

4.3 Label Vertices Randomly

We have proved in Chapter 3 that for any graph $G = (V, E)$,

$$\alpha(G) \geq \sum_{v \in V} \frac{1}{1 + d(v)}. \quad (4.2)$$

This result has an elegant probabilistic proof due to Alon and Spencer (1992) as follows.

Proof of (4.2). Label vertices in $V(G)$ randomly. Define a set

$$I = \{v \in V : \ell(v) < \ell(w) \text{ for any } w \in N(v)\},$$

where $\ell(v)$ is the label of v . Note that I is a random set. Let X_v be the indicator random variable for $v \in I$ and $X = \sum_{v \in V} X_v$, clearly $X = |I|$. For each v ,

$$E(X_v) = Pr[v \in I] = \frac{1}{1 + d(v)},$$

since $v \in I$ if and only if v is the least element among v and its neighbors. Hence

$$E(X) = \sum_{v \in V} \frac{1}{1 + d(v)}$$

and so there exists a specific ordering $<$ with

$$|I| \geq \sum_{v \in V} \frac{1}{1 + d(v)}.$$

It is clear that I is an independent set thus $\alpha(G) \geq |I|$. $\square$

Shearer (1998) extended the idea in the above proof to obtain a result as follow. Let $m \geq 2$ be a fixed integer, then there exists a constant $c = c(m) > 0$ such that if G is a graph that contains no $C_3, C_5, \dots, C_{2m+1}$, then

$$\alpha(G) \geq c \left(\sum_{v \in V} d(v)^{1/(m-1)} \right)^{(m-1)/m}.$$

However, we need such result that can apply to graphs that have only one forbidden odd cycle.

Let us introduce some notions before presenting results. Given a graph $G = (V, E)$, set

$$N_i(v) = \{w \in V : d(w, v) = i\},$$

which is the set of all the vertices of distance i from vertex v in G , and set $d_i(v) = |N_i(v)|$. Thus $d_1(v) = d(v)$. We do not distinguish the subset $N_i(v)$ and the subgraph of G induced by $N_i(v)$ when there is no danger of confusion. The graph G is called (m, k) -colorable if $N_i(v)$ is k -colorable for any vertex v and any $i \leq m$, that is, there is an assignment of k colors on vertices of $N_i(v)$ so that no two adjacent vertices receive the same color.

Lemma 4.4 *Let $G = (V, E)$ be a $(1, k)$ -colorable graph. Then*

$$\alpha(G) \geq \frac{1}{k} \sum_{v \in V} \frac{d_1(v)}{1 + d_1(v) + d_2(v)}.$$

Proof. Randomly label the vertices of G with a permutation of integers $1, 2, \dots, N$, where $N = |V|$. Let X be the set of all the vertices v such that the minimum label on the vertices in $\{v\} \cup N_1(v) \cup N_2(v)$ is on a vertex in $N_1(v)$. Then the probability that X contains a vertex v is $\frac{d_1(v)}{1 + d_1(v) + d_2(v)}$. So the expected size of X is $\sum_{v \in V} \frac{d_1(v)}{1 + d_1(v) + d_2(v)}$. It follows that for certain fixed permutation of integers from 1 to N , we have $|X| \geq \sum_{v \in V} \frac{d_1(v)}{1 + d_1(v) + d_2(v)}$. We aim to prove that there is an independent set in this X with size at least $|X|/k$.

To this end, we define a relation R on X as follows. Let $u, v \in X$. Call u and v satisfy the relation R if the minimum label on $\{u\} \cup N_1(u) \cup N_2(u)$ is precisely the same as that on the vertices in $\{v\} \cup N_1(v) \cup N_2(v)$. Clearly R is an equivalence relation, and thus X can be partitioned into certain equivalence classes $X_1, X_2, \dots, X_p$ for some positive integer p . For each $1 \leq i \leq p$, by the definition of relation R , all vertices in X_i share a neighbor v_i in common, in which for any $u \in X_i$, the label of v_i is the minimum label on vertices in $\{u\} \cup N_1(u) \cup N_2(u)$. Hence $X_i \subseteq N_1(v_i)$ and clearly $v_i \neq v_j$ for $i \neq j$. We claim that there is no edge between X_i and X_j whenever $1 \leq i \neq j \leq p$. To justify it, assume the contrary: some $w_i \in X_i$ is adjacent to some $w_j \in X_j$. Since $w_i \in X$ and since $v_j \in N_1(w_i) \cup N_2(w_i)$, by the definition of X we see that the label on v_i is less than that on v_j . Similarly, by considering w_j , we conclude that the label on v_j is less than that on v_i , yielding a contradiction.

Since $X_i \subseteq N_1(v_i)$ for each $1 \leq i \leq p$ is k -colorable there is an independent set Y_i in X_i with $|Y_i| \geq |X_i|/k$. It follows from the above claim that $\cup_{i=1}^p Y_i$ is an independent set with size at least $\sum_{i=1}^p |X_i|/k = |X|/k$, as desired. $\square$

Note For a triangle-free graph G , the proof is easier since X itself is independent, so there is no need to introduce the equivalence relation.

Lemma 4.5 *Let $G = (V, E)$ be an (m, k) -colorable graph. Then for any $1 \leq \ell \leq m+1$, we have*

$$\alpha(G) \geq \frac{1}{2k} \sum_{v \in V} \frac{1 + d_1(v) + \cdots + d_{\ell-1}(v)}{1 + d_1(v) + \cdots + d_{\ell}(v)}.$$

Proof. The proof goes along the same line as that of the preceding lemma, so we only give a sketch here.

Randomly label the vertices of G with a permutation of the integers $1, 2, \dots, N$, where $N = |V|$. Let X be the set of all the vertices v such that the minimum label on the vertices in $\cup_{j=0}^{\ell} N_j(v)$ is on a vertex $\cup_{j=0}^{\ell-1} N_j(v)$. Then for certain fixed permutation of the integers from 1 to N , we have

$$|X| \geq \sum_{v \in V} \frac{1 + d_1(v) + \cdots + d_{\ell-1}(v)}{1 + d_1(v) + \cdots + d_{\ell}(v)}.$$

We aim to prove that there is an independent set in this X with size at least $|X|/(2k)$.

To this end, define an equivalence relation R on X such that u and v satisfy R if the minimum label on the vertices in $\cup_{j=0}^{\ell} N_j(u)$ is precisely the same as that on the vertices in $\cup_{j=0}^{\ell} N_j(v)$. Then X can be partitioned into certain equivalence classes $X_1, X_2, \dots, X_p$ for some integer $p \geq 1$. It can be shown that for each $1 \leq i \leq p$, there exists a vertex v_i such that

- the distance between each vertex in X_i and v_i is at most $\ell - 1$; and
- v_i is the vertex with the minimum label in $\cup_{j=0}^{\ell} N_j(v)$ for each $v \in X_i$.

Based on these v_i , we can deduce that there is no edge between X_i and X_j whenever $1 \leq i \neq j \leq p$. Now partition each X_i into subsets $X_{i,j}$, $1 \leq j \leq \ell - 1$, such that the distance between every vertex in $X_{i,j}$ and v_i is j . Since each $X_{i,j}$ contains an independent set $Y_{i,j}$ of size at least $|X_{i,j}|/k$, where $1 \leq i \leq p$ and $0 \leq j \leq \ell - 1$. Thus one of $\cup_{i=1}^p \cup_{\text{odd } j} Y_{i,j}$ and $\cup_{i=1}^p \cup_{\text{even } j} Y_{i,j}$ is an independent set with size at least $\frac{1}{2k} \cup_{i=1}^p \cup_{j=0}^{\ell-1} X_{i,j} = \frac{|X|}{2k}$, completing the proof. $\square$

Note For a graph G that contains no $C_3, C_5, \dots, C_{2m+1}$, it is $(m, 1)$ -colorable.

Theorem 4.9 *Let $m \geq 2$ and $k \geq 1$ be integers and let $G = (V, E)$ be an (m, k) -colorable graph, then*

$$\alpha(G) \geq \frac{1}{k2^{(m-1)/m}} \left(\sum_{v \in V} d(v)^{1/(m-1)} \right)^{(m-1)/m}.$$

Proof. Applying Lemma 4.4 and Lemma 4.5 repeatedly, we have

$$\begin{aligned} \alpha(G) \geq \frac{1}{k(m-1)} \sum_{v \in V} & \left(\frac{d_1(v)}{1 + d_1(v) + d_2(v)} \right. \\ & + \frac{1 + d_1(v) + d_2(v)}{2(1 + d_1(v) + d_2(v) + d_3(v))} \\ & + \cdots + \left. \frac{1 + d_1(v) + \cdots + d_{m-1}(v)}{2(1 + d_1(v) + \cdots + d_m(v))} \right). \end{aligned}$$

Since the arithmetic mean is no less than the geometric mean, we obtain

$$\alpha(G) \geq \frac{1}{k2^{(m-2)/(m-1)}} \sum_{v \in V} \left(\frac{d_1(v)}{1 + d_1(v) + \cdots + d_m(v)} \right)^{1/(m-1)}.$$

By the condition that $N_i(v)$ is k -colorable, there is an independent set in $N_i(v)$ with size $\alpha_i(v) \geq d_i(v)/k$, and by the fact that there is no edge between $N_i(v)$ and $N_j(v)$ whenever $i - j = 0 \pmod{2}$,

$$\begin{aligned} 2\alpha(G) & \geq 1 + \alpha_1(v) + \cdots + \alpha_m(v) \\ & \geq \frac{1}{k} [1 + d_1(v) + \cdots + d_m(v)]. \end{aligned}$$

Therefore

$$\alpha(G) \geq \frac{1}{k2^{(m-2)/(m-1)}} \sum_{v \in V} \left(\frac{d_1(v)}{2k\alpha(G)} \right)^{1/(m-1)},$$

the desired statement follows. $\square$

Note The theorem 4.9 is sharp for graph $G = K_{n,n}$ for which $k = 1$.

4.4 Odd Cycle vs. Large K_n

Erdős, Faudree, Rousseau and Schelp (1978) proved that

$$r(C_{2m+1}, K_n) \leq cn^{1+1/m},$$

where $c = c(m) > 0$ is a constant. We will improve their bound. A relation they established can be stated as follows.

Lemma 4.6 *Let $m \geq 1$ be an integer and let $G = (V, E)$ be a graph that contains no C_{2m+1} . Then it is $(m, 2m - 1)$ -colorable.*

Proof. For a fixed vertex v , we need to verify that $N_i(v)$ is $(2m - 1)$ -colorable. The proof is based on the construction of a spanning tree T , of the component of G that contains v . As the construction going on, we label all vertices of G , which yields an order $<$ on $V(G)$. First of all, for any $u \in N_i(v)$ and $w \in N_j(v)$ with $i < j$, we define $u < w$. We label the vertices in $N_1(v)$ in an arbitrary way. Assuming that the vertices in $N_i(v)$ have been labelled, such that $N_i(v) = \{u_1, u_2, \dots, u_\ell\}$, where u_j has label j , we then proceed to $N_{i+1}(v)$ as follows. Take the vertex $u_1 \in N_i(v)$, label all neighbors of u_1 in $N_{i+1}(v)$ arbitrarily, say $w_1, \dots, w_r$, and each is labelled as its index, and make $w_1, \dots, w_r$ adjacent with u_1 in T . Then consider neighbors of u_2 in $N_{i+1}(v) \setminus \{w_1, \dots, w_r\}$, which have no label yet, label them arbitrarily as $\{w_{r+1}, w_{r+2}, \dots\}$, and make them adjacent to u_2 in T . Goes this way unless all vertices in $N_{i+1}(v)$ have a label, and each is adjacent with exactly one vertex of $N_i(v)$ in T . Clearly, if vertices y and z in $N_{i+1}(v)$ are adjacent in T to vertices u and v in $N_i(v)$, respectively, and if $u < v$, then $y < z$.

Assign each edge uv a direction from u to v if $u < v$. We now claim that there is no directed path of length $2m - 1$ in $N_i(v)$. Let

$$d^* = \max_j d_T(v_j, v_{j+1}) = d_T(v_s, v_{s+1}).$$

Consider the relationship between the construction of T and the labelling of $N_1(v), \dots, N_i(v)$, we know that in fact $d^* = d_T(v_r, v_t)$ for any $r \leq s < s + 1 \leq t$. Moreover, for $i \leq m$, whatever the value of d^* , if there exists a directed path $(v_r, v_{r+1}, \dots, v_t)$ in $N_i(v)$ of length $2m - 1$, then the unique path connecting some v_k and v_ℓ ($r \leq k < \ell \leq t$) in T with the path would form a cycle C_{2m+1} . Since G contains no C_{2m+1} , so the subgraph induced by $N_i(v)$, $i \leq m$, in which each edge has direction from smaller vertex to bigger vertex, contains no directed path of length $2m - 1$.

We now assign each vertex $u \in N_i(v)$ with a color that is the maximum length of directed sub-path beginning from u in $N_i(v)$. Clearly, any two adjacent vertices receive distinct colors. Since the maximum length of directed path in $N_i(v)$ is at most $2m - 1$, thus $N_i(v)$ is $(2m - 1)$ -colorable. $\square$

Combining the above lemma and Theorem 4.9 in last section, we have the following result.

Theorem 4.10 *Let $m \geq 2$ be an integer. If a graph $G = (V, E)$ is C_{2m+1} -free, then*

$$\alpha(G) \geq \frac{1}{(2m-1)2^{(m-1)/m}} \left(\sum_{v \in V} d(v)^{1/(m-1)} \right)^{(m-1)/m}.$$

In particular, if G is C_5 -free, then $\alpha(G) \geq \sqrt{Nd/18}$, where N is the order of G and d is its average degree.

□

Remark 1 If G is regular of degree d , then

$$\alpha(G) \geq c(m)N^{1-1/m}d^{1/m},$$

which is also valid for $m = 1$ with $c(1) = 1$. So Theorem 4.10 can be referred as a generalization of a trivial fact that the independence number of a triangle-free graph is at least its average degree.

Remark 2 From the proof of Lemma 4.6, we see that if a graph G of order N is C_{2m} -free, then it is $(m-1, 2m-2)$ -colorable. So we have

$$\alpha(G) \geq c \left(\sum_{v \in V} d(v)^{1/(m-2)} \right)^{(m-2)/(m-1)}. \quad (4.3)$$

However, it will be shown that the average degree of G is at most $cN^{1/m}$ in Chapter 8. From a result in Chapter 3, we then have $\alpha(G) \geq cN^{1-1/m} \log N$. If G is regular, the above Theorem 4.10 just yields $\alpha(G) \geq cN^{1-1/m}$. So the inequality (4.3) for C_{2m} -free graphs is weaker than that given by the main result in Chapter 8.

We now improve the upper bound $c(m)n^{1+1/m}$ of Erdős et. al (1978) of $r(C_{2m+1}, K_n)$ in following theorem. This improvement is independently due to Li and Zang (2001,2003), and Sudkov (2002).

Theorem 4.11 *Let $m \geq 1$ be an fixed integer. Then for all sufficiently large n ,*

$$r(C_{2m+1}, K_n) \leq c \left(\frac{n^{m+1}}{\log n} \right)^{1/m},$$

where $c = c(m) > 0$ is a constant.

Proof. *Case 1.* $m = 1$. The desired upper bound for $r(C_3, K_n) = r(3, n)$ has obtained in Chapter 3.

Case 2. $m \geq 2$. Suppose that G is a C_{2m+1} -free graph on $N = r(C_{2m+1}, K_n) - 1$ vertices with $\alpha(G) \leq n - 1$. We aim to show that there exists some constant $c = c(m) > 0$ so that $r(C_{2m+1}, K_n) \leq c \left(\frac{n^{m+1}}{\log n} \right)^{1/m}$ holds for all sufficiently large n . Set

$$X = \{v \in V(G) : d(v) > n^{1/m}(\log n)^{(m-1)/m}\}.$$

We consider two subcases depending on $|X|$.

Subcase 2.1. $|X| \geq N/2$. By Theorem 4.10,

$$\begin{aligned} n &> c_1 \left(\sum_{v \in V} d(v)^{1/(m-1)} \right)^{(m-1)/m} \\ &\geq c_1 \left(\sum_{v \in X} d(v)^{1/(m-1)} \right)^{(m-1)/m} \\ &\geq c_1 \left[\frac{N}{2} \left(n^{1/m}(\log n)^{(m-1)/m} \right)^{1/(m-1)} \right]^{(m-1)/m} \\ &\geq c_2 N^{(m-1)/m} n^{1/m^2} (\log n)^{(m-1)/m^2}, \end{aligned}$$

yielding $N + 1 \leq c_3 n^{1+1/m} / (\log n)^{1/m}$, where c_1 and henceforth c_i are positive constants depending on m only.

Subcase 2.2. $|X| < N/2$. Denote by H for the subgraph of G induced by $V \setminus X$. Then the order of H is at least $N/2$, and the average degree d of H is at most $n^{1/m}(\log n)^{(m-1)/m}$. By Theorem 3.4 in Chapter 3, for large n ,

$$\begin{aligned} n &> c_4 \frac{N \log d}{2} \\ &> c_5 N \frac{\log n}{n^{1/m}(\log n)^{(m-1)/m}} \\ &= c_5 N \frac{(\log n)^{1/m}}{n^{1/m}}, \end{aligned}$$

yielding $N + 1 \leq c_6 n^{1+1/m} / (\log n)^{1/m}$.

Now set $c = \max\{c_3, c_6\}$. Then in both subcases we have

$$r(C_{2m+1}, K_n) = N + 1 \leq c \frac{n^{1+1/m}}{(\log n)^{1/m}},$$

completing the proof. $\square$

The above proof is due to Li and Zang that contains a probabilistic algorithm (in proof of Theorem 4.9). The following proof is due to Sudakov that in fact contains a deterministic algorithm.

The Second Proof of Theorem 4.11. Let

$$N = \left\lfloor \frac{an^{1+1/m}}{(\log n)^{1/m}} \right\rfloor, \quad \text{and} \quad d = bn^{1/m}(\log n)^{(m-1)/m}.$$

where $a = a(m)$ and $b = b(m)$ are constants to be chosen. Let G be a graph of order N that contains no C_{2m+1} . We aim to verify that $\alpha(G) \geq n$, which implies that $r(C_{2m+1}, K_n) \leq N$. Starting with $G' = G$ and $I = \emptyset$. If G' has a vertex of degree greater than d we do the following iterative procedure, otherwise, we stop.

Pick a vertex v with $d(v) = d_1(v) > d$. Since

$$d_{m+1}(v) = d_1(v) \frac{d_2(v)}{d_1(v)} \cdots \frac{d_{m+1}(v)}{d_m(v)} \leq N,$$

there exists some i , $1 \leq i \leq m$, such that

$$\frac{d_{i+1}(v)}{d_i(v)} \leq \left(\frac{N}{d} \right)^{1/m} \leq \left(\frac{an}{b \log n} \right)^{1/m}.$$

Take the smallest i with this property, then

$$\frac{d_{i-1}(v)}{d_i(v)} \leq \left(\frac{d}{N} \right)^{1/m} < 1$$

for large n . By Lemma 4.6, $N_i(v)$ is $(2m-1)$ -colorable hence it contains an independent set I' of size $|I'| \geq d_i(v)/(2m-1)$. Enlarge I to $I \cup I'$ and remove $N_{i-1}(v) \cup N_i(v) \cup N_{i+1}(v)$ from G' . The number of removed vertices is

$$\begin{aligned} d_{i-1}(v) + d_i(v) + d_{i+1}(v) &= \left(\frac{d_{i-1}(v)}{d_i(v)} + 1 + \frac{d_{i+1}(v)}{d_i(v)} \right) d_i(v) \\ &\leq \left(2 + \left(\frac{an}{b \log n} \right)^{1/m} \right) d_i(v) \\ &\leq (2m-1) \left(\left(2 + \frac{an}{b \log n} \right)^{1/m} \right) |I'| \\ &= A|I'| \end{aligned}$$

for any large n , where $A = (2m-1) \left(2 + \left(\frac{an}{b \log n} \right)^{1/m} \right)$. Clearly all neighbors of I' have been removed thus I is an independent set after each step, and the ratio between the number of removed vertices and $|I|$ is at most A .

Let $G_1 = (V_1, E_1)$ be a graph in the end of the process, and $G_2 = (V_2, E_2)$ be the graph induced by all removed vertices $V_2 = V \setminus V_1$. We distinguish two cases depending on $|V_2|$.

Case 1. $|V_2| > N/2$. G_2 has an independent set I of size at least

$$|I| > \frac{N/2}{A} \geq (1 - o(1))n \frac{a}{2(2m-1)} \left(\frac{b}{a} \right)^{1/m} > n$$

for large n if a and b satisfy

$$\frac{a}{2(2m-1)} \left(\frac{b}{a} \right)^{1/m} > 1 \quad (4.4)$$

Case 2. $|V_2| \leq N/2$. In this case, $|V_1| > N/2$. Since no vertices in G_1 has degree greater than d , so the average degree of G_1 is at most d , and each subgraph induced by a neighbor of G_1 does not contain a path of length $2m-1$ since G_1 does not contain C_{2m+1} . By Theorem 3.4 in Chapter 3, we have

$$\alpha(G) \geq \alpha(G_1) \geq \frac{N \log(d/(2m)) - 1}{2} \geq \frac{a}{2bm} n > n,$$

if a and b satisfy

$$\frac{a}{2bm} > 1. \quad (4.5)$$

To obtain the desired constants a and b , let us look at the case where the equality holds for (4.4) and (4.5). then $a/b = 2m$ and $a = 2(2m-1)(2m)^{1/m}$. To get what we want, just perturb this solution a little bit, When a and b are slightly larger, such that both (4.4) and (4.5) are satisfied. Thus for all sufficiently large n ,

$$r(C_{2m+1}, K_n) \leq (1 + o(1))2(2m-1)(2m)^{1/m} \left(\frac{n^{m+1}}{\log n} \right)^{1/m},$$

completing the proof. $\square$

From the above proof, $r(C_5, K_n) \leq (12 + o(1))n^{3/2}/\sqrt{\log n}$. Since the above proof uses just “half” of the information, so perhaps the following slightly better bound for $m = 2$ with a short proof is more pleasing.

Theorem 4.12 *Let $n \geq e^{e^2}$. Then*

$$r(C_5, K_n) \leq 6 \frac{n^{3/2}}{\sqrt{\log n}}.$$

Proof. Let G be a graph on $N = r(C_5, K_n) - 1$ vertices that contains no C_5 and $\alpha(G) \leq n - 1$. We consider two cases depending on the value of d , the average degree of G .

Case 1. $d > 3\sqrt{n \log n}$. By Theorem 4.10, we have $\alpha(G) \geq \sqrt{Nd/18}$. It follows that $n - 1 > \sqrt{3N\sqrt{n \log n}/18}$, implying

$$N + 1 \leq \frac{6(n - 1)^2}{\sqrt{n \log n}} + 1 \leq \frac{6n^{3/2}}{\sqrt{\log n}}.$$

Case 2. $d \leq 3\sqrt{n \log n}$. Since G is C_5 -free, each neighborhood of G does not contain any path of length 3, according to Theorem 3.4 in Chapter 3, we have

$$n - 1 \geq N \frac{\log(3\sqrt{n \log n}/3) - 1}{3\sqrt{n \log n}} \geq N \frac{\sqrt{\log n}}{6\sqrt{n}}.$$

It follows that

$$N + 1 \leq \frac{6(n - 1)\sqrt{n}}{\sqrt{\log n}} + 1 \leq \frac{6n^{3/2}}{\sqrt{\log n}},$$

completing the proof. $\square$

4.5 Pick Vertices Randomly

To illustrate the method, this section contains two results with proofs of picking vertices randomly. The first is weaker than that $\alpha(G) \geq N/(d + 1)$, which we have obtained. The second may be related to Ramsey graphs for $r(3, n)$.

Theorem 4.13 *Let G be a graph with N vertices and average degree $d \geq 1$, then*

$$\alpha(G) \geq \frac{N}{2d}.$$

Proof. We want to find in G an independent set at least $N/(2d)$ vertices. We begin by picking a set at random. Let S be a random subset of $V(G)$

where each vertex of G belongs to S with probability $1/d$ independently. That is to say, S is defined by

$$\Pr[v \in S] = \frac{1}{d},$$

where all these events are mutually independent over $v \in V(G)$. Let $X(S)$ be the number of vertices in S and let $Y(S)$ be the number of edges in the subgraph induced by S . Note that for one of the $Nd/2$ edges of G , the probability is $p^2 = 1/d^2$ that both of its vertices belong to S . By linearity of expectation,

$$E(X) = Np = \frac{N}{d} \quad \text{and} \quad E(Y) = \frac{Nd}{2}p^2 = \frac{N}{2d},$$

so

$$E(X - Y) = E(X) - E(Y) = \frac{N}{2d}.$$

Thus there exists a set S satisfying

$$X(S) - Y(S) \geq \frac{N}{2d}.$$

Now S is not quite what we want as the subgraph of G induced by S may have edges. However, if we delete one vertex from each edge, then at most $Y(S)$ vertices are deleted, we obtain a new set with at least $N/(2d)$ vertices and whose induced subgraph has no edges. $\square$

This simple proof can give a nontrivial result for hypergraphs. Let $\mathcal{G} = (V, \mathcal{E})$ be a r -uniform hypergraph of order N . A subset $S \subseteq V$ is called an *independent* set if it does not contain any edge of $\mathcal{G}$. The degree d_v of a vertex $v \in V$ is the number of edges that contains v . Let d be the average degree of $\mathcal{G}$, then $Nd = \sum_{v \in V} d_v = r|\mathcal{E}|$. Denote by $\alpha(\mathcal{G})$ for the independence number of $\mathcal{G}$. Then for any hypergraph $\mathcal{G}$ with average degree $d \geq 1$,

$$\alpha(\mathcal{G}) \geq \left(1 - \frac{1}{r}\right) \frac{1}{d^{1/(r-1)}}.$$

The proof is similar. Define a random subset $S \subseteq V$ by $\Pr(v \in S) = p$ for some $0 \leq p \leq 1$ with all these events are mutually independent over $v \in V$. For the similarly defined $X(S)$ and $Y(S)$, there exists a subset S with

$$X(S) - Y(S) \geq E(X) - E(Y) = Np - \frac{Nd}{r}p^r.$$

The desired lower bound follows by taking $p = 1/d^{1/(r-1)}$.

A *dominating set* of a graph $G = (V, E)$ is a set $U \subseteq V$ such that $U \cup N(U) = V$, where $N(U) = \cup_{u \in U} N(u)$. The *domination number* is the smallest cardinality among all dominating sets of G . Let $\beta(G)$ denote the domination number of G . Clearly

$$\alpha(G) \geq \beta(G)$$

since any maximal independent set is a dominating set. We would like to ask what G look like if G is triangle-free and $\alpha(G)$ and $\beta(G)$ are close?

Recall that in Chapter 3, we have that for any triangle-free graph G with N vertices and average degree d , then

$$\alpha(G) \geq Nf(d),$$

where $f(x) \sim \log x/x$ as $x \rightarrow \infty$. A similar bound for domination number due to Alon (1990) is as follows, in which the involved function has the same asymptotical form.

Theorem 4.14 *Let G be a graph with N vertices and minimum degree $\delta \geq 1$, then*

$$\beta(G) \leq N \frac{1 + \log(\delta + 1)}{\delta + 1}.$$

Proof. Let us pick a vertex v of G with probability p randomly and independently, where $p = \log(\delta + 1)/(\delta + 1)$. Let X be the random set of picked vertices and let $Y = V \setminus (X \cup N(X))$. The set $X \cup Y_X$ is clearly a dominating set of G which can be picked with the cardinality as small as its expected number. The expected values of $|X|$ is Np . A vertex v belongs to Y if and only if neither v nor any neighbor belongs to X , so

$$E(|Y|) = \sum_{v \in V} (1 - p)^{1+d(v)} \leq N(1 - p)^{1+\delta}.$$

Thus

$$E(|X| + |Y|) \leq N [p + (1 - p)^{1+\delta}].$$

Since

$$(1 - p)^{\delta+1} \leq e^{-p(\delta+1)} = e^{-\log(\delta+1)} = \frac{1}{\delta+1},$$

then we have

$$E(|X| + |Y|) \leq N \frac{1 + \log(\delta + 1)}{\delta + 1},$$

as desired. □

A triangle-free graph G is called *maximal* or *critical* when adding any edge would yield a triangle. Form the definition, there must exist a Ramsey graph on $N = r(3, n) - 1$ vertices which is maximally triangle-free with $\alpha(G) \leq n - 1$. Thus $\beta(G) \leq n - 1$ since $\beta(G) \leq \alpha(G)$ holds for any graph. We shall conclude this chapter with the following problem.

Problem 4.1 *Does there exist a family $\{G_n\}$ of graphs satisfying the following conditions:*

- (1) *Each G_n is maximally triangle-free, and $\Delta(G) \sim \delta(G) \sim n$;*
- (2) *$\alpha(G_n) \leq c\beta(G_n)$ for some constant c .*

The first condition is natural but the second is crucial. We are expecting that the existence of above family would yield good lower bound for $r(3, n)$.

4.6 Independence and Chromatic Number ★

We have obtained some lower bounds for independence numbers of graphs with some forbidden subgraphs. However, it is often that a lower bound of independence number is associated with an upper bound of chromatic number. With the naive relation

$$\alpha(G)\chi(G) \geq N$$

for any graph G of order N , any upper bound of $\chi(G)$ gives a lower bound of $\alpha(G)$. Let the maximum degree, minimum degree and average degree of G be Δ , δ , d , respectively. Then $\chi(G) \leq \Delta + 1$ implies that $\alpha(G) \geq \frac{N}{\Delta + 1}$. In fact, we have a stronger form $\alpha(G) \geq \frac{N}{1 + d}$, and we do not expect that $d + 1$ becomes an upper bound of $\chi(G)$ since the edges may be concentrated in a small subset. We list these associated bounds in the following table for those readers who may have interests, and most of these bounds have proved by probabilistic method. In the table, $o(1)$ is a small term tending to zero as the corresponding parameter Δ , or δ , or d tending to infinity.

Condition	Lower bound of $\alpha(G)$	Upper bound of $\chi(G)$
general	$N/(d + 1)$	$\Delta + 1$
no C_3	$(1 - o(1))N \log d/d$	$C\Delta/\log \Delta$
$\Delta(N(v)) < m$	$(1 - o(1))N \log(d/m)/d$	$C\Delta/\log(\Delta/m)$
no C_3, C_4	$(1 - o(1))N \log d/d$	$(1 + o(1))\Delta/\log \Delta$
no C_{2m+1}	$cN^{1-1/m}\delta^{1/m}$	

Table: Lower bound of Independence Number and the associated upper bound of chromatic number

The upper bound of chromatic number for triangle-free graphs is due to Johansson (1986); that for graphs with each neighborhood has a bounded maximum degree is due to Alon, Krivelevich and Sudakov (1999); that for graph with forbidden subgraphs C_3 and C_4 is due to Kim (1995). A good source for these bounds of chromatic numbers proved by probabilistic method is the book *Graph Coloring and the Probabilistic Method* by Molloy and Reed (2002).

Problem 4.2 Can the constant C in upper bound of $\chi(G)$ for graphs with sparse neighborhoods in above table be improved as $1 + o(1)$?

Problem 4.3 Prove or disprove $\chi(G) \leq C \left(\frac{N}{\delta}\right)^{1/m}$ for graphs G containing no C_{2m+1} .

Exercises

4.1 Show that the complement $\overline{G}_p$ of a random graph G_p is exactly G_q , where $q = 1 - p$.

4.2 Give a graph F of order at least two such that $\mathcal{A}(F)$ is a singleton.

4.3 Verify Theorem 4.1 for graph $F = K_{1,3}$.

4.4 Let v be a fixed vertex of $\mathcal{G}(n, p)$ and let $d(v)$ be the degree of v . Compute the probability of the event $d(v) = k$ and its expectation and variance.

4.5 For distinct vertices u and v , show

$$\Pr(d(u) \leq k_1, d(v) \leq k_2) \leq \Pr(d(u) \leq k_1) \Pr(d(v) \leq k_2).$$

4.6 (1) Show that the probability that a fixed subset S of $\mathcal{G}(n, 1/2)$ is contained in the neighborhood of a vertex is $(n - |S|)2^{-|S|}$.

(2) Show that almost every graph G in $\mathcal{G}(n, 1/2)$ has $\Delta(G) \geq n/2 + \sqrt{n}$ and $\delta(G) \leq n/2 - \sqrt{n}$.

4.7 Try to obtain asymptotically optimal constants in Theorem 4.7.

4.8 Denote by W_m for $K_1 + C_m$. Using that fact that $r(C_m, K_n)$ is super-linear on n , show

$$r(W_m, K_n) \leq c r(C_m, K_n) \frac{n}{\log n},$$

where $c = c(m)$ is a constant.

4.9 Call a term $1/(1 + d_v)$ in $\alpha(G) \geq \sum_v 1/(1 + d_v)$ as the ratio of independence. Using this concept, explain Lemmas 4.4 and 4.5 intuitively.

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Chapter 5

Concentration

5.1 The Chernoff's Inequality

The probability space we consider in graph Ramsey theory has only finite many possible outcomes, and the random variable is often nonnegative. Let X be a random variable, the expected value of X is defined to be $E(X) = \sum_i a_i \Pr(X = a_i)$, where the summation is taken over all values a_i that X can take.

Theorem 5.1 (Markov's Inequality) *Let $a > 0$ and let X be a nonnegative random variable, then*

$$\Pr(X \geq a) \leq \frac{E(X)}{a}.$$

Proof. Suppose that $\{a_i\}$ is the set of all values that X takes. Then

$$\begin{aligned} E(X) &= \sum_i a_i \Pr(X = a_i) \\ &\geq \sum_{a_i \geq a} a_i \Pr(X = a_i) \\ &\geq a \sum_{a_i \geq a} \Pr(X = a_i) = a \Pr(X \geq a), \end{aligned}$$

as required. $\square$

In particular, if X only takes nonnegative integer values and $E(X) < 1$, then we have $\Pr(X \geq 1) < 1$ hence $\Pr(X = 0) > 0$. This is exactly what we used to get a lower bound of $r(n, n)$ in the last chapter.

For a positive integer k , the k th moment of a real-valued random variable X is defined to be $E(X^k)$, and so the first moment is simply the expected value. Denote by $\mu = E(X)$, and define the variance of X as $E((X - \mu)^2)$, and denote by σ^2 for the variance. Call σ as the standard deviation of X .

Theorem 5.2 (Chebyshev's Inequality) *Let X be a random variable and let a be a positive number, then*

$$\Pr(|X - \mu| \geq a) \leq \frac{\sigma^2}{a^2}.$$

Proof. By Markov's inequality, for any $a > 0$,

$$\begin{aligned} \sigma^2 &= E((X - \mu)^2) \\ &\geq a^2 \Pr((X - \mu)^2 \geq a^2) \\ &= a^2 \Pr(|X - \mu| \geq a). \end{aligned}$$

It follows by the required statement. $\square$

In important, the second moment $E(X^2)$ is second to the first moment $E(X)$.

Lemma 5.1 (Second Moment Method) *If X is a random variable, then*

$$\Pr(X = 0) \leq \frac{\sigma^2}{\mu^2} = \frac{E(X^2) - \mu^2}{\mu^2},$$

where $\mu = E(X)$. In particular, $\Pr(X = 0) \rightarrow 0$ if $E(X^2)/\mu^2 \rightarrow 1$.

The proof follows from Chebyshev's Inequality and the trivial fact that $\Pr(X = 0) \leq \Pr(|X - \mu| \geq \mu)$ immediately. Intuitively, if σ grows more slowly than μ grows, then $\Pr(X = 0) \rightarrow 0$ since σ "pulls" X close to μ .

The Chebyshev's inequality is in fact the Markov's inequality on random variable $|X - \mu|$. However, Chebyshev's inequality states the probability of a random variable X apart from $E(X)$ is bounded. When this is the case, we say that X is *concentrated*. A concentration bounds is used to show that a random variable is very close to its expected value with high probability, so it behaves approximately as one may "expect" it to. When S_n is the sum of n independent variables, each variable equals to 1 with probability p and -1 with probability $1 - p$, respectively, the bound can be sharper. Such random variables are bounded in Chernoff's inequality. Most of the results in this chapter may be found in, or immediately derived from, the seminal paper of Chernoff (1952). While our proofs are self-contained. A set of random variables $X_1, X_2, \dots$ are said to be mutually independent

means each X_i is independent of any Boolean expression formed from other $(X_j)'s$. In any form of Chernoff bounds, we assume that

Assumption A : *On the independence of variables in Chernoff bound*
Let $X_1, X_2, \dots$ be mutually independent variables and they have the same binomial distribution. Set

$$S_n = \sum_{i=1}^n X_i.$$

All concentration bounds in the remaining part of this section are Chernoff bounds of different forms, which estimate the probability of

$$\Pr(S_n \geq n(\mu + \delta)),$$

where $\mu = E(X_i)$. The symmetric bound on $\Pr(S_n \leq n(\mu - \delta))$ can be obtained similarly.

Theorem 5.3 *Under Assumption A, suppose*

$$\Pr(X_i = 1) = \Pr(X_i = -1) = \frac{1}{2}$$

for $i = 1, 2, \dots$. Then for any $\delta > 0$,

$$\Pr(S_n \geq n\delta) < \exp\{-n\delta^2/2\},$$

and for any $a > 0$,

$$\Pr(S_n \geq a) < \exp\{-a^2/(2n)\}.$$

Proof. Let $\lambda > 0$ be arbitrary. Then

$$E(e^{\lambda X_i}) = \frac{e^{\lambda} + e^{-\lambda}}{2}.$$

Note that

$$\begin{aligned} E(e^{\lambda S_n}) &= E(e^{\lambda X_1})E(e^{\lambda X_2}) \dots E(e^{\lambda X_n}) \\ &= \left(\frac{e^{\lambda} + e^{-\lambda}}{2} \right)^n = \left(\sum_{j=0}^{\infty} \frac{\lambda^{2j}}{(2j)!} \right)^n \\ &< \left(\sum_{j=0}^{\infty} \frac{1}{j!} \left(\frac{\lambda^2}{2} \right)^j \right)^n = e^{n\lambda^2/2}, \end{aligned}$$

where we use the fact that $(2j)! \geq 2^j j!$ for all $j \geq 0$ with strict inequality when $j \geq 2$. Now by Markov's inequality,

$$\begin{aligned} \Pr(S_n \geq n\delta) &= \Pr(e^{\lambda S_n} \geq e^{\lambda n\delta}) \\ &\leq \frac{E(e^{\lambda S_n})}{e^{\lambda n\delta}} \\ &< \exp\{n(\lambda^2/2 - \lambda\delta)\}, \end{aligned}$$

for all $\lambda > 0$. Setting $\lambda = \delta$, we obtain the desired result. $\square$

For large n , the central limit theorem implies that S_n is approximately normal with zero mean and standard deviation $\sqrt{n}$. For any fixed u ,

$$\lim_{n \rightarrow \infty} \Pr(S_n \geq u\sqrt{n}) = \int_u^\infty \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt < e^{-u^2/2}.$$

However, the Chernoff bound holds for all positive n and a .

Since X_i is often an indicator variable of some random event, so X_i takes 1 when the event appears and 0 otherwise. The following form of Chernoff bound may be used in more cases.

Theorem 5.4 *Under Assumption A, suppose*

$$\Pr(X_i = 1) = \Pr(X_i = 0) = \frac{1}{2}$$

for $i = 1, 2, \dots$. Then for any $\delta > 0$,

$$\Pr(S_n \geq n(1 + \delta)/2) < \exp\{-n\delta^2/2\},$$

or

$$\Pr(S_n \geq n(1/2 + \delta)) < \exp\{-2n\delta^2\}.$$

Proof. Set $Y_i = 2X_i - 1$ and $T_n = \sum_{i=1}^n Y_i = 2S_n - n$. Then

$$\Pr(Y_i = 1) = \Pr(Y_i = -1) = \frac{1}{2},$$

and $\{Y_i\}$ satisfies Assumption A. Note that $T_n \geq n\delta$ if and only if $S_n \geq n(1 + \delta)/2$. Applying Theorem 5.3 to $\{Y_i\}$ and T_n , we have

$$\Pr(S_n \geq n(1 + \delta)/2) = \Pr(T_n \geq n\delta) < \exp\{-n\delta^2/2\}$$

as claimed. $\square$

Under Assumption A, suppose

$$\Pr(X_i = 1) = p, \quad \text{and} \quad \Pr(X_i = 0) = 1 - p$$

for $i = 1, 2, \dots$. Then we say that the sum $S_n = \sum_{i=1}^n X_i$ has binomial distribution, denoted by $B(n, p)$. Involved in Theorem 5.4 is special binomial distribution $B(n, 1/2)$. For general case, the calculation is slightly more complicated, but the technique is the same. As usual, denote by q for $1 - p$.

Theorem 5.5 *Under Assumption A, suppose*

$$\Pr(X_i = 1) = p \quad \text{and} \quad \Pr(X_i = 0) = q$$

for $i = 1, 2, \dots$. Then there exists $\delta_0 = \delta_0(p) > 0$ so that if $0 < \delta < \delta_0$, then

$$\Pr(S_n \geq n(p + \delta)) < \exp\{-n\delta^2/(3pq)\}.$$

Proof. Denote by a for $p + \delta$. By the same argument as used before,

$$\begin{aligned} \Pr(S_n \geq na) &= \Pr(e^{\lambda S_n} \geq e^{\lambda na}) \\ &\leq \frac{1}{e^{\lambda na}} E(e^{\lambda S_n}) \\ &= \frac{1}{e^{\lambda na}} (pe^\lambda + q)^n \\ &= (pe^{\lambda(1-a)} + qe^{-\lambda a})^n \end{aligned}$$

for all $\lambda > 0$. Let $c = 1 - a = q - \delta > 0$. By taking $\lambda = \log(aq/cp)$, we have

$$\begin{aligned} \min_{\lambda > 0} (pe^{\lambda c} + qe^{-\lambda a}) &= e^{-\lambda a} (pe^\lambda + q) \\ &= \left(\frac{cp}{aq}\right)^a \frac{q}{c} = \left(\frac{p}{a}\right)^a \left(\frac{q}{c}\right)^c. \end{aligned}$$

Setting $0 < \delta < 1 - p$, and expanding in powers of δ , with the fact that

$$\log(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} + O(x^4),$$

we find

$$\begin{aligned} \log\left(\frac{p}{a}\right)^a &= (p + \delta) \log\left(1 - \frac{\delta}{p + \delta}\right) \\ &= -\delta - \frac{\delta^2}{2(p + \delta)} - \frac{\delta^3}{3(p + \delta)^2} + o(\delta^3), \end{aligned}$$

and

$$\begin{aligned}\log\left(\frac{q}{c}\right)^c &= (q - \delta) \log\left(1 + \frac{\delta}{q - \delta}\right) \\ &= \delta - \frac{\delta^2}{2(q - \delta)} + \frac{\delta^3}{3(q - \delta)^2} + o(\delta^3).\end{aligned}$$

Adding them by terms,

$$\begin{aligned}-\frac{\delta^2}{2} \left(\frac{1}{p + \delta} + \frac{1}{q - \delta} \right) &= -\frac{\delta^2}{2} \left(\frac{1}{p(1 + \delta/p)} + \frac{1}{q(1 - \delta/q)} \right) \\ &= -\frac{\delta^2}{2} \left(\frac{1}{pq} - \frac{(q - p)\delta}{p^2 q^2} + o(\delta) \right) \\ &= \frac{-\delta^2}{2pq} + \frac{(q - p)\delta^3}{2p^2 q^2} + o(\delta^3),\end{aligned}$$

and

$$\begin{aligned}\frac{\delta^3}{3} \left(\frac{1}{(q - \delta)^2} - \frac{1}{(p - \delta)^2} \right) &= \frac{\delta^3}{3} \left(\frac{1}{q^2} - \frac{1}{p^2} + o(1) \right) \\ &= \frac{-(q - p)\delta^3}{3p^2 q^2} + o(\delta^3),\end{aligned}$$

we have

$$\log \left[\left(\frac{p}{a} \right)^a \left(\frac{q}{c} \right)^c \right] = \frac{-\delta^2}{2pq} + \frac{(q - p)\delta^3}{6p^2 q^2} + o(\delta^3) < \frac{-\delta^2}{3pq}.$$

Thus

$$\Pr(S_n \geq n(p + \delta)) < \exp\{-n\delta^2/(3pq)\},$$

completing the proof. $\square$

From above proof for $p > q$ and Theorem 5.4 for $p = q = 1/2$, we see that if $p \geq 1/2$, the bound can be slightly better as

$$\Pr(S_n > n(p + \delta)) < \exp\{-n\delta^2/(2pq)\}.$$

We now write out a symmetric form for Theorem 5.5, for Theorem 5.3 and Theorem 5.4, we omit them.

Theorem 5.6 *Under Assumption A, suppose*

$$\Pr(X_i = 1) = p \quad \text{and} \quad \Pr(X_i = 0) = q$$

for $i = 1, 2, \dots$. Then there exists $\delta_0 = \delta_0(p) > 0$ so that if $0 < \delta < \delta_0$, then

$$\Pr(S_n \leq n(p - \delta)) < \exp\{-n\delta^2/(3pq)\}.$$

Therefore

$$\Pr(|S_n - np| > n\delta) < 2 \exp\{-n\delta^2/(3pq)\}.$$

From the above proof, we have

$$\Pr(S_n \geq na) \leq \exp\left\{n \left(a \log \frac{p}{a} + (1-a) \log \frac{q}{1-a}\right)\right\}.$$

Set $k = na$, since $a = p + \delta > p$ then $k > np$, and

$$\Pr(S_n \geq k) \leq \exp\left\{n \left((k/n) \log \frac{p}{k/n} + (1 - k/n) \log \frac{q}{1 - k/n}\right)\right\}.$$

Let $H(x)$ signify the entropy function

$$H(x) = x \log \frac{p}{x} + (1-x) \log \frac{q}{1-x},$$

then

$$\Pr(S_n \geq k) \leq \exp\{nH(k/n)\},$$

which is valid also for $k = np$ since $H(p) = 0$. The following form of Chernoff's inequality was used by Beck (1983).

Theorem 5.7 *Under Assumption A, suppose*

$$\Pr(X_i = 1) = p \quad \text{and} \quad \Pr(X_i = 0) = q$$

for $i = 1, 2, \dots$. If $k \geq np$, then

$$\Pr(S_n \geq k) \leq \left(\frac{np}{k}\right)^k \left(\frac{nq}{n-k}\right)^{n-k}.$$

Consequently,

$$\Pr(S_n \geq k) \leq \left(\frac{npe}{k}\right)^k.$$

Proof. The right hand side of the first inequality is just $\exp\{nH(k/n)\}$. For the second inequality, simply note that

$$\left(\frac{nq}{n-k}\right)^{n-k} \leq \left(\frac{n}{n-k}\right)^{n-k} = \left(1 + \frac{k}{n-k}\right)^{n-k} < e^k.$$

Thus the required result follows. $\square$

We conclude this section with bounding $\Pr(S_n \geq k)$ by $\Pr(S_n = k)$. Denote by $b(j; n, p)$ for the probability of $\Pr(S_n = j)$, i.e.,

$$b(j; n, p) = \Pr(S_n = j) = \binom{n}{j} p^j (1-p)^{n-j}.$$

Theorem 5.8 $\star$ Under Assumption A, suppose

$$\Pr(X_i = 1) = p \quad \text{and} \quad \Pr(X_i = 0) = q$$

for $i = 1, 2, \dots$. If $k \geq np$, then

$$\Pr(S_n \geq k) \leq b(k; n, p) \frac{(k+1)(1-p)}{(k+1) - (n+1)p},$$

and if $k \leq np$,

$$\Pr(S_n \leq k) \leq b(k; n, p) \frac{(n-k+1)p}{(n+1)p - k}.$$

Proof. To prove the first result, note that for any $j \geq k$,

$$\begin{aligned} \frac{b(j+1; n, p)}{b(j; n, p)} &= \frac{(n-j)p}{(j+1)(1-p)} \\ &\leq \frac{(n-k)p}{(k+1)(1-p)}. \end{aligned}$$

Denote by r for $\frac{(n-k)p}{(k+1)(1-p)}$, then

$$\begin{aligned} \Pr(S_n \geq k) &= \sum_{j=k}^n b(j; n, p) \\ &\leq b(k; n, p) [1 + r + r^2 + \dots] \\ &= b(k; n, p) \frac{1}{1-r} \\ &= b(k; n, p) \frac{(k+1)(1-p)}{(k+1) - (n+1)p}. \end{aligned}$$

A similar proof holds for the second bound. $\square$

When p is sufficiently small, we can deal with factor $(1-p)^k$ in $b(k; n, p)$ as follows.

Property 5.1 ★ If $0 \leq p \leq 1/2$, then

$$\exp(-kp^2) \leq e^{kp}(1-p)^k \leq \exp(-\frac{1}{2}kp^2).$$

In particular, if $kp^2 = o(1)$, then $(1-p)^k \sim e^{-kp}$.

Proof. Note that for $|x| < 1$, $\log(1-x) = -(x + \frac{x^2}{2} + \frac{x^3}{3} + \dots)$, thus

$$\begin{aligned} \log(e^{kp}(1-p)^k) &= kp - k \left(p + \frac{p^2}{2} + \frac{p^3}{3} + \dots \right) \\ &= -k \left(\frac{p^2}{2} + \frac{p^3}{3} + \dots \right) \\ &\leq \frac{-kp^2}{2}. \end{aligned}$$

To obtain the second bound, note that $2(1-p) \geq 1$,

$$\begin{aligned} \log(e^{kp}(1-p)^k) &= -k \left(\frac{p^2}{2} + \frac{p^3}{3} + \dots \right) \\ &\geq \frac{-k}{2} (p^2 + p^3 + \dots) \\ &= \frac{-kp^2}{2(1-p)} \\ &\geq -kp^2, \end{aligned}$$

the required bound follows. □

5.2 Chernoff Bound for Different Distributions ★

In applications, we may need to consider a sum of random variables such as $X = \sum_{i=1}^n X_i$, where X_i may have different distributions.

Theorem 5.9 Let $X_1, \dots, X_n$ be mutually independent random variables such that for $1 \leq i \leq n$, $\Pr[X_i = 1] = p_i$ and $\Pr[X_i = 0] = 1 - p_i$, where $0 < p_i < 1$. Then for $X = \sum_{i=1}^n X_i$, and for $\delta > 0$,

$$\Pr[X > (1 + \delta)\mu] < \left(\frac{e^\delta}{(1 + \delta)^{1+\delta}} \right)^\mu,$$

where $\mu = E[X] = \sum_{i=1}^n p_i$. In particular, for $0 < \delta \leq 1$

$$\Pr[X > (1 + \delta)\mu] < \exp\left(-\frac{\delta^2\mu}{4}\right).$$

A more general form of the above theorem is a Chernoff bound for weighted random variables in different distributions as follows.

Theorem 5.10 *Let $X_1, \dots, X_n$ be mutually independent random variables such that for $1 \leq i \leq n$, $\Pr[X_i = 1] = p_i$ and $\Pr[X_i = 0] = 1 - p_i$, where $0 < p_i < 1$. Then for $X = \sum_{i=1}^n w_i X_i$, where $w_1, \dots, w_n$ are positive constants, and for $\delta > 0$,*

$$\Pr[X > (1 + \delta)\mu] < \left(\frac{e^\delta}{(1 + \delta)^{1+\delta}}\right)^{\mu/w},$$

where $\mu = E[X] = \sum_{i=1}^n w_i p_i$, and $w = \max\{w_i : 1 \leq i \leq n\}$. In particular, for $0 < \delta \leq 1$

$$\Pr[X > (1 + \delta)\mu] < \exp\left(-\frac{\delta^2\mu}{4w}\right).$$

Remark. If $\mu/w \rightarrow \infty$ as $n \rightarrow \infty$, then $\Pr[X > (1 + \delta)\mu] \rightarrow 0$.

We need a simple lemma as follows.

Lemma 5.2 *For any real numbers $0 < p, w \leq 1$, and $t > 0$,*

$$p(e^{tw} - 1) + 1 < \exp(wp(e^t - 1)).$$

Proof. Set

$$f(t) = \exp(wp(e^t - 1)) - p(e^{tw} - 1) - 1.$$

Then $f(0) = 0$ and

$$\begin{aligned} f'(t) &= \exp(wp(e^t - 1))wpe^t - wpe^{tw} \\ &\geq wp(e^t - e^{tw}) \geq 0, \end{aligned}$$

and for small t , $f'(t) > 0$, which yield the required statement.

Proof of Theorem 5.10. We shall separate the proof for the first bound into two cases depending on the value of w .

Case 1: $w = 1$.

For any real number $t > 0$,

$$\Pr[X > (1 + \delta)\mu] = \Pr[\exp(tX) > \exp(t(1 + \delta)\mu)].$$

Applying the Markov inequality gives

$$\Pr[X > (1 + \delta)\mu] \leq \frac{E[\exp(tX)]}{\exp(t(1 + \delta)\mu)}.$$

From the assumption on independency of $X_1, \dots, X_n$ and Lemma 5.2, we have

$$\begin{aligned} E[\exp(tX)] &= E[\exp(\sum_{i=1}^n tw_i X_i)] = E[\prod_{i=1}^n \exp(tw_i X_i)] \\ &= \prod_{i=1}^n E[\exp(tw_i X_i)] = \prod_{i=1}^n [p_i \exp(tw_i) + 1 - p_i] \\ &= \prod_{i=1}^n [p_i(\exp(tw_i) - 1) + 1] < \prod_{i=1}^n [\exp(w_i p_i (e^t - 1))] \\ &= \exp\left(\sum_{i=1}^n w_i p_i (e^t - 1)\right) = \exp((e^t - 1)\mu). \end{aligned}$$

Therefore

$$\Pr[X > (1 + \delta)\mu] < \frac{\exp((e^t - 1)\mu)}{\exp(t(1 + \delta)\mu)} = (\exp(e^t - 1 - t(1 + \delta)))^\mu.$$

Then the first bound with $w = 1$ follows by taking $t = \log(1 + \delta) > 0$, where the logarithmic function is on natural base.

Case 2: $w \neq 1$.

Set $w'_i = w_i/w$, $X' = \sum_{i=1}^n w'_i X_i = X/w$, and $\mu' = E[X'] = \sum_{i=1}^n w'_i p_i = \mu/w$. Then $w' = \max\{w'_i : 1 \leq i \leq n\} = 1$. Applying what just proved in case 1 on X' , we obtain

$$\begin{aligned} \Pr[X > (1 + \delta)\mu] &= \Pr[X' > (1 + \delta)\mu'] \\ &< \left(\frac{e^\delta}{(1 + \delta)^{1+\delta}}\right)^{\mu'} = \left(\frac{e^\delta}{(1 + \delta)^{1+\delta}}\right)^{\mu/w}, \end{aligned}$$

completing the proof for the first bound.

To obtain the second bound, note that for any $1 \geq \delta > 0$,

$$(1 + \delta) \log(1 + \delta) > \delta + \delta^2/4,$$

thus

$$\begin{aligned} \frac{e^\delta}{(1 + \delta)^{1+\delta}} &= \frac{e^\delta}{\exp((1 + \delta) \log(1 + \delta))} \\ &< \frac{e^\delta}{\exp(\delta + \delta^2/4)} = \exp(-\delta^2/4), \end{aligned}$$

as desired.

Corollary 5.1 *Let $Y_1, \dots, Y_n$ be mutually independent random variables such that for $1 \leq i \leq n$, $\Pr[Y_i = w_i] = p_i$ and $\Pr[Y_i = 0] = 1 - p_i$, where $0 < p_i < 1$, and $w_1, \dots, w_n$ are positive constants. Then for $Y = \sum_{i=1}^n Y_i$, and for $\delta > 0$,*

$$\Pr[Y > (1 + \delta)\mu] < \left(\frac{e^\delta}{(1 + \delta)^{1+\delta}} \right)^{\mu/w},$$

where $\mu = E[Y] = \sum_{i=1}^n w_i p_i$, and $w = \max\{w_i : 1 \leq i \leq n\}$.

5.3 Lower Bounds of $r_k(K_{m,n})$ and $r(K_{m,n}, K_n)$

Recall that $r_k(G)$ is the smallest integer N such that in any k -coloring of edges of K_N , there is a monochromatic G . Chung, Erdős, and Graham (1975) propose a problem to determine $r_k(K_{m,n})$. We now give a lower bound for it as k and m are fixed and $n \rightarrow \infty$.

Theorem 5.11 (Li, Tang and Zang) *For any fixed positive integers k and m , as $n \rightarrow \infty$,*

$$r_k(K_{m,n}) \geq (1 - o(1))k^m n.$$

Proof. Fix $\epsilon > 0$, we turn to show that $r_k(K_{m,n}) \geq (k^m - \epsilon)n$ for large enough n . To this end, set

$$N = \lfloor (k^m - \epsilon/2)n \rfloor - m.$$

Since for all sufficiently small (independent of n) $\delta > 0$,

$$n \geq \frac{N + m}{k^m - \epsilon/2} > \frac{N}{k^m - \sqrt{\delta}} > (k^{-m} + \delta)N = (p + \delta)N,$$

where $p = k^{-m}$, we can fix a small $\delta > 0$ which meets the requirement of Theorem 5.5. Now let us randomly and independently color the edges of K_{N+m} with k colors, such that each edge is assigned in each color with probability $1/k$. Consider a fixed color, say color A , and an arbitrary but fixed set U of m vertices. Let $v_1, v_2, \dots, v_N$ be the N vertices outside U . For each j , define a random variable X_j such that $X_j = 1$ if the edges between v_j and U are all in color A and 0 otherwise. Then $\Pr(X_j = 1) = k^{-m} = p$. Set $S_N = \sum_{j=1}^N X_j$. Clearly S_N has the binomial distribution $B(N, p)$ and the event $S_N \geq n$ means that there is a monochromatic $K_{m,n}$ in color A (in which U is the m -vertex part). Hence

$$\Pr(\exists \text{ monochromatic } K_{m,n}) \leq k \binom{N+m}{m} \Pr(S_N \geq n).$$

By virtue of Chernoff bound (Theorem 5.5)

$$\Pr(S_N \geq n) \leq \Pr(S_N \geq (p + \delta)N) \leq \exp\{-N\delta^2/(3pq)\}.$$

So the probability that there exists monochromatic $K_{m,n}$ tends to zero as $N \rightarrow \infty$, which guarantees the existence of an edge-coloring of K_{N+m} with no monochromatic $K_{m,n}$. Hence $r_k(K_{m,n}) > N + m > (k^m - \epsilon)n$ for all sufficiently large n . $\square$

In Chapter 3, we proved that for any fixed $m \geq 1$,

$$r(K_m + \overline{K_n}, K_n) \leq (1 + o(1)) \frac{n^{m+1}}{(\log n)^{m-1}}.$$

We will see that the upper bound is very close to truth. In fact, an application of Chernoff bound will yield a lower bound of $r(K_{m,n}, K_n)$ as follows.

Theorem 5.12 (Li and Rousseau, 1997) *Let integer $m \geq 2$ be fixed. Then there exists a constant $c = c(m) > 0$ such that*

$$r(K_{m,n}, K_n) \geq c \frac{n^{m+1}}{(\log n)^m}.$$

Proof. The lower bound is obtained through a simple application of Chernoff bound (Theorem 5.7). With

$$N = \left\lfloor \frac{n^{m+1}}{3(2m \log n)^m} \right\rfloor,$$

let $G(N, p)$ be a random graph of order N and edge probability $p = (2m \log n)/n$. The probability that m chosen vertices in $G(N, p)$ are connecting with another fixed vertex is p^m . So the probability that they have at least n common neighbors is $\Pr(S \geq n)$, where S has the binomial distribution $B(N - m, p^m)$. Then $n > Np^m$ and Theorem 5.7 yields

$$\begin{aligned} \Pr(K_{m,n} \subseteq G(N, p)) &\leq \binom{N}{m} \left(\frac{(N - m)p^m e}{n} \right)^n \\ &< \frac{N^m}{m!} \left(\frac{Np^m e}{n} \right)^n \\ &< c_1 \frac{n^{m(m+1)}}{(\log n)^{m^2}} \left(\frac{e}{3} \right)^n, \end{aligned}$$

where $c_1 = c_1(m) > 0$ is a constant. Hence $\Pr(K_{m,n} \subseteq G(N, p)) \rightarrow 0$. At the time, by standard estimates that $\binom{N}{n} \leq (Ne/n)^n$ and $1 - p < e^{-p}$, we

obtain a bound of the probability that $G(N, p)$ has an independent set of size at least n as follows

$$\begin{aligned} \Pr(\alpha(G(N, p)) \geq n) &\leq \binom{N}{n} (1-p)^{n(n-1)/2} \\ &\leq \left(\frac{Ne}{n} e^{-p(n-1)/2} \right)^n \\ &\leq \left(\frac{c_2}{3(2m \log n)^m} \right)^n, \end{aligned}$$

where $c_2 = c_2(m) > 0$ is a constant, so $\Pr(\alpha(G(N, p)) \geq n) \rightarrow 0$. Hence the probability that $G(N, p)$ contains neither $K_{m,n}$ as a subgraph nor an independent set of size n is positive (in fact, close to 1). Thus $r(K_{m,n}, K_n) > N$. $\square$.

We shall review to this problem in Chapter 8, where we will show that the lower bound in above theorem is the right order of $r(K_{m,n}, K_n)$.

Exercises

5.1 Let X be a random variable, the its variance $\sigma^2 = E(|X - \mu|)$ and expectation $\mu = E(X)$ satisfy

$$\sigma^2 = E(X^2) - \mu^2.$$

5.2 With $p = (1 - \epsilon) \log n/n$, how large can m be such that almost every graph in $\mathcal{G}(n, p)$ has at least m isolated vertices.

5.3 What is the asymptotically optimal constant in Theorem 5.12 from the proof?

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Chapter 6

Some Properties of Random Graphs ★

6.1 Some Behavior of Almost All Graphs

We digress a bit from extremal graphs to random graphs in this chapter. We do so for the beauty of the results and for the better understanding the nature of the existence given by the “almost all” arguments.

Given a graph property A , it is often associated with a family Q of graphs as

$$Q = Q(A) = \{G : G \text{ has } A\}.$$

Slightly abusing notation, we do not distinguish the property A and the family Q if no danger of confusion. We say that almost all (a.a.) graphs in $\mathcal{G}(n, p)$ have property Q if $\lim_{n \rightarrow \infty} \Pr[G_p \in Q] = 1$. In this case we also say that almost surely that $G_p \in \mathcal{G}(n, p)$ has property Q . We always suppose that n is large. We begin at a classic result of Erdős that almost all graphs seem to behave strangely even though they are sparse.

We will use $S|_G$ signify for the subgraph of G induced by vertex set S , and $d(G)$ for the average degree of G .

Theorem 6.1 (Erdős, 1962) *For all $k \geq 1$, there exist positive constants c and ϵ such that almost all graphs in $\mathcal{G}(n, p)$ with $p = c/n$ satisfy that $d(G) \geq k$ and $\chi(G) \geq k$, and yet $d(S|_G) < 3$ and $\chi(S|_G) \leq 3$ for any vertex subset S with $|S| \leq \epsilon n$.*

Proof. In order to simplify the notation, we will prove that $d(G) \geq k - 1$ instead of $d(S|_G) \geq k$. Let

$$H(x) = -\log x^x(1-x)^{1-x}, \quad 0 < x < 1,$$

and let constants c and ϵ satisfy

$$c > 2k^2 H(1/k) \quad \text{and} \quad \frac{c^{3/2} e^{5/2}}{3^{3/2}} \epsilon^{1/2} < 1.$$

Set $p = c/n$ and consider $G = G_p$ in $\mathcal{G}(n, p)$. We show that all most graphs in this space satisfy the conditions in Theorem. If $d(G) \leq k-1$ or $\chi(G) \leq k$, then $\alpha(G) \geq n/k$. The expected numbers of such independent sets is

$$\binom{n}{n/k} (1-p)^{\binom{n/k}{2}}.$$

From $m! \sim \sqrt{2\pi m} (m/e)^m$, we estimate that

$$\begin{aligned} \binom{n}{n/k} &= \frac{n!}{(n/k)!(n-n/k)!} \\ &\sim \frac{k}{\sqrt{2\pi n(k-1)}} \left((1/k)^{1/k} (1-1/k)^{1-1/k} \right)^{-n} \\ &< \exp\{nH(1/k)\}, \end{aligned}$$

and

$$(1-p)^{\binom{n/k}{2}} \leq \exp\left\{-\frac{pn}{2k} \left(\frac{n}{k} - 1\right)\right\} = \exp\left\{-\frac{cn}{2k^2} (1 - o(1))\right\}.$$

Therefore,

$$\binom{n}{n/k} (1-p)^{\binom{n/k}{2}} \leq \exp\left\{-n \left(\frac{c}{2k^2} - H\left(\frac{1}{k}\right) - o(1)\right)\right\},$$

which tends to zero by the condition satisfied by c .

Suppose some set S with $t \leq \epsilon n$ vertices such that $d(S|_G) > 3$ or $\chi(S|_G) \geq 4$, we claim that that $S|_G$ would have at least $3t/2$ edges. The claim is trivial if $d(S|_G) > 3$. If $\chi(S|_G) \geq 4$, then we would have a minimal such set S . For any $v \in S$, there would be a (proper) 3-coloring of $S \setminus \{v\}$. If v has two or fewer neighbors in $S|_G$ then it would be extended to a 3-coloring of S . Hence the minimum degree of $S|_G$ is at least 3 and it follows by the claim. The probability that some $t \leq \epsilon n$ vertices have at least $3t/2$ edges is less than

$$\sum_{4 \leq t \leq \epsilon n} \binom{n}{t} \binom{\binom{t}{2}}{3t/2} \left(\frac{c}{n}\right)^{3t/2} = \sum_{4 \leq t \leq n^{1/4}} + \sum_{n^{1/4} < t \leq \epsilon n} = s_1 + s_2.$$

We bound

$$\binom{n}{t} \leq \left(\frac{en}{t}\right)^t \quad \text{and} \quad \binom{\binom{t}{2}}{3t/2} \leq \left(\frac{et}{3}\right)^{3t/2}.$$

So each term is at most

$$\left(\frac{en}{t}\right)^t \left(\frac{et}{3}\right)^{3t/2} \left(\frac{c}{n}\right)^{3t/2} = \left(\frac{c^{3/2}e^{5/2}t^{1/2}}{3^{3/2}n^{1/2}}\right)^t.$$

Hence the first summation

$$s_1 \leq n^{1/4} \left(\frac{c^{3/2}e^{5/2}n^{1/4}}{3^{3/2}n^{1/2}}\right)^4 \rightarrow 0.$$

Each term in the second summation is at most

$$\left(\frac{c^{3/2}e^{5/2}t^{1/2}}{3^{3/2}n^{1/2}}\right)^t \leq \left(\frac{c^{3/2}e^{5/2}}{3^{3/2}}\epsilon^{1/2}\right)^{n^{1/4}}.$$

Let δ be the bracketed term $\frac{c^{3/2}e^{5/2}}{3^{3/2}}\epsilon^{1/2}$, then $\delta < 1$ by the choice of ϵ . Thus the second summation

$$s_2 \leq \epsilon n \delta^{n^{1/4}} \rightarrow 0.$$

So almost surely no such set S exists, completing the proof. $\square$

By the last theorem, we know that in random graphs, the neighbors of an average vertex distribute evenly in every part of the vertex set. So their clique numbers and independence numbers are small, and chromatic number are big. Let us enjoy a result of Erdős style that he used this central observation to disprove a conjecture with almost all graphs.

A suspended path in graph G is a path $(x_0, x_1, \dots, x_k)$ in which $x_1, \dots, x_{k-1}$ have degree two in G . A graph H is a *subdivision* of G if H is obtained from G replacing edges of G by suspended paths, that is to say, H is obtained by adding vertices on the edges of G . Hajos conjectured that every graph G with $\chi(G) \geq r$ contains a subdivision of K_r . Catlin disproved the conjecture by a constructive proof, but Erdős's disproof was more powerful. Let $\gamma(G)$ denote the largest r such that G contains a subdivision of K_r . Hojas conjecture is equivalent to that $\gamma(G) \geq \chi(G)$.

Theorem 6.2 *Almost all graphs $G_p \in \mathcal{G}(n, 1/2)$ satisfy*

$$\chi(G) \geq \frac{n}{2 \log_2 n}, \quad \text{and} \quad \gamma(G) \leq \sqrt{6n}.$$

Proof. Set $k = \lfloor 2 \log_2 n \rfloor$. Since

$$\Pr(\alpha(G) \geq k) \leq \binom{n}{k} 2^{-\binom{k}{2}} < \left(\frac{e\sqrt{2}n}{k2^{k/2}} \right)^k \rightarrow 0,$$

and the fact

$$\alpha(G)\chi(G) \geq n$$

for any graph G , the first statement follows immediately. Set $r = \lceil \sqrt{6n} \rceil$. Then $n \leq r^2/6$. There are

$$\binom{n}{r} \leq \left(\frac{en}{r} \right)^r \leq \left(\frac{er}{6} \right)^r$$

potential K_r subdivisions, one for each r -element subset of $V(G)$. If we fix such a subset X , then we notice that since each subdivided edge has to use a distinct vertex of $V(G) \setminus X$. There are $\binom{r}{2}$ suspended paths in a subdivision, and at most $n - r$ of them are of length two or more, which are “really” subdivided edges. So the number of edges in subgraph induced by X is at least

$$\binom{r}{2} - (n - r) \geq \binom{r}{2} + r - \frac{r^2}{6} \geq \frac{2}{3} \binom{r}{2}$$

edges. But the number of edges in subgraph induced by X , denoted by $e(X)$, has binomial distribution $B(N, 1/2)$, where $N = \binom{r}{2}$. From Chernoff bound in Chapter 5,

$$\Pr(e(X) \geq N(1 + \delta)/2) \leq \exp\{-N\delta^2/2\},$$

by taking $\delta = 1/3$ hence $\frac{2}{3} \binom{r}{2} = \binom{r}{2}(1 + \delta)/2$, we have

$$\Pr\left(e(X) \geq \frac{2}{3} \binom{r}{2}\right) \leq \exp\{-N\delta^2/2\} = \exp\left\{-\frac{1}{18} \binom{r}{2}\right\}.$$

Thus we bound the probability that our random graph G contains a subdivision of K_r as follows.

$$\begin{aligned} \Pr(\gamma(G) \geq r) &\leq \sum_X \Pr\left(e(X) \geq \frac{2}{3} \binom{r}{2}\right) \\ &\leq \binom{n}{r} \exp\left\{-\frac{1}{18} \binom{r}{2}\right\} \\ &= \left(\frac{er \exp\{-(r-1)/36\}}{6} \right)^r, \end{aligned}$$

which tends to zero as $n \rightarrow \infty$. $\square$

Since for almost all G ,

$$\chi(G) - \gamma(G) \geq \frac{n}{2 \log_2 n} - \sqrt{6n} \rightarrow \infty$$

as $n \rightarrow \infty$, so Hajos conjecture is false, and almost all graphs in $\mathcal{G}(n, 1/2)$ are counterexamples. Further more, the gap between the truth and the conjecture is big.

For fixed $0 < p \leq 1$, most graphs in $\mathcal{G}(n, p)$ are dense. One of the most striking phenomena on parameter concentration was discovered by Bollobás and Erdős (1976) for dense random graphs. Let $d = d(n, p)$ be the maximum integer with

$$\binom{n}{d} p^{\binom{d}{2}} \geq \log n.$$

So $d \sim 2 \log_{1/p} n$ for fixed $1 > p > 0$. Then

$$\Pr [d \leq \omega(G_p) \leq d + 1] \rightarrow 1.$$

Namely, the clique numbers of a.a. graphs in $\mathcal{G}(n, p)$ are concentrated at two values. Since the complementary space of $\mathcal{G}(n, p)$ is $\mathcal{G}(n, q)$ with $q = 1 - p$, so the independence numbers of a.a. graphs in $\mathcal{G}(n, p)$ are concentrated at two values, which are asymptotically $2 \log_{1/q} n$. Bollobás (1988) proved that the trivial inequality $\alpha(G)\chi(G) \geq n$ becomes an asymptotical equality by determining the chromatic number for $G_p \in \mathcal{G}(n, p)$ by showing

$$\Pr \left(\frac{n}{2 \log_{1/q} n} < \chi(G_p) < (1 + \epsilon) \frac{n}{2 \log_{1/q} n} \right) \rightarrow 1$$

where $1 > p > 0$ and $\epsilon > 0$ are fixed.

We shall see that almost all graphs are nearly regular.

Theorem 6.3 *Let $0 \leq p \leq 1$ be fixed. Then almost all graphs G in $\mathcal{G}(n, p)$ satisfy that*

$$\Delta(G) \sim \delta(G) \sim (n - 1)p.$$

Proof. Let G be a random graph in $\mathcal{G}(n, p)$ and let v be a fixed vertex of G . Then $\deg(v)$ has binomial distribution $B(n - 1, p)$. From Chernoff's Theorems in Chapter 5, we have

$$\Pr(|\deg(v) - (n - 1)p| > (n - 1)\epsilon) < 2 \exp\{-(n - 1)\epsilon^2/(2pq)\},$$

which tends to zero as $n \rightarrow \infty$. $\square$

6.2 Threshold Functions

We have seen that random graphs in $\mathcal{G}(n, p)$ behave sensitively on $p = p(n)$. A monumental discovery of Erdős and Rényi (1960) was that many natural graph theoretic properties become true in a very narrow range of $p = p(n)$.

A property Q is said to be *monotone increasing* if G has Q , then any graph from G by adding new edges has Q . The *monotone decreasing* property can be defined similarly. Thus the property of being connected is monotone increasing and that of being triangle-free is monotone decreasing. As mentioned in Section 1, a property Q is associated with a family of graphs. We call this family to be monotone increasing if so is Q . Also we do not distinguish the property and its associated family.

Lemma 6.1 *Let Q be a monotone increasing property, then $\Pr(G_p \in Q)$ is an increasing function on p .*

Proof. Let $0 \leq p_1(n) \leq p_2(n) \leq 1$, we shall verify

$$\Pr(G_{p_1} \in Q) \leq \Pr(G_{p_2} \in Q).$$

Let

$$Q = \{H_1, H_2, \dots, H_\ell\}$$

be the family of all spanning graphs on vertex set $[n]$ that have property. For $1 \leq j \leq \ell$, we have $\Pr(G_p \supset H_j) = p^{e(H_j)}$ hence $\Pr(G_p \not\supset H_j) = 1 - p^{e(H_j)}$. By the fact that Q is increasing, we obtain

$$\Pr(G_p \notin Q) = \prod_{j=1}^{\ell} (1 - p^{e(H_j)}),$$

yielding that

$$\Pr(G_p \in Q) = 1 - \prod_{j=1}^{\ell} (1 - p^{e(H_j)}).$$

The desired inequality follows since $p_1 \leq p_2$. $\square$

Let Q be a monotone increasing property. Erdős and Rényi defined a function $f(n)$ with $0 \leq f(n) \leq 1$ as *threshold function* for Q if

$$\lim_{n \rightarrow \infty} \Pr[G_p \in Q] = \begin{cases} 0 & \text{if } p = f(n)/\omega(n) \\ 1 & \text{if } p = f(n)\omega(n), \end{cases}$$

where $0 < \omega(n) < 1/f(n)$ is a function which tends to infinity with n , as slowly as desired. For example, if $f(n) = \log n/n$, we may take $\omega(n) =$

$\log \log n$. Note that if $f(n)$ is a threshold function for Q , so is $cf(n)$ with $c > 0$.

Obviously, the definition of $f(n)$ being a threshold function for a monotone increasing property Q is equivalent to

$$\lim_{n \rightarrow \infty} \Pr[G_p \in Q] = \begin{cases} 0 & \text{if } p \ll f(n) \\ 1 & \text{if } p \gg f(n), \end{cases}$$

where $p \ll f(n)$ means $p = o(f(n))$.

The definition of the threshold function for a monotone decreasing property is similar. The definition means the situation whether or not a.a. G_p have Q changes suddenly even though $p = p(n)$ changes slightly in the moment.

Let $X = X(G)$ be a non-negative integral parameter of graph G . Since

$$\Pr(X \geq 1) = \sum_{k \geq 1} \Pr(X = k) \leq E(X),$$

so $E(X) \rightarrow 0$ implies that a.a. graphs in $\mathcal{G}(n, p)$ satisfy $X = 0$. And in many cases $E(X) \rightarrow \infty$ implies that a.a. graphs in $\mathcal{G}(n, p)$ satisfy $X \geq 1$, which can be shown by Chebyshev's inequality often. For example, let X be the number of triangles in $G_p \in \mathcal{G}(n, p)$. Then

$$E(X) = \binom{n}{3} p^3 \sim \frac{1}{6} (np)^3.$$

As we will see in the next theorem that $f(n) = 1/n$ truly is a threshold function for triangle-freeness. Let $p = c/n$ and let $c \rightarrow 0$ and $c \rightarrow \infty$ signify $\omega(n)$ in the denominator and numerator in the definition, respectively. When c reaches and passes 1, the structure of G_p radically. This is called the *double jump* because the structure of G_p is significantly different for $c \ll 1$, $c \sim 1$ and $c \gg 1$.

Let us recall the Second Moment Method in last chapter.

Lemma 6.2 (Second Moment Method) *If X is a random variable, then*

$$\Pr(X = 0) \leq \frac{E(X^2) - \mu^2}{\mu^2},$$

where $\mu = E(X)$. In particular, $\Pr(X = 0) \rightarrow 0$ if $E(X^2)/\mu^2 \rightarrow 1$.

A graph G with average degree d is called *balanced* if no subgraph of it has average degree greater than d . Complete graphs, cycles and trees are balanced.

Theorem 6.4 *Let F be a balanced graph with $k \geq 2$ vertices and $\ell \geq 1$ edges and let Q be the property that a graph contains F as a subgraph. Then $f(n) = 1/n^{k/\ell}$ is a threshold function for Q .*

Proof. To simplify the notation, we shall use $p = \frac{\gamma}{n^{k/\ell}}$ with $\gamma \rightarrow 0$ and $\gamma \rightarrow \infty$ as before. Let $X = X(G)$ be the number of copies of F contained in $G = G_p \in \mathcal{G}(n, p)$. Denote by a for the number of graphs isomorphic to F on fixed k labelled vertices. Then

$$\mu = E(X) = \binom{n}{k} ap^\ell.$$

With the simple facts that $1 \leq a \leq k!$, k and ℓ are fixed, we have $E(X) \leq n^k p^\ell = \gamma^\ell$ and $E(X) \sim \frac{a}{k!} n^k p^\ell = \frac{a}{k!} \gamma^\ell$. Thus

$$c_1 \gamma^\ell \leq c_2 n^k p^\ell \leq \mu = E(X) \leq \gamma^\ell,$$

where c_1 and c_2 henceforth c_i are positive constants.

When $\gamma \rightarrow 0$, by Markov's inequality,

$$\Pr(G_p \in Q) = \Pr(X \geq 1) \leq E(X) \rightarrow 0.$$

We then want to show that $\Pr(G_p \in Q) = \Pr(X \geq 1) \rightarrow 1$ as $\gamma \rightarrow \infty$. We turn to the Second Moment Method for help since the Markov's inequality does not work in this case.

For any k labelled vertices in $[n]$, we have $a = k!/|\mathcal{A}|$, where $\mathcal{A}$ is the automorphism group of F . Then there are total of $a \binom{n}{k}$ potential copies of F on $[n]$. Denote by

$$\mathcal{F} = \{F_1, F_2, \dots\}$$

for the family of these copies. Denote by $F_i \cup F_j$ for the graph with vertex set $V(F_i) \cup V(F_j)$ and edge set $E(F_i) \cup E(F_j)$. The two critical observations are that most pairs F_i and F_j have no vertices in common, and if they have $s \geq 1$ common vertices and t edges, then $t/s \leq \ell/k$ since F is balanced.

Let X_i be the indicator function of F_i , then

$$E(X_i) = \Pr(X_i = 1) = \Pr(G \supset F_i).$$

Since $X = \sum_i X_i$, thus

$$E(X^2) = \sum_{i,j} E(X_i X_j) = \sum_{i,j} \Pr(G \supset F_i \cup F_j),$$

where the sum is taken over all pairs $\{i, j\}$ with $F_i, F_j \in \mathcal{F}$. Set

$$A_s = \sum_{|V(F_i) \cap V(F_j)|=s} \Pr(G \supset F_i \cup F_j),$$

where the sum is taken over all pairs F_i and F_j as they have s vertices in common. Then $E(X^2) = \sum_{s=0}^k A_s$. Note that if $V(F_i) \cap V(F_j) = \emptyset$, then

$$\Pr(G \supset F_i \cup F_j) = \Pr(G \supset F_i) \Pr(G \supset F_j)$$

since the independency of the events. We thus have the fact $A_0 < \mu^2$ which follows from

$$\begin{aligned} A_0 &= \sum_{V(F_i) \cap V(F_j) = \emptyset} \Pr(G \supset F_i \cup F_j) \\ &= \sum_{V(F_i) \cap V(F_j) = \emptyset} \Pr(G \supset F_i) \Pr(G \supset F_j) \\ &< \left(\sum_i \Pr(G \supset F_i) \right)^2 = \left(\sum_i E(X_i) \right)^2 \\ &= E^2(X) = \mu^2. \end{aligned}$$

For $s \geq 1$ it is expected that A_s is much less than μ^2 . Fix F_i , counting F_j that has s common vertices with F_i which contains at most $t \leq s\ell/k$ since F is balanced, we have

$$\begin{aligned} &\sum_{j: |V(F_i) \cap V(F_j)|=s} \Pr(G \supset F_i \cup F_j) \\ &\leq \sum_{0 \leq t \leq s\ell/k} \binom{k}{s} \binom{n-k}{k-s} p^{2\ell-t} \\ &\leq c_3 n^{k-s} \sum_{0 \leq t \leq s\ell/k} p^{2\ell-t} \end{aligned}$$

since k, s, ℓ are fixed and t is bounded. From the fact that there are $a \binom{n}{k}$ elements in $\mathcal{F}$, we obtain

$$\begin{aligned} A_s &\leq a \binom{n}{k} c_3 n^{k-s} \sum_{0 \leq t \leq s\ell/k} p^{2\ell-t} \\ &\leq c_4 n^{2k-s} \sum_{0 \leq t \leq s\ell/k} p^{2\ell-t} \leq c_4 (n^k p^\ell)^2 n^{-s} \sum_{0 \leq t \leq s\ell/k} p^{-t} \\ &\leq \frac{c_5 \gamma^{2\ell} n^{-s}}{p^{s\ell/k}} = \frac{c_5 \gamma^{2\ell}}{(np^{\ell/k})^s} \leq \frac{c_6 \gamma^{2\ell}}{\gamma^{s\ell/k}} \leq \frac{c_7 \mu^2}{\gamma^{s\ell/k}}, \end{aligned}$$

where we used the fact that $n^k p^\ell$, γ^ℓ and μ have the same order. So for

$s \geq 1$, we have $A_s/\mu^2 \leq c_7/\gamma^{s\ell/k}$, and

$$\frac{E(X^2)}{\mu^2} = \frac{A_0 + \sum_{s=1}^k A_s}{\mu^2} \leq 1 + \sum_{s=1}^k \frac{c_7}{\gamma^{s\ell/k}} \leq 1 + \frac{c_8}{\gamma^{\ell/k}}.$$

By the Second Moment Method,

$$\begin{aligned} \Pr(X = 0) &\leq \Pr(|X - \mu| \geq \mu) \leq \frac{\sigma^2}{\mu^2} \\ &= \frac{E(X^2) - \mu^2}{\mu^2} \leq c_8 \frac{1}{\gamma^{\ell/k}}, \end{aligned}$$

which tends to zero as $\gamma \rightarrow \infty$. $\square$

In traditional definition of threshold function $f(n)$ of Erdős and Rényi $p_1/p_2 \rightarrow 0$ with $p_1(n) = f(n)/\omega(n)$ and $p_2(n) = f(n)\omega(n)$. In many case this is not necessary. Perhaps it is just needed that p_1 is slightly less than p_2 . A more precise definition of threshold function is as follows.

Let Q be a monotone increasing property of graphs. A function $p_\ell = p_\ell(n)$ is called a *lower threshold function* (ltf) if almost no graphs in $\mathcal{G}(n, p_\ell)$ have Q ; a function $p_u = p_u(n)$ is called an *upper threshold function* (utf) if almost all graphs in $\mathcal{G}(n, p_u)$ have Q .

A realistic situation is very interesting. In a conference, a pair of mathematicians unknown each other can found a common mathematician friend, even they are regular, and none of them is familiar with many mathematicians. The following result called distance two theorem gives us a good explanation for this small world phenomenon.

Theorem 6.5 *For any function $\omega(n) \rightarrow \infty$ with $\omega(n) < \log n$, set*

$$p_\ell = \sqrt{\frac{2 \log n - \omega(n)}{n}}, \quad \text{and} \quad p_u = \sqrt{\frac{2 \log n + \omega(n)}{n}}.$$

Then p_ℓ and p_u are ltf and utf for the property of graph having distance two.

Proof. Enumerate of all pairs of vertices $\{u, v\}$ of $\mathcal{G}(n, p)$ as $e_1, e_2, \dots, e_m$ with $m = \binom{n}{2}$. For $e_k = \{u, v\}$, let $d(u, v)$ be the distance between u and v . Define

$$X_k = X_k(G) = \begin{cases} 0 & d(u, v) \leq 2 \\ 1 & \text{otherwise,} \end{cases}$$

and $X = \sum_{k=1}^m X_k$. A non-complete graph G has distance two if and only if $X = 0$. Since the event $d(u, v) \geq 3$ for a pair of non-adjacent vertices is

equivalent to that neither u nor v is adjacent to any other $n - 2$ vertices, so

$$E(X_k) = \Pr(X_k = 1) = (1 - p)(1 - p^2)^{n-2}.$$

Set $\mu = E(X)$, then

$$\mu = E(X) = \binom{n}{2}(1 - p)(1 - p^2)^{n-2}.$$

(i) Let $p = p_u = \sqrt{(2 \log n + \omega(n))/n}$. We have

$$\begin{aligned} \mu &\sim \frac{n^2}{2}(1 - p^2)^n \sim \frac{n^2}{2}e^{-np^2} \\ &= \frac{1}{2}e^{-\omega(n)} \rightarrow 0. \end{aligned}$$

Thus $\Pr(X \geq 1) \leq E(X) \rightarrow 0$, which proves that a.a. graphs in $\mathcal{G}(n, p)$ have distance two.

(ii) Let $p = p_\ell = \sqrt{(2 \log n - \omega(n))/n}$. Suppose that $\omega(n) < \log \log n$ without loss of generality since the property of distance two is monotone increasing. Consider

$$E(X^2) = \sum_{i,j} E(X_i X_j) = A_0 + A_1 + A_2,$$

where $A_s = \sum_{|e_i \cap e_j|=s} E(X_i X_j)$, in which the sum is taken over all pairs $\{i, j\}$ with e_i and e_j having s vertices in common. Clearly

$$\mu = E(X) \sim \frac{n^2}{2}e^{-np^2} = \frac{e^{\omega(n)}}{2} \rightarrow \infty,$$

and

$$A_0 = \sum_{|e_i \cap e_j|=0} E(X_i X_j) \leq \binom{n}{2} \binom{n-2}{2} (1 - p^2)^{2(n-2)} < \mu^2.$$

Also

$$A_2 = \sum_{k=1}^m E(X_k) = \mu.$$

We now estimate A_1 that should not be big since A_0 counts most of pairs. For $e_i = \{u, v\}$ and $e_j = \{v, w\}$ with $|e_i \cap e_j| = 1$, the event $d(u, v) \geq 3$ and $d(v, w) \geq 3$ is equivalent to the following events happen:

B_1 : u and v , v and w are non-adjacent;

B_2 : none of vertices other than u, v, w is adjacent to all of the three vertices.

B_3 : none of distinct vertices is adjacent to both u and v , another is not adjacent to both v and w ;

We have

$$\begin{aligned} A_1 &= \sum_{|e_i \cap e_j|=1} E(X_i X_j) = \Pr(B_1 B_2 B_3) \leq \Pr(B_3) \\ &= 3 \binom{n}{3} (1-p^2)^{2n-7} \\ &\sim \frac{n^3}{2} (1-p^2)^{2n} \sim \frac{n^3}{2} e^{-2np^2} = \frac{1}{2n} e^{2\omega(n)} \rightarrow 0. \end{aligned}$$

Hence

$$\sigma^2 = E(X^2) - \mu^2 = A_0 + A_1 + A_2 - \mu^2 < \mu + 1,$$

which and the Second Moment Method yield

$$\Pr(X = 0) \leq \Pr(|X - \mu| \geq \mu) \leq \frac{\sigma^2}{\mu^2} \rightarrow 0,$$

proving that almost no graphs in $\mathcal{G}(n, p)$ has distance two with $p = p_\ell$. $\square$

The further solution for diameter of random graphs is as follows. Let $\text{diam}(G)$ be the diameter of G and let $d \geq 2$ be an integer. If

$$p = n^{1/d-1} (\log(n^2/x))^{1/d},$$

then

$$\Pr(\text{diam}(G_p) = d) \rightarrow e^{-x/2}$$

and

$$\Pr(\text{diam}(G_p) = d+1) \rightarrow 1 - e^{-x/2}$$

as $n \rightarrow \infty$. See Bollobás (2001) for details. The above limit distribution implies that

$$p_{\ell,u} = n^{1/d-1} (2 \log n \pm \omega(n))^{1/d}$$

are ltf and utf of graphs being distance d .

The distance two graphs are of interest in graph Ramsey theory. Let G_n be a Ramsey graph of order $n = r(3, k) - 1$. We suppose that G_n is edge maximal for triangle-freeness. Then G_n is a distance two graph. Since the order of n is $k^2 / \log k$, so the maximum degree of G_n is bounded above by $k \leq c\sqrt{n \log n}$. It is likely that the minimum degree of G_n has order $\sqrt{n \log n}$ hence the order of its edge density is $\sqrt{\log n/n}$ as that in above theorem.

Problem 6.1 Let G_n be a Ramsey graph of order $n = r(3, k) - 1$ that is edge maximal. Determine the orders of the minimum and maximum degrees of G_n as $k \rightarrow \infty$.

The following result gives a precise form of the threshold functions for the property of being connected. The deeper version of the result will be in the next section.

Theorem 6.6 Let $\omega(n) \rightarrow \infty$ be a function with $\omega(n) < \log n$. Set

$$p_\ell = \frac{\log n - \omega(n)}{n} \quad \text{and} \quad p_u = \frac{\log n + \omega(n)}{n}.$$

Then p_ℓ and p_u are ltf and utf for graph in $\mathcal{G}(n, p)$ with the property of being connected.

Proof. Let Q be the family of connected graphs. Since Q is monotone increasing, we may assume that $\omega(n) \leq \log \log n$ without loss of generality. Let $X_k = X_k(G)$ be the number of components of $G \in \mathcal{G}(n, p)$ that have exactly k vertices. We first prove that $p = p_\ell$ is a ltf for Q . Denote by μ for $E(X_1)$. Note that $(1 - p)^n \sim e^{-np}$ since $np^2 \rightarrow 0$, we have

$$\begin{aligned} \mu &= E(X_1) = n(1 - p)^{n-1} \\ &\sim n(1 - p)^n \sim ne^{-np} = e^{\omega(n)} \rightarrow \infty. \end{aligned}$$

This may indicate that $\Pr(X_1 = 0) \rightarrow 0$ and $\Pr(X_1 \geq 1) \rightarrow 1$, so a.a. graphs have isolated vertices hence they are disconnected. In order to use the Second Moment Method, we need to estimate the variance $\sigma^2(X_1)$ hence $E(X_1^2)$. We first have

$$E[X_1(X_1 - 1)] = n(n - 1)(1 - p)^{2n-3},$$

which is the expected number of order pairs of isolated vertices. There are $n(n - 1)$ ordered pairs of vertices, and such a pair are isolated if and only if they are neither adjacent each other nor adjacent to any other $n - 2$ vertices, which count $2n - 3$ edges. Then

$$\begin{aligned} E(X_1^2) &= E[X_1(X_1 - 1)] + E(X_1) \\ &= \mu + n(n - 1)(1 - p)^{2n-3}. \end{aligned}$$

We thus have

$$\begin{aligned} \sigma^2(X_1) &= E[(X_1 - \mu)^2] = E(X_1^2) - \mu^2 \\ &= \mu + n(n - 1)(1 - p)^{2n-3} - n^2(1 - p)^{2n-2} \\ &\leq \mu + pn^2(1 - p)^{2n-3}. \end{aligned}$$

Since $p = (\log n - \omega(n))/n$ with $\log \log n \geq \omega(n) \rightarrow \infty$ and $1 - p \leq e^{-p}$,

$$\begin{aligned} pn^2(1-p)^{2n-3} &\leq (1+o(1))(\log n)ne^{-2np} \\ &= (1+o(1))(\log n)ne^{-2\log n+2\omega(n)} \\ &= (1+o(1))\frac{\log n}{n}e^{2\omega(n)} \rightarrow 0. \end{aligned}$$

We thus obtain

$$\sigma^2(X_1) \leq \mu + 1.$$

This and the Second Moment Method give

$$\Pr(G_p \in Q) \leq \Pr(X_1 = 0) \leq \Pr(|X_1 - \mu| \geq \mu) \leq \frac{\sigma^2}{\mu^2} \rightarrow 0,$$

proving that p_ℓ is a ltf for property of being connected.

Now let $p = p_u = (\log n + \omega(n))/n$. Note that if G is not connected then it must contain a component of order at most $\lfloor n/2 \rfloor$. So

$$\Pr(G_p \notin Q) = \Pr\left(\sum_{k=1}^{\lfloor n/2 \rfloor} X_k \geq 1\right) \leq E\left(\sum_{k=1}^{\lfloor n/2 \rfloor} X_k\right) = \sum_{k=1}^{\lfloor n/2 \rfloor} E(X_k).$$

Since if a set with k vertices induces a component, then any vertex in it is not adjacent to any vertex out of it. Thus

$$E(X_k) \leq \binom{n}{k}(1-p)^{k(n-k)},$$

where we ignore condition that the set is connected. Therefor,

$$\Pr(G_p \notin Q) \leq \sum_{k=1}^{\lfloor n/2 \rfloor} \binom{n}{k}(1-p)^{k(n-k)}.$$

Let us split the sum into two parts S_1 and S_2 . Note that $e^{k^2 p} \rightarrow 1$ uniformly for $k \leq n^{3/4}$, and $e^{-knp} = (n^{-1}e^{-\omega(n)})^k$, we have

$$\begin{aligned} S_1 &= \sum_{1 \leq k \leq n^{3/4}} \binom{n}{k}(1-p)^{k(n-k)} \\ &\leq \sum_{1 \leq k \leq n^{3/4}} \left(\frac{en}{k}\right)^k e^{-knp} e^{k^2 p} \\ &\leq (1+o(1)) \sum_{1 \leq k \leq n^{3/4}} \frac{e^{(1-\omega(n))k}}{k^k} \end{aligned}$$

$$\begin{aligned}
&\leq (1 + o(1)) \sum_{1 \leq k \leq n^{3/4}} e^{(1-\omega(n))k} \\
&\leq (1 + o(1)) e^{1-\omega(n)} \rightarrow 0.
\end{aligned}$$

Note that for $n^{3/4} < k \leq n/2$, we have $\binom{n}{k} \leq (en/k)^k \leq (en^{1/4})^k$ and

$$(1-p)^{k(n-k)} \leq (1-p)^{kn/2} \leq e^{-knp/2} < \frac{1}{n^{k/2}},$$

hence

$$\begin{aligned}
S_2 &= \sum_{n^{3/4} < k \leq \lfloor n/2 \rfloor} \binom{n}{k} (1-p)^{k(n-k)} \\
&\leq \sum_{n^{3/4} < k \leq n/2} \left(\frac{e}{n^{1/4}} \right)^k \leq (1 + o(1)) \left(\frac{e}{n^{1/4}} \right)^{n^{3/4}} \rightarrow 0.
\end{aligned}$$

Thus $S_1 + S_2 \rightarrow 0$, proving that a.a. graphs in $\mathcal{G}(n, p)$ are connected. $\square$

6.3 Poisson Limit

In order to show that a.a graphs in $\mathcal{G}(n, p)$ are disconnected for $p = (\log n - \omega(n))/n$, we in fact have proved that a.a. graphs in $\mathcal{G}(n, p)$ have isolated vertices. We shall have some deeper results in this section. Let us begin with a basic identity called *inclusion-exclusion formula*.

In a probability space Ω , let $X_1, X_2, \dots, X_\ell$ be 0-1 random variables and set

$$X = X_1 + X_2 + \dots + X_\ell.$$

As usual, denote by $[\ell]$ for $\{1, 2, \dots, \ell\}$. Define $S_0 = 1$ and

$$S_r = \sum_{[\ell]^{(r)}} \Pr(X_{i_1} \dots X_{i_r} = 1),$$

where the sum is taken over elements $\{i_1, \dots, i_r\} \in [\ell]^{(r)}$, the family of all r -subsets of $[\ell]$. Note the elements ω such that $X_{i_1} \dots X_{i_r} = 1$ are what such that $X_{i_1} = 1, \dots, X_{i_r} = 1$, and $S_r = 0$ for $r > \ell$. For general r ,

$$S_r = \sum_{\omega \in \Omega} \binom{X(\omega)}{r} \Pr(\omega)$$

as an element ω of the sample space for which $X(\omega) = t$ contributes to $\binom{t}{r}$ of the terms defining S_r . Here and in what follow, we write the formulas appropriate for finite sample space. Following standard notation, we define falling factorials by $(X)_0 = 1$ and

$$(X)_r = X(X-1) \cdots (X-r+1).$$

Then

$$S_r = \sum_{\omega \in \Omega} \frac{(X)_r}{r!} \Pr(\omega) = \frac{E((X)_r)}{r!}.$$

The quantity $E((X)_r)$ is called the r th factorial moment of X .

Theorem 6.7 (Inclusion-Exclusion Formula) *For each integer $k \geq 0$,*

$$\Pr(X = k) = \sum_{r \geq 0} (-1)^r \binom{k+r}{r} S_{k+r}.$$

Moreover, for each integer $m \geq 0$,

$$\sum_{r=0}^{2m-1} (-1)^r \binom{k+r}{r} S_{k+r} \leq \Pr(X = k) \leq \sum_{r=0}^{2m} (-1)^r \binom{k+r}{r} S_{k+r}.$$

Proof. It is easy to see

$$\begin{aligned} \Pr(X = 0) &= \Pr(X_1 = 0, \dots, X_\ell = 0) \\ &= 1 - \Pr(\exists i, X_i = 1) = S_0 - S_1 + S_2 - \cdots. \end{aligned}$$

For general k , using

$$S_{k+r} = \sum_{\omega \in \Omega} \binom{X(\omega)}{k+r} \Pr(\omega),$$

and interchanging orders of the summation, we obtain

$$\sum_{r \geq 0} (-1)^r \binom{k+r}{r} S_{k+r} = \sum_{\omega \in \Omega} \left\{ \sum_{r \geq 0} (-1)^r \binom{k+r}{r} \binom{X(\omega)}{k+r} \right\} \Pr(\omega).$$

For a fixed ω hence fixed $X = X(\omega)$, note that

$$\sum_{r \geq 0} (-1)^r \binom{k+r}{r} \binom{X}{k+r} = \binom{X}{k} \sum_{r \geq 0} (-1)^r \binom{X-k}{r}.$$

If $X < k$, all terms vanish. If $X = k$, then one term ($r = 0$) contributes and the sum is 1. Finally, if $X > k$, the sum vanishes since

$$\sum_{r \geq 0} (-1)^r \binom{X-k}{r} = (1-1)^{X-k} = 0.$$

Thus

$$\sum_{r \geq 0} (-1)^r \binom{k+r}{r} \binom{X}{k+r} = \begin{cases} 1 & \text{if } X = k \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\sum_{r \geq 0} (-1)^r \binom{k+r}{r} S_{k+r} = \sum_{\omega \in \Omega} \delta_{k, X(\omega)} \Pr(\omega) = \Pr(X = k),$$

where δ_{ij} is the Kronecker delta. To verify the second result, note for all $s \geq 0$ and $n \geq 1$,

$$\sum_{r=0}^s (-1)^r \binom{n}{r} = (-1)^s \binom{n-1}{s},$$

which can be proved easily by induction on s , hence

$$\begin{aligned} \sum_{r=0}^s (-1)^r \binom{k+r}{r} \binom{X}{k+r} &= \binom{X}{k} \sum_{r=0}^s (-1)^r \binom{X-k}{r} \\ &= \begin{cases} 0 & \text{if } X < k \\ 1 & \text{if } X = k \\ (-1)^s \binom{X}{k} \binom{X-k-1}{s} & \text{if } X > k. \end{cases} \end{aligned}$$

We thus have

$$\begin{aligned} &\sum_{r=0}^s (-1)^r \binom{k+r}{r} S_{k+r} \\ &= \sum_{\omega \in \Omega} \left\{ \sum_{r=0}^s (-1)^r \binom{k+r}{r} \binom{X(\omega)}{k+r} \right\} \Pr(\omega) \\ &= \sum_{X(\omega)=k} \Pr(\omega) + \sum_{X(\omega)>k} (-1)^s \binom{X}{k} \binom{X-k-1}{s} \Pr(\omega) \\ &= \Pr(X = k) + (-1)^s \sum_{X(\omega)>k} \binom{X}{k} \binom{X-k-1}{s} \Pr(\omega). \end{aligned}$$

In the last line elements ω such that $X(\omega) > k$ make a positive or negative contribution depending on whether s is even or odd. $\square$

Suppose that we have defined a sequence of probability spaces and that in the space $\Omega = \Omega_n$ we have the preceding situation with $\ell = \ell(n)$. If $E((X)_r) \rightarrow \mu^r$ as $n \rightarrow \infty$, we can make a precise statement about the limiting distribution of $X = \sum_{i=1}^{\ell} X_i$. In probability theory, we call a random variable Y to have Poisson distribution if it takes non-negative integral values and $\Pr(Y = k) = \frac{\mu^k}{k!} e^{-\mu}$ for some constant $\mu > 0$, which is the expectation of Y (and the variance of Y).

Theorem 6.8 (Poisson Limit) *Suppose that there is a positive number μ such that*

$$\lim_{n \rightarrow \infty} S_r = \frac{\mu^r}{r!},$$

which is equivalent to $\lim_{n \rightarrow \infty} E((X)_r) = \mu^r$, for each fixed integer $r \geq 0$. Then

$$\lim_{n \rightarrow \infty} \Pr(X = k) = \frac{\mu^k}{k!} e^{-\mu}.$$

Namely, the limiting distribution of X is Poisson with mean μ .

Proof. Refer to the inequalities in the last theorem we have

$$\frac{1}{k!} \sum_{r=0}^{2m-1} (-1)^r \frac{E((X)_{k+r})}{r!} \leq \Pr(X = k) \leq \frac{1}{k!} \sum_{r=0}^{2m} (-1)^r \frac{E((X)_{k+r})}{r!}.$$

Note that if m is fixed, we can make the limit $n \rightarrow \infty$ term by term to get

$$\frac{1}{k!} \sum_{r=0}^{2m-1} (-1)^r \frac{\mu^{k+r}}{r!} \leq \lim_{n \rightarrow \infty} \Pr(X = k) \leq \frac{1}{k!} \sum_{r=0}^{2m} (-1)^r \frac{\mu^{k+r}}{r!}.$$

Since m is arbitrary, the result follows. $\square$

Theorem 6.9 *For any fixed real number x , let*

$$p = \frac{\log n + x}{n},$$

and let X be the number of isolated vertices in a graph of $\mathcal{G}(n, p)$, then

$$\lim_{n \rightarrow \infty} \Pr[X = k] = \frac{\mu^k}{k!} e^{-\mu},$$

where $\mu = e^{-x}$. In particular, the limiting probability that graph in $\mathcal{G}(n, p)$ has no isolated vertices is $\exp(-e^{-x})$.

Proof. Define X_i as an indicator that the vertex i is an isolated vertex, and define $X = \sum_{i=1}^n X_i$. Then X counts the number of isolated vertices in G_p and

$$S_1 = E(X) = n(1-p)^{n-1} \rightarrow e^{-x}$$

as $n \rightarrow \infty$. More generally,

$$S_r = \binom{n}{r} (1-p)^{r(n-r)+\binom{r}{2}} \sim \frac{n^r}{r!} (1-p)^{rn} \rightarrow \frac{\mu^r}{r!},$$

where $\mu = e^{-x}$. The limiting distribution of X follows from Poisson limit theorem as desired. $\square$

Corollary 6.1 *Suppose $\log n \geq \omega(n) \rightarrow \infty$. Then $p_\ell = \frac{\log n - \omega(n)}{n}$ and $p_u = \frac{\log n + \omega(n)}{n}$ are ltf and utf for graph in $\mathcal{G}(n, p)$ having no isolated vertices, respectively.*

We shall show that for the same $p = (\log n + x)/n$,

$$\lim_n \Pr(G_p \text{ has no isolates}) = \lim_n \Pr(G_p \text{ is connected}) = \exp(-e^{-x}).$$

So in almost every graph when the last isolated vertex disappears, the graph G_p becomes connected in the evolution of random graph as x increases. Slightly before it is connected, a giant component with only a bounded number vertices outside has formed. In fact, the giant component are formed by larger components and the smaller components have bigger chances to survive.

Theorem 6.10 *For any fixed real number x , let*

$$p = \frac{\log n + x}{n},$$

and let A denote the event that outside of at most one non-trivial component, all vertices are isolated. Then

$$\lim_{n \rightarrow \infty} \Pr(A) = 1$$

and

$$\lim_{n \rightarrow \infty} \Pr[G_p \text{ is connected}] = \exp(-e^{-x}).$$

Proof. We begin by identifying the following events in $\mathcal{G}(n, p)$.

A: Outside of at most one non-trivial component, G_p has only isolated vertices.

B: G_p has no isolated vertices.

C: G_p is connected.

Then $C = A \cap B$ and

$$\Pr(B) = \Pr(C) + \Pr(\bar{A} \cap B).$$

To prove that $\Pr(C) \rightarrow \exp(-e^{-x})$ as $n \rightarrow \infty$, it suffices to show that $\Pr(\bar{A}) \rightarrow 0$ since we have known that $\Pr(B) \rightarrow \exp(-e^{-x})$. Let $X \subseteq [n]$ be the vertex set of the largest component of G and let $Y = V \setminus X$. We do not distinguish a vertex set and the subgraph induced by this set if no danger of confusion. If $\bar{A}$ holds, then for some $X \subseteq [n]$ with $|X| \geq 2$,

1. X is connected;
2. Y contains edge;
3. There is no $X - Y$ edges.

Note that these events are independent. Let the probabilities of the events 1, 2 and 3 be denoted P_X , P_Y and P_{XY} , respectively and let $|X| = k$ and $|Y| = m = n - k$. By distinguishing that $k \leq \lfloor n/2 \rfloor$ or $m \leq \lfloor n/2 \rfloor$, we have

$$\Pr(\bar{A}) \leq \sum_{k=2}^{\lfloor n/2 \rfloor} \binom{n}{k} P_X P_{XY} + \sum_{m=2}^{\lfloor n/2 \rfloor} \binom{n}{m} P_Y P_{XY}. \quad (6.1)$$

To bound $\Pr(\bar{A})$, we use the following facts:

1. $P_X \leq k^{k-2} p^{k-1}$;
2. $P_Y = 1 - (1 - p)^{\binom{m}{2}}$
3. $P_{XY} = (1 - p)^{mk}$.

The first fact follows since X must contain one of the k^{k-2} possible spanning trees. Consider then first term on the right hand side of (6.1), we have

$$\sum_{k=2}^{\lfloor n/2 \rfloor} \binom{n}{k} P_X P_{XY} \leq \sum_{k=2}^{\lfloor n/2 \rfloor} \binom{n}{k} k^{k-2} p^{k-1} (1 - p)^{k(n-k)}.$$

The term corresponding to any fixed $k \geq 2$ can be bounded above as

$$\begin{aligned} c_1 n^k p^{k-1} (1 - p)^{kn} &\leq c_1 n^k p^{k-1} e^{-knp} \\ &= c_1 e^{-kx} p^{k-1} \rightarrow 0, \end{aligned}$$

where c_1 henceforth c_i are positive constants. For any $k \leq n/2$,

$$(1-p)^{n-k} \leq e^{-(n-k)p} \leq e^{-np/2} = \frac{e^{-x/2}}{\sqrt{n}}.$$

Thus

$$\begin{aligned} \binom{n}{k} k^{k-2} p^{k-1} (1-p)^{k(n-k)} &\leq \left(\frac{en}{k}\right)^k k^{k-2} p^{k-1} \left(\frac{e^{-x/2}}{\sqrt{n}}\right)^k \\ &= \frac{1}{k^2 p} \left(enp \frac{e^{-x/2}}{\sqrt{n}}\right)^k \\ &\leq \frac{n}{\log n + x} \left(\frac{c_2 \log n}{\sqrt{n}}\right)^k. \end{aligned}$$

It follows that

$$\sum_{k=2}^{\lfloor n/2 \rfloor} \binom{n}{k} P_X P_{XY} \leq o(1) + \sum_{k \geq 4} \frac{n}{\log n + x} \left(\frac{c_2 \log n}{\sqrt{n}}\right)^k \rightarrow 0$$

as $n \rightarrow \infty$. Now setting

$$K = \lfloor \sqrt[4]{n} \rfloor, \quad \text{and} \quad M = \lceil 2\sqrt{n} \exp(1-x/2) \rceil,$$

we shall separate the second term on the right hand side of (6.1) into three parts by K and M . Using the facts that

$$P_{XY} = (1-p)^{m(n-m)} \leq e^{-m(n-m)p} \leq e^{-mnp/2} = \left(\frac{e^{-x/2}}{\sqrt{n}}\right)^m$$

for $m \leq \lfloor n/2 \rfloor$ and $\binom{n}{m} \leq (en/m)^m$, we have

$$\begin{aligned} \sum_{m=M}^{\lfloor n/2 \rfloor} \binom{n}{m} P_Y P_{XY} &\leq \sum_{m=M}^{\lfloor n/2 \rfloor} \binom{n}{m} P_{XY} \\ &\leq \sum_{m \geq M} \left(\frac{ene^{-x/2}}{m\sqrt{n}}\right)^m \\ &\leq \sum_{m \geq M} \frac{1}{2^m} = \frac{1}{2^{M-1}} \rightarrow 0 \end{aligned}$$

since $M \rightarrow \infty$ as $n \rightarrow \infty$. On the other hand, if $m < M$, we have $e^{mp} \leq 2$ for all large n since $mp \leq Mp \rightarrow 0$, so

$$P_{XY} = (1-p)^{m(n-m)} \leq (e^{-np} e^{mp})^m \leq \left(\frac{2e^{-x}}{n}\right)^m.$$

It follows that

$$\begin{aligned} \sum_{m=K}^{M-1} \binom{n}{m} P_Y P_{XY} &\leq \sum_{m=K}^{M-1} \binom{n}{m} P_{XY} \\ &\leq \sum_{m \geq K} \frac{(2e^{-x})^m}{m!} \rightarrow 0 \end{aligned}$$

since the sum in the last line is the tail of a convergent series.

Finally, for all $m < K$,

$$P_Y \leq 1 - (1-p)^{\binom{K}{2}},$$

which tends to zero uniformly on $m < M$, and it follows that

$$\sum_{m=2}^{K-1} \binom{n}{m} P_Y P_{XY} = o\left(\sum_{m=2}^{K-1} \binom{n}{m} P_{XY}\right) \leq o\left(\sum_{m \geq 2} \frac{(2e^{-x})^m}{m!}\right) \rightarrow 0.$$

Combining the results, we find that $\Pr(\bar{A}) \rightarrow 0$, completing the proof. $\square$

Corollary 6.2 *Suppose that $\log n \geq \omega(n) \rightarrow \infty$. Then $p_\ell = \frac{\log n - \omega(n)}{n}$ and $p_u = \frac{\log n + \omega(n)}{n}$ are ltf and utf for graph in $\mathcal{G}(n, p)$ of being connected, respectively.*

Exercises

6.1 Explain the neighbors of a vertex of a random graph are likely to be “spreading” by Theorem 6.1.

6.2 A historic result of Erdős (1959) is that for any k and ℓ , there is a graph G with girth $g(G) > \ell$ and $\chi(G) > k$. Prove this result by the following steps.

(1) Fix $0 < \theta < 1/\ell$, let $p = n^{\theta-1}$. Consider random graphs in $\mathcal{G}(n, p)$. Let X be the number of cycles of length at most ℓ in G . Show that $\Pr(X \geq n/2) = o(1)$ by $E(X) = o(1)$.

(2) Set $m = 3n^{1-\theta} \log n$. Show $\Pr(\alpha(G) \geq m) = o(1)$.

(3) For any graph G with $X(G) < n/2$ and $\alpha(G) < m$, delete a vertex from each cycle of length at most ℓ . The resulting graph G^* satisfies $\chi(G^*) \geq n^\theta / (6 \log n)$.

6.3 Prove that $p = n^{-1/\rho(H)}$ is a threshold function for the appearance of H as a subgraph of G_p , where $\rho(H) = \max\{e(F)/|V(F)| : F \subseteq H\}$.

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Chapter 7

Local Lemma

7.1 The Lovász Local Lemma

Let $A_1, A_2, \dots, A_n$ be the events in a probability space $(\Omega, \mathcal{F}, \Pr)$. In combinatorial application such as a coloring of edges of K_N , any A_i is a “bad” event. We wish to show $\Pr(\cap \bar{A}_i) > 0$ so there is a point (coloring) which is good. In proofs in Chapter 4 for lower bound of $r(m, n)$, A_S is the event that S is monochromatic for an n -set S . The event A_S is “bad”. Then $\cap \bar{A}_S$ is the event in which none of n -sets is monochromatic. $\Pr(\cap \bar{A}_S) > 0$ means that there must be a coloring in which no “bad” thing happens. That is, there is a edge coloring of K_N , which determines a graph G such that G contains no K_n and whose complement contains no K_n either.

It is a trivial fact if $A_1, A_2, \dots, A_n$ satisfy

$$\sum_{i=1}^n \Pr(A_i) < 1, \quad (7.1)$$

then

$$\Pr(\cap \bar{A}_i) > 0. \quad (7.2)$$

On the other hand, if $A_1, \dots, A_n$ are mutually independent events and $\Pr(A_i) = x_i < 1$ for $i \geq 1$ then (7.2) holds as

$$\Pr(\cap \bar{A}_i) = \prod_{i=1}^n (1 - x_i) > 0.$$

with no further restriction on probabilities $\Pr(A_i)$.

The local lemma may be understood in term of taking advantage of *partial independence* of the events $A_1, A_2, \dots, A_n$ so that (7.2) can be ensured

with far weaker bounds on the probabilities $\Pr(A_i)$ than that are needed for (7.1).

The argument to follow uses *conditional probability*. Recall that for events A and B with $\Pr(B) > 0$, the conditional probability $\Pr(A|B)$ is given by

$$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)}.$$

If A and B are independent then $\Pr(A|B) = \Pr(A)$. Two events A and B are independent if and only if $\Pr(A \cap B) = \Pr(A)\Pr(B)$.

A graph D on vertices $[n] = \{1, 2, \dots, n\}$ (the set of indices for the events A_i) is called *dependency graph* for events $A_1, A_2, \dots, A_n$ if for all i , event A_i is mutually independent of all A_j with $\{i, j\} \notin E(D)$. That is, A_i is independent of any Boolean function of these A_j . This is slightly stronger than that A_i is dependent with A_j for $\{i, j\} \in E(D)$ even though they are same condition in most of applications in graph Ramsey theory.

The following is the general form of the Lovász Local Lemma given by Spencer (1977), whose original form was in Erdős and Lovász (1975).

Theorem 7.1 *Let $A_1, A_2, \dots, A_n$ be the events in the probability space $(\Omega, \mathcal{F}, \Pr)$ and suppose that there exist real numbers $x_1, x_2, \dots, x_n$ such that $0 < x_i < 1$ and for $(i = 1, 2, \dots, n)$,*

$$\Pr(A_i) \leq x_i \prod_{\{i, j\} \in E(D)} (1 - x_j).$$

Then

$$\Pr(\cap_{i=1}^n \overline{A_i}) \geq \prod_{i=1}^n (1 - x_i) > 0.$$

Note. If i is an isolated vertex in D , that is, the event A_i is mutually independent of all other events, then $\{j : \{i, j\} \in E(D)\} = \emptyset$. In this case $\prod_{j \in \emptyset} (1 - x_j) = 1$ as usual.

Proof. The desired result follows directly from the following claim.

Claim. For $S \subset [n]$, set

$$\mathcal{C}_S = \cap_{j \in S} \overline{A_j}.$$

(For the case $S = \emptyset$, we take $\mathcal{C}_S$ to be Ω). If $i \notin S$, then

$$\Pr(A_i | \mathcal{C}_S) \leq x_i.$$

Proof of the claim. The proof is by induction on $|S|$ for fixed i with $i \notin S$. If $|S| = 0$ the desired result is immediate since the hypothesis of the

local lemma yields

$$\begin{aligned}\Pr(A_i|\mathcal{C}_S) &= \Pr(A_i|\Omega) = \Pr(A_i) \\ &\leq x_i \prod_{\{i,j\} \in E(D)} (1 - x_j) \\ &\leq x_i.\end{aligned}$$

Now assume that $|S| \geq 1$ and form a partition $S = (S_1, S_2)$, where

$$S_1 = \{j \in S : \{i, j\} \in E(D)\} \quad \text{and} \quad S_2 = S \setminus S_1.$$

Noticing that $\mathcal{C}_S = \mathcal{C}_{S_1} \cap \mathcal{C}_{S_2}$, we write

$$\Pr(A_i|\mathcal{C}_S) = \frac{\Pr(A_i \cap \mathcal{C}_S)}{\Pr(\mathcal{C}_S)} = \frac{\Pr(A_i \cap \mathcal{C}_{S_1}|\mathcal{C}_{S_2})}{\Pr(\mathcal{C}_{S_1}|\mathcal{C}_{S_2})}, \quad (7.3)$$

and bound the numerator and denominator separately. First, since A_i and $\mathcal{C}_{S_2}$ are independent,

$$\begin{aligned}\Pr(A_i \cap \mathcal{C}_{S_1}|\mathcal{C}_{S_2}) &\leq \Pr(A_i|\mathcal{C}_{S_2}) \\ &= \Pr(A_i) \\ &\leq x_i \prod_{j \in S_1} (1 - x_j).\end{aligned}$$

To bound the denominator of (7.3), we use the induction hypothesis. If $|S_1| = 0$ then $\mathcal{C}_{S_1} = \Omega$ and

$$\Pr(\mathcal{C}_{S_1}|\mathcal{C}_{S_2}) = \Pr(\Omega|\mathcal{C}_{S_2}) = 1$$

and the claim follows. Otherwise, suppose $S_1 = \{j_1, j_2, \dots, j_r\}$, where $r \geq 1$. Let $\mathcal{D}_0, \mathcal{D}_1, \dots, \mathcal{D}_r$ be the events defined recursively by

$$\mathcal{D}_0 = \mathcal{C}_{S_2} = \bigcap_{j \in S_2} \bar{A}_j$$

and for $k = 1, 2, \dots, r$,

$$\mathcal{D}_k = \mathcal{D}_{k-1} \cap \bar{A}_{j_k} = \left(\bigcap_{j \in S_2} \bar{A}_j \right) \cap \left(\bigcap_{t=1}^k \bar{A}_{j_t} \right).$$

They start with $\mathcal{D}_0 = \mathcal{C}_{S_2}$ and end with $\mathcal{D}_r = \mathcal{C}_S$. Note that for each $k = 0, 1, \dots, r-1$, the event $\mathcal{D}_k$ has a form of $\mathcal{C}_T$ for some set $T \subseteq S$ with $|T| < |S|$ and the fact that $\Pr(\bar{A}_j|\mathcal{D}_k) = 1 - \Pr(A_j|\mathcal{D}_k)$. Using the induction hypothesis on $\mathcal{C}_T$ repeatedly, we have

$$\begin{aligned}\Pr(\mathcal{C}_{S_1}|\mathcal{C}_{S_2}) &= \frac{\Pr(\mathcal{C}_S)}{\Pr(\mathcal{D}_0)} = \frac{\Pr(\mathcal{D}_r)}{\Pr(\mathcal{D}_0)} \\ &= \frac{\Pr(\mathcal{D}_r)}{\Pr(\mathcal{D}_{r-1})} \cdots \frac{\Pr(\mathcal{D}_1)}{\Pr(\mathcal{D}_0)} \\ &= \Pr(\bar{A}_{j_r}|\mathcal{D}_{r-1}) \cdots \Pr(\bar{A}_{j_1}|\mathcal{D}_0) \\ &= (1 - \Pr(A_{j_r}|\mathcal{D}_{r-1})) \cdots (1 - \Pr(A_{j_1}|\mathcal{D}_0)) \\ &\geq \prod_{j \in S_1} (1 - x_j).\end{aligned}$$

Combining this with what proved we have established the claim.

Note that $\cap_{i=k+1}^n \bar{A}_i$ has a form of $\mathcal{C}_S$ with $k \notin S$. In view of the claim just established,

$$\begin{aligned} \Pr(\cap_{i=1}^n \bar{A}_i) &= \Pr(\bar{A}_1 | \cap_{i=2}^n \bar{A}_i) \Pr(\cap_{i=2}^n \bar{A}_i) \\ &= \Pr(\bar{A}_1 | \cap_{i=2}^n \bar{A}_i) \Pr(\bar{A}_2 | \cap_{i=3}^n \bar{A}_i) \cdots \Pr(\bar{A}_n | \Omega) \\ &= (1 - \Pr(A_1 | \cap_{i=2}^n \bar{A}_i)) \cdots (1 - \Pr(A_n | \Omega)) \\ &\geq \prod_{i=1}^n (1 - x_i). \end{aligned}$$

This completes the proof of the local lemma. $\square$

The original local lemma in Erdős and Lovász (1975) appeared in a form like the following one, which is now called the symmetric form.

Theorem 7.2 *If each of the events of $A_1, \dots, A_n$ has probability p or less and each vertex in the dependence graph has degree at most d and if*

$$e(d+1)p < 1,$$

where e is natural logarithm base, then $\Pr(\cap_{i=1}^n \bar{A}_i) > 0$, and if $d \geq 1$,

$$\Pr(\cap_{i=1}^n \bar{A}_i) \geq \left(\frac{d}{1+d} \right)^n > 0.$$

Proof. As remarking at the beginning of this chapter, the result is immediate if $d = 0$ with no further restriction on $\Pr(A_i)$. Otherwise, taking $x_i = 1/(1+d)$ for $i = 1, 2, \dots, n$, we have

$$\Pr(A_i) \leq x_i \prod_{\{i,j\} \in E(D)} (1 - x_j),$$

since the left hand side is at most p and the right side is at least

$$\frac{1}{d+1} \left(1 - \frac{1}{d+1} \right)^d > \frac{1}{e(d+1)}.$$

To see that the last inequality truly holds, it suffices to assume that for $d \geq 1$, use the fact that $(1 + 1/d)^d < e$. Thus the conclusion of the local lemma holds and this yields

$$\Pr(\cap_{i=1}^n \bar{A}_i) \geq \prod_{i=1}^n (1 - x_i) = \left(\frac{d}{d+1} \right)^n > 0.$$

$\square$

We also need the following form of local lemma due to Spencer who used it to obtain his lower bound for non-diagonal classic Ramsey numbers. This form is slightly more convenient for some applications.

Theorem 7.3 (Spencer, 1977) *Let $A_1, A_2, \dots, A_n$ be events in a probability space. If there exist positive numbers $y_1, y_2, \dots, y_n$ such that for $i = 1, 2, \dots, n$,*

$$y_i \Pr(A_i) < 1,$$

and

$$\log y_i \geq - \sum_{ij \in E(D)} \log(1 - y_j \Pr(A_j)),$$

then $\Pr(\cap \overline{A_i}) > 0$.

Proof. We may suppose that for each i the probability $\Pr(A_i)$ is positive. Let x_i be as in the general form of the local lemma and set

$$y_i = \frac{x_i}{\Pr(A_i)}$$

for $i = 1, 2, \dots, n$. The hypothesis of the local lemma

$$\Pr(A_i) \leq x_i \prod_{\{i,j\} \in E(D)} (1 - x_j)$$

will take the form

$$y_i \geq \prod_{\{i,j\} \in E(D)} \frac{1}{1 - y_j \Pr(A_j)}.$$

So taking logarithms, we get

$$\log y_i \geq - \sum_{\{i,j\} \in E(D)} \log(1 - y_j \Pr(A_j)) \quad (7.4)$$

for $i = 1, 2, \dots, n$. Thus if there exist positive numbers $y_1, y_2, \dots, y_n$ such that

$$y_i \Pr(A_i) < 1,$$

and (7.4) holds, then

$$\Pr(\cap_{i=1}^n \overline{A_i}) > 0.$$

□

7.2 Applications of the Local Lemma

Local lemma is one of important mathematical contributions of Lovász. It has a lot applications in recent years. The following theorem improves Theorem 3 in Chapter 4 with another factor $\sqrt{2}$. This is negligible when viewed in the light of the gap between the upper bound and lower bounds, but we do what we can. The progress on this difficult problem has been slow. For comment, see Spencer (1987).

Theorem 7.4 *As $n \rightarrow \infty$,*

$$r(n, n) \geq (1 - o(1)) \frac{\sqrt{2}}{e} n 2^{n/2}.$$

Proof. Consider the random graph space $\mathcal{G}(N, 1/2)$ or consider to color each edge of K_N with probability $1/2$, randomly and independently. Let S be a subset of size n of $V(K_N)$ and let A_S signify the event “ S is monochromatic”, S ranging over the n -subsets. Define a graph D with vertex set consisting all such S and connect vertices S and T in D if and only if $|S \cap T| \geq 2$. Then A_S is mutually independent of all A_T with T not adjacent to S , since the A_T give information only about edges outside of S . Hence D is a dependency graph. We apply the local lemma with

$$p = \Pr(A_S) = 2^{1-\binom{n}{2}}.$$

And for any S , its degree d in D can be bounded as

$$d = |\{T : |S \cap T| \geq 2\}| < \binom{n}{2} \binom{N}{n-2}.$$

If $ep(d+1) < 1$ then $\Pr(\cap \bar{A}_S) > 0$ thus $r(n, n) > N$. So we want

$$e \binom{n}{2} \binom{N}{n-2} 2^{1-\binom{n}{2}} < 1.$$

As we did it before, the left hand side is less than

$$\frac{en^2}{2} \left(\frac{eN}{n-2} \right)^{n-2} \frac{2}{2^{n(n-1)/2}},$$

which is

$$\frac{en^2}{2} \left(\frac{n}{n-2} \right)^{n-2} \left(\frac{eN}{\sqrt{2}n2^{n/2}} \right)^{n-2}.$$

For any $\epsilon > 0$, if we take $N = \lceil (1 - \epsilon) \frac{\sqrt{2}}{e} n 2^{n/2} \rceil$, then the above tends to zero. $\square$

We now apply the local lemma to give a lower bound of $r(m, n)$, which improves that obtained in Chapter 4.

Theorem 7.5 (Spencer, 1977) *For any fixed integer $m \geq 3$, there exists a constant $c = c(m) > 0$ such that*

$$r(m, n) \geq (c - o(1)) \left(\frac{n}{\log n} \right)^{(m+1)/2},$$

as $n \rightarrow \infty$.

Proof: Let us color the edges of K_N in red and blue randomly and independently, in which each edge is colored red with probability p and blue with probability $q = 1 - p$. For each subset S of vertices of K_N with $|S| = m$, let A_S signify the event that S spans a monochromatic red K_m . For each subset T of vertices of K_N with $|T| = n$, let B_T signify the event that T spans a monochromatic blue graph K_n . Then $P(A_S) = p^{m(m-1)/2}$ and $P(B_T) = q^{n(n-1)/2}$. Denote by $(N)_m$ for the falling factorial $N(N-1)\cdots(N-m+1)$ and by ℓ_m for $\binom{N}{m} = (N)_m/m!$. There are ℓ_m events of form A_S and ℓ_n events of form B_T . Obviously, two events are dependent if and only if their subgraphs have edges in common. Hence, each A event is mutually independent of all but at most $\binom{m}{2}(N-2)_{m-2} < \frac{m(m-1)}{2}N^{m-2}$ other A events and mutually independent of all but at most ℓ_n of the B events; each B event is mutually independent of all but at most $\binom{n}{2}(N-2)_{n-2} < \frac{n(n-1)}{2}N^{n-2}$ of the A events and mutually independent of all but at most ℓ_n other B events.

We aim to prove that there exist positive numbers a and b satisfying Spencer form of Local Lemma, namely, $ap^{e(K_m)} < 1$ and $bq^{e(K_n)} < 1$ hold with $y_i = a$ for each A event and $y_i = b$ for each B event. Specifically,

$$\log a \geq -e(K_m)N^{m-2} \log(1 - ap^{e(K_m)}) - \ell_n \log(1 - bq^{e(K_n)}), \quad (7.5)$$

$$\log b \geq -e(K_n)N^{n-2} \log(1 - ap^{e(K_m)}) - \ell_n \log(1 - bq^{e(K_n)}). \quad (7.6)$$

If such a and b are available, then there exists a two-coloring of K_N in which there is neither red K_m nor blue K_n ; in other words $r(m, n) > N$. To this end, set $\beta = \frac{2}{m+1} > 0$ and set

$$\begin{aligned} p &= c_1 N^{-\beta}, \\ a &= c_2, \\ b &= \exp \{c_3 N^\beta (\log N)^2\}, \\ n &= c_4 N^\beta \log N, \end{aligned}$$

where c_1, c_2, c_3 and c_4 are constants to be chosen. Observe that

$$\ell_n = \frac{(N)_n}{n!} \leq N^n = \exp \{c_4 N^\beta (\log N)^2\}.$$

Using the basic inequality $1 - x < e^{-x}$ for $x > 0$, we have $(1 - x)^k < e^{-kx}$ for $k > 0$ and

$$\begin{aligned} q^{e(K_n)} &= (1 - p)^{e(K_n)} \\ &\leq \exp \left\{ -(1 - o(1)) \frac{n^2}{2} p \right\} \\ &= \exp \left\{ -\left(\frac{1}{2} - o(1)\right) c_1 c_4^2 N^\beta (\log N)^2 \right\}. \end{aligned}$$

Thus

$$\ell_n b q^{e(K_n)} \leq \exp \left\{ \left[c_4 + c_3 - \left(\frac{1}{2} - o(1)\right) c_1 c_4^2 \right] N^\beta (\log N)^2 \right\}.$$

If we choose the constants c_1, c_2, c_3 and c_4 such that

$$c_3 < \frac{1}{2} c_1 c_4^2 - c_4, \quad (7.7)$$

then $\ell_n b q^{e(K_n)} \rightarrow 0$ as $n \rightarrow \infty$ hence as $N \rightarrow \infty$. So $b q^{e(K_n)} \rightarrow 0$, and from the fact $\log(1 - x) \sim x$ as $x \rightarrow 0$, the second terms in the right sides of the inequalities (7.5) and (7.6)

$$\ell_n \log(1 - b q^{e(K_n)}) \sim \ell_n b q^{e(K_n)} \rightarrow 0.$$

Clearly

$$N^{m-2} a p^{e(K_m)} = c_1^{e(K_m)} c_2 N^{-\beta} \rightarrow 0$$

as $N \rightarrow \infty$, so $a p^{e(K_m)} \rightarrow 0$ and

$$-e(K_m) N^{m-2} \log(1 - a p^{e(K_m)}) \sim e(K_m) N^{m-2} a p^{e(K_m)} \rightarrow 0.$$

Thus (7.5) is satisfied for all large n hence for all large N if we choose

$$a = c_2 > 1. \quad (7.8)$$

Finally, note that the first term in (7.6)

$$\begin{aligned} -e(K_n) N^{m-2} \log(1 - a p^{e(K_m)}) &\sim e(K_n) N^{m-2} a p^{e(K_m)} \\ &\sim \frac{n^2}{2} N^{m-2} a p^{e(K_m)} \\ &= \frac{1}{2} c_1^{e(K_m)} c_2 c_4^2 N^\beta (\log N)^2. \end{aligned}$$

Thus (7.6) is satisfied for large n hence for all large N if we choose

$$c_3 > \frac{1}{2} c_1^{e(K_m)} c_2 c_4^2. \quad (7.9)$$

To get the desired constants c_1, c_2, c_3, c_4 , let us first look at the case where the equality holds for (7.7), (7.8), and (7.9), then $c_2 = 1$ and

$$\frac{1}{2} c_1 c_4^2 - c_4 = c_3 = \frac{1}{2} c_1^{e(K_m)} c_4^2.$$

Therefore

$$c_4 = \frac{2}{c_1 - c_1^{e(K_m)}}.$$

Since the function $f(x) = x - x^{e(K_m)}$ is concave on $[0, 1]$ and $f(0) = f(1) = 0$, f achieves a positive maximum value at some $x_m \in (0, 1)$, which can be determined by setting $f'(x_m) = 0$. So the minimum value of c_4 , denoted by $\overline{c}_4$, occurs at $c_1 = x_m$. To get what we want, just perturb this solution a little bit. With c_1 is slightly less than x_m and c_4 slightly greater than $\overline{c}_4$, we have

$$\frac{1}{2} c_1 c_4 - 1 > \frac{1}{2} c_1^{e(K_m)} c_4.$$

Thus we can choose appropriate $c_2 > 1$ and c_3 such that

$$\frac{1}{2} c_1 c_4^2 - c_4 > c_3 > \frac{1}{2} c_1^{e(K_m)} c_2 c_4^2.$$

So the inequalities (7.7), (7.8) and (7.9) are satisfied simultaneously. We then conclude that $r(m, n) > N$. Since $n = c_4 N^\beta \log N$, $\log n \sim \beta \log N$. Thus

$$N = \left(\frac{n}{c_4 \log N} \right)^{1/\beta} > (c - o(1)) \left(\frac{n}{\log n} \right)^{(m+1)/2},$$

where

$$c = \left(\frac{f(x_m)}{2} \beta \right)^{1/\beta} = \left(\frac{f(x_m)}{m+1} \right)^{(m+1)/2},$$

which is independent of n . This completes the proof. $\square$

7.3 A Generalization for Large Dense Graphs

The present section is devoted to generalize the result to the Ramsey numbers involving large dense graphs. We shall first prove the following theorem, which yields not only Spencer's lower bound of $r(K_m, K_n)$ but also the Erdős-Faudree-Rousseau-Schelp(1987) and Krivelevich(1995) lower bound of the form $r(F, K_n)$. We need to say a bit on automorphism group of a graph G on n vertices. It is straightforward to verify that (under the operation of composition) the set of all automorphisms of G forms a group, denoted by $\mathcal{A}(G)$ and referred to as the automorphism group of G . Clearly, $|\mathcal{A}(G)| \leq n!$ and the equality holds if and only if $G = K_n$ or $G = \overline{K}_n$.

Theorem 7.6 (Li and Zang, 2003) *Let integer $m \geq 3$ be fixed, and let constants $\delta > 0$ and $\alpha \geq 0$ be fixed. Let $\{G_n\}$ be a family of graphs, in which G_n has n vertices and*

$$e(G_n) \geq (\delta - o(1)) \frac{n^2}{(\log n)^\alpha}$$

edges as $n \rightarrow \infty$. If F is a graph on m vertices, then there exists a constant $c = c(m, \delta) > 0$ such that

$$r(F, G_n) \geq (c - o(1)) \left(\frac{n}{(\log n)^{\alpha+1}} \right)^{(e(F)-1)/(m-2)}.$$

Proof: Clearly, it suffices to consider the case in which $e(F) \geq 2$ and $e(G_n) \sim \delta n^2 / (\log n)^\alpha$. Let us color the edges of K_N randomly and independently, where each edge is colored red with probability p and colored blue with probability $q = 1 - p$. For each subgraph S of K_N that is isomorphic to F , let A_S be the event that S spans a monochromatic red graph F . For each subgraph T of K_N that is isomorphic to G_n , let B_T be the event that T spans a monochromatic blue graph G_n . Then $P(A_S) = p^{e(F)}$ and $P(B_T) = q^{e(G_n)}$. Since F has m vertices, there are $(N)_m / |\mathcal{A}(F)|$ events of form A_S . Similarly, there are $(N)_n / |\mathcal{A}(G_n)|$ events of form B_T . Obviously, two events are dependent if and only if the corresponding subgraphs have edges in common. Hence, each A event is mutually independent of all but at most $e(F)(N-2)_{m-2} < e(F)N^{m-2}$ other A events and mutually independent of all but at most $\ell(G_n)$ of the B events, where $\ell(G_n) = (N)_n / |\mathcal{A}(G_n)|$; each B event is mutually independent of all but at most $e(G_n)(N-2)_{m-2} < e(G_n)N^{m-2}$ of the A events and mutually independent of all but at most $\ell(G_n)$ other B events. It is trivial that $\ell(G_n) < N^n$.

We aim to prove that there exist positive numbers a and b satisfying Spencer form of local lemma, namely, $ap^{e(F)} < 1$ and $bq^{e(G_n)} < 1$ such that

$$\log a \geq -e(F)N^{m-2} \log(1 - ap^{e(F)}) - \ell(G_n) \log(1 - bq^{e(G_n)}),$$

$$\log b \geq -e(G_n)N^{m-2} \log(1 - ap^{e(F)}) - \ell(G_n) \log(1 - bq^{e(G_n)}).$$

To this end, set $\beta = \frac{m-2}{e(F)-1} > 0$ and set

$$\begin{aligned} p &= c_1 N^{-\beta} \\ a &= c_2 \\ b &= \exp \{c_3 N^\beta (\log N)^{\alpha+2}\} \\ n &= c_4 N^\beta (\log N)^{\alpha+1}, \end{aligned}$$

where c_1, c_2, c_3 and c_4 are constants to be chosen. We omit the details of the proof which are similar to that in the last section. $\square$

Corollary 7.1 *For fixed $m \geq 3$ as $n \rightarrow \infty$,*

$$c_m \left(\frac{n}{\log n} \right)^{(m+1)/2} \leq r(m, n) \leq (1 + o(1)) \frac{n^{m-1}}{(\log n)^{m-2}},$$

where $c_m = c(m) > 0$ is constant.

Note Only for the case $m = 3$, the lower bound was improved by Kim (1995) with a $\log n$ factor so that the exact order of magnitude of $r(3, n)$ is known to be $n^2/(\log n)$.

The following result improves the lower bound obtained by deletion method as $r(C_m, K_n) \geq c(n/\log n)^{m/(m-1)}$.

Corollary 7.2 *Let integer $m \geq 3$ be fixed. Then there exists constant $c = c(m) > 0$ so that*

$$r(C_m, K_n) \geq c \left(\frac{n}{\log n} \right)^{(m-1)/(m-2)}$$

for all large n .

Corollary 7.3 *For any fixed integers $k \geq m \geq 2$, there exist constant $c = c(m, k) > 0$ so that*

$$r(K_{m,k}, K_n) \geq c \left(\frac{n}{\log n} \right)^{(mk-1)/(m+k-2)}$$

for all large n .

We will prove that $r(K_{m,k}, K_n) \leq c \left(\frac{n}{\log n} \right)^m$ later. Note that the exponent $(mk-1)/(m+k-2)$ in the lower bound can be arbitrarily close to the exponent m in the upper bound for fixed large k .

Exercises

7.1 Let $\Omega = \{1, 2, 3, 4\}$ and let the probability of a singleton is $1/4$. Set events $A_i = \{i, 4\}$ for $i = 1, 2, 3$. Show A_1, A_2, A_3 are pairwise independent but not mutually independent. How we consider the mutual independency of the events in graph for using Local Lemma?

7.2 Using the Local Lemma, give a lower bound for $r(K_{n,n})$, compare it with that by basic probabilistic method.

7.3 Give a lower bound and an upper bound for $r(K_3, K_{n,n})$.

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Chapter 8

Turán Numbers

8.1 Turán Number of H with $\chi(H) \geq 3$

For a graph H , the *Turán number*, denoted by $ex(n; H)$, is defined as the maximum number of edges in a graph on n vertices that does not contain the given graph H .

Let $\chi(H) = k \geq 3$ and let $n = \sum_{i=1}^{k-1} n_i$ such that $|n_i - n_j| \leq 1$. Thus n_i is either $\lfloor \frac{n}{k-1} \rfloor$ or $\lceil \frac{n}{k-1} \rceil$. We call the complete $(k-1)$ -partite graph $K(n_1, \dots, n_{k-1})$ as the Turán graph of order n , and denote by $t_{k-1}(n)$ for the number of edges of it. Simple calculations show that

$$\frac{k-2}{k-1} \binom{n}{2} \leq t_{k-1}(n) \leq \frac{(k-2)n^2}{2(k-1)}.$$

Write $n = s(k-1) + r$ where $0 \leq r < k-1$. In Turán graph $K(n_1, \dots, n_{k-1})$, there are $k-1-r$ parts of size $s = \lfloor n/(k-1) \rfloor = (n-r)/(k-1)$ and the remaining r parts of size $s+1$. It is easy to verify

$$t_{k-1}(n) = \frac{(k-2)n^2 - r(k-1-r)}{2(k-1)}.$$

Clearly

$$ex(n; H) \geq t_{k-1}(n)$$

since the Turán graph does not contain H as $\chi(H) = k$. We shall show that the equality holds strictly if H is a complete graph and asymptotically if H is a general graph. We begin with a slightly less precise theorem with a proof extended from Mantel ingenious argument for $k = 3$ in 1907.

Theorem 8.1 *Let n and k be positive integers with $k \geq 2$. Then*

$$t_{k-1}(n) \leq ex(n; K_k) \leq \frac{(k-2)n^2}{2(k-1)}.$$

Furthermore both inequalities become equalities if n is a multiple of $k-1$.

Proof. The first inequality follows from the fact that the Turán graph does not contain K_k , and the second is equivalent to show that if a graph G of order n contains no K_k , then $|E(G)| \leq \frac{(k-2)n^2}{2(k-1)}$. Assign to each vertex $v \in V(G)$ a weight $w(v) \geq 0$ so that $\sum_{v \in V(G)} w(v) = 1$. Let

$$S(w) = \sum_{uv \in E(G)} w(u)w(v),$$

where the sum is taken over unordered pairs of end vertices of all edges. Suppose that w has been chosen so as to maximize S as $S_{\max}$. Then for $uv \notin E(G)$ we may **claim** that either $w(u) = 0$ or $w(v) = 0$. To see this, we suppose that

$$\sum_{x \in N(u)} w(x) \geq \sum_{y \in N(v)} w(y).$$

Then

$$S(w) = w(u) \sum_{x \in N(u)} w(x) + w(v) \sum_{y \in N(v)} w(y) + S_1,$$

where S_1 is independent of $w(u)$ and $w(v)$. If $w(u)$ and $w(v)$ were both positive, we could increase $w(u)$ by some amount, and decrease $w(v)$ by the same amount, and S would increase, this is impossible. This claim means that the vertices x with $w(x) > 0$ belong to some clique of G . Label all these positive weights as $w_1, w_2, \dots, w_t$, where t is the order of the clique. Then

$$S(w) = \sum_{i < j} w_i w_j = \frac{1}{2} \left[\left(\sum_{i=1}^t w_i \right)^2 - \sum_{i=1}^t w_i^2 \right] = \frac{1}{2} \left[1 - \sum_{i=1}^t w_i^2 \right]$$

with $\sum_{i=1}^t w_i = 1$. From Cauchy-Schwarz inequality,

$$1 = \sum_{i=1}^t w_i \cdot 1 \leq \sqrt{\sum_{i=1}^t w_i^2 \sum_{i=1}^t 1} = \sqrt{t \sum_{i=1}^t w_i^2},$$

we get the minimum value of $\sum_{i=1}^t w_i^2$ to be $1/t$ as w_i 's are equal, so

$$S_{\max} = \frac{t-1}{2t}.$$

On the other hand, by assigning $w(x) = 1/n$ to each vertex x of G we get $S = |E(G)|/n^2$, thus

$$\frac{|E(G)|}{n^2} \leq \frac{t-1}{2t} \leq \frac{k-2}{2(k-1)}$$

since $t \leq k-1$, and the desired upper bound follows immediately.

If n is a multiple, then $t_{k-1}(n) = (k-2)n^2/(2(k-1))$ hence both inequalities in the theorem become equalities. $\square$

Corollary 8.1 *Let n and k be positive integers with $k \geq 2$. Then*

$$ex(n; K_k) \leq \frac{(k-2)n^2}{2(k-1)}.$$

As mentioned in Chapter 3, we shall show that the Turán lower bound on independence number of graph G with n vertices and average degree d

$$\alpha(G) \geq \frac{n}{1+d} \quad (8.1)$$

can be implied easily from above corollary. The short proof is as follows.

Proof of (8.1) by Corollary 8.1. Let us consider the complement graph $\bar{G}$ of G . Denote by $\bar{d}$ for its average degree, then $d + \bar{d} = n-1$. Set $k = \alpha(G) + 1$, then $k = \omega(\bar{G}) + 1$ and $\bar{G}$ contains no K_k . Applying Theorem 8.1 to $\bar{G}$, we have

$$\frac{n\bar{d}}{2} = |E(\bar{G})| \leq \frac{(k-2)n^2}{2(k-1)},$$

yielding $k-1 \geq n/(n-\bar{d})$ hence

$$\alpha(G) = \omega(\bar{G}) = k-1 \geq \frac{n}{n-\bar{d}} = \frac{n}{1+d},$$

as desired. $\square$

In Chapter 3, we deduced (8.1) directly from a slightly stronger bound as

$$\alpha(G) \geq \sum_{v \in V(G)} \frac{1}{1 + \deg(v)}. \quad (8.2)$$

This bound in fact implies Corollary 8.1, too.

Second Proof of the upper bound in Theorem 8.1. First note a well known result that for positive numbers $a_1, \dots, a_n$ as

$$\frac{a_1 + \dots + a_n}{n} \geq (a_1 \dots a_n)^{1/n} \geq \frac{n}{a_1^{-1} + \dots + a_n^{-1}}.$$

If G does not contain K_k , then $k-1 \geq \omega(G) = \alpha(\overline{G})$. Applying (8.2) to $\overline{G}$, we have

$$\begin{aligned} k-1 &\geq \alpha(\overline{G}) \geq \sum_{v \in V(G)} \frac{1}{1 + \deg_{\overline{G}}(v)} \\ &= \sum_{v \in V(G)} \frac{1}{n - \deg(v)} \geq \frac{n^2}{\sum_{v \in V(G)} (n - \deg(v))} \\ &= \frac{n^2}{n^2 - 2|E(G)|}. \end{aligned}$$

It follows that $|E(G)| \leq \frac{(k-2)n^2}{2(k-1)}$ as required. $\square$

When n is not a multiple of $k-1$, the lower bound in Theorem 8.1 is still an equality, but the upper bound is not.

Theorem 8.2 *Let k and n be integers with $k \geq 2$ and*

$$n = s(k-1) + r,$$

where $0 \leq r < k-1$. Then

$$ex(n; K_k) = t_{k-1}(n) = \frac{(k-2)n^2 - r(k-1-r)}{2(k-1)}.$$

Furthermore, if a graph with n vertices and $ex(n; K_k)$ edges that contains no K_k , then it is the Turán graph.

Proof. From the fact $ex(n; K_2) = 0$, we may assume that $k \geq 3$. We shall use induction on $s = \lfloor n/(k-1) \rfloor$. The case $s = 0$ corresponds to $n = r < k-1$, and in this case it is obvious since $ex(n; K_k) = \binom{n}{2}$. The equality holds and the graph makes the equality is K_n . Now we assume that $s \geq 1$ and G is a graph with $n = s(k-1) + r$ vertices that contains no K_k . It suffices to prove $|E(G)| \leq t_{k-1}(n)$.

We may suppose that G has the maximum possible number of edges subject to this condition. Then G must contain K_{k-1} as a subgraph, otherwise we could add an edge and the resulting graph would still not contain

K_k . Pick such a K_{k-1} on vertex set X , let $Y = V(G) \setminus X$. Slightly abusing notation, we use Y for the subgraph of G induced by Y . Then Y has $n - k + 1 = (s - 1)(k - 1) + r$ vertices, and by induction, the number of edges in Y satisfies

$$\begin{aligned} |E(Y)| &\leq t_{k-1}(n - k + 1) \\ &= \frac{(k - 2)(n - k + 1)^2 - r(k - 1 - r)}{2(k - 1)} \\ &= t_{k-1}(n) - (k - 2)(n - k + 1) - \binom{k - 1}{2}. \end{aligned}$$

Since G contains no K_k , no vertex in Y is adjacent to all the vertices of X . Thus

$$|E(G)| \leq |E(Y)| + (k - 2)(n - k + 1) + \binom{k - 1}{2} \leq t_{k-1}(n),$$

completing the induction.

If G has $t_{k-1}(n)$ edges, an inductive argument shows that Y must have $t_{k-1}(n - k + 1)$ edges and each vertex in Y must be adjacent to $k - 2$ vertices of X . It follows that G is the Turán graph. $\square$

Since $ex(n; H) \geq t_{k-1}(n)$ for a graph H with $\chi(H) = k$, so a graph F with n vertices and $t_{k-1}(n)$ edges need not contain H or K_k . However, one more edge in F will force it to contain a K_k . A deep result of Erdős and Stone, published in 1946, that if $\epsilon > 0$ is fixed, then ϵn^2 more edges ensure that F contains not only K_k , but a $K_k(t)$ with any large t hence any fixed graph H as n is large. The Erdős-Stone theorem is a fundamental theorem in extremal graph theory. Sharpening this result, Bollobás and Erdős in 1973 proved that the speed of $t \rightarrow \infty$ can be at least $c\epsilon \log n$ as $n \rightarrow \infty$, where $c = c(k)$ is a constant.

Lemma 8.1 *Let $k \geq 2$ be an integer and let $1 > \epsilon > 0$. Then there is an integer $n_0 = n_0(k, \epsilon)$ such that if F is a graph of order $n \geq n_0$ with*

$$\delta(F) \geq \left(\frac{k - 2}{k - 1} + \epsilon \right) n,$$

then F contain a $K_k(t)$ for some $t \geq \frac{\epsilon \log n}{2^{k-2}(k-2)!}$.

Proof $\star$. We use the induction on k . Suppose $k = 2$. Let F be a graph with n vertices and minimum degree at least ϵn that does not contain a $K_2(t)$ with $t = \lceil \epsilon \log n \rceil$. As before, we say that a set S is covered by a

vertex x if x is adjacent to every vertex in S . Every vertex of F covers at least $\binom{\epsilon n}{t}$ sets of t vertices, and no set of t vertices is covered by t vertices. Therefore,

$$n \binom{\epsilon n}{t} \leq (t-1) \binom{n}{t}.$$

This inequality is invalid for $t = \lceil \epsilon \log n \rceil$ with large n , since then

$$\begin{aligned} \frac{(t-1) \binom{n}{t}}{n \binom{\epsilon n}{t}} &< \frac{(t-1)n^t}{n(\epsilon n)^t (1 - 1/(\epsilon n)) \dots (1 - (t-1)/(\epsilon n))} \\ &< \frac{t}{n\epsilon^t (1 - t/(\epsilon n))^t} < \frac{2t}{n\epsilon^t} = \frac{2t}{n} (1/\epsilon)^t \\ &< \frac{2t}{n} e^{\epsilon \log n \log(1/\epsilon)} \leq \frac{2\epsilon \log n}{n} n^{1/e} \rightarrow 0, \end{aligned}$$

where we use the facts that $(1 - t/(\epsilon n))^t \rightarrow 1$ and $\epsilon \log(1/\epsilon) \leq 1/e < 1$. This contradiction proves the assertion for $k = 2$, so we assume that $k \geq 2$ and assertion is valid for k . Let F be a graph with n vertices and minimum degree at least $(\frac{k-1}{k} + \epsilon)n$. Note that the existence of such a F implies $0 < \epsilon < 1/k$. Since

$$\delta(F) \geq \left(\frac{k-1}{k} + \epsilon \right) n > \left(\frac{k-2}{k-1} + \frac{1}{k(k-1)} \right) n,$$

by the induction hypothesis F contains a $K_k(t')$ on vertex set X with $t' = \lceil c(k) \log n \rceil$ vertices in each class, where

$$c(k) = \frac{1/(k(k-1))}{2^{k-2} (k-2)!} = \frac{1}{2^{k-2} k!}.$$

Let Y be the set of vertices in $V(F) \setminus X$ adjacent to at least $(\frac{k-1}{k} + \frac{\epsilon}{2})|X|$ vertices of X . We shall prove that

$$|Y| \geq \epsilon n.$$

To see this, let us consider number of edges between X and $V(F) \setminus X$, denoted by f . Clearly, each vertex in X is adjacent to at least $\delta(F) - |X| + 1$ vertices in $V(F) \setminus X$, so

$$f > |X| \left(\left(\frac{k-1}{k} + \epsilon \right) n - |X| \right).$$

Also, for a vertex $v \in (V(F) \setminus X) \setminus Y$, it is adjacent at most $(\frac{k-1}{k} + \frac{\epsilon}{2})|X|$ vertices in X , so

$$f \leq |X||Y| + (n - |X| - |Y|) \left(\frac{k-1}{k} + \frac{\epsilon}{2} \right) |X|$$

$$< |X||Y| + (n - |Y|) \left(\frac{k-1}{k} + \frac{\epsilon}{2} \right) |X|.$$

The above lower bound and upper bound of f give that

$$(2 - k\epsilon)|Y| + 2k|X| > k\epsilon n.$$

Since $|X| = kt' = o(n)$, so for large n we have

$$|Y| \geq \frac{(k\epsilon - o(1))n}{2 - k\epsilon} > \frac{k\epsilon n}{2} \geq \epsilon n.$$

Set

$$t = \left\lceil \frac{\epsilon \log n}{2^{k-1}(k-1)!} \right\rceil \leq \left\lceil \frac{k\epsilon t'}{2} \right\rceil.$$

Then

$$\left\lceil \left(\frac{k-1}{k} + \frac{\epsilon}{2} \right) |X| \right\rceil = \left\lceil (k-1)t' + \frac{k\epsilon}{2}t' \right\rceil = (k-1)t' + t.$$

That is to say that each vertex in Y is adjacent to at least $(k-1)t' + t$ vertices in X , showing that each such vertex covers at least one $K_k(t)$ in existing $K_k(t')$. There are $\binom{t'}{t}^k$ such $K_k(t)$ in this $K_k(t')$, so there is a set $Z \subseteq Y$ with $|Z| \geq |Y|/\binom{t'}{t}^k$, such that every vertex of Z covers the same $K_k(t)$. To complete the proof, we have to verify that $|Z| \geq t$. Note the facts that $\binom{t'}{t} \leq (et'/t)^t$ from the Stirling's formula, and $t/(et') > k\epsilon/(2e) \geq \epsilon/e$,

$$\begin{aligned} |Z| &\geq \epsilon n \left(\frac{t}{et'} \right)^{tk} \geq \epsilon n \left(\frac{\epsilon}{e} \right)^{tk} \\ &\geq \epsilon n \exp \left(\frac{k\epsilon \log n \log(\epsilon/e)}{2^{k-1}(k-1)!} \right). \end{aligned}$$

Note that $k \leq (k-1)! 2^{k-1}$ and $\epsilon \log(\epsilon/e) > \log(1/2e)/2 = -1 + \eta > -1$ since $\epsilon < 1/2$, where $\eta = \log(e/2)/2 > 0$. We thus have $|Z| \geq \epsilon n^\eta > \log n \geq t$ for large n , hence we get a subgraph $K_{k+1}(t)$ in F , completing the induction on $k+1$ hence the proof. $\square$

The condition on minimum degree should be weakened to one on the number of edges of a graph.

Lemma 8.2 *Let c, ϵ be positive real numbers and let $n > 3/\epsilon$ be an integer. If F is a graph with n vertices and at least $(c+\epsilon)\binom{n}{2}$ edges, then it contains a subgraph H with $n' \geq \epsilon^{1/2}n$ vertices and $\delta(H) \geq cn'$.*

Proof. The condition $e(F) \geq (c+\epsilon)\binom{n}{2}$ implies that $0 < \epsilon < c+\epsilon \leq 1$. If the assertion is not valid, then there is a sequence of graphs $\{G_i : \ell \leq j \leq n\}$, where $\ell = \lfloor \epsilon^{1/2}n \rfloor$ and the order of G_j is j , such that

$$G = G_n \supseteq G_{n-1} \supseteq \cdots \supseteq G_{\ell+1} \supseteq G_\ell$$

and the only vertex of G_j not in G_{j-1} has degree less than cj in G_j for $n \geq j \geq \ell + 1$. Then

$$\begin{aligned} e(G_\ell) &> (c+\epsilon)\binom{n}{2} - \sum_{j=\ell+1}^n cj \\ &= \epsilon\binom{n}{2} + c\binom{\ell+1}{2} - cn \\ &> \epsilon\binom{n}{2} > \binom{\ell}{2}, \end{aligned}$$

since $\binom{\ell+1}{2} \geq n$ with $\epsilon^{1/2}n - 1 < \ell \leq \epsilon^{1/2}n$, $0 < \epsilon < 1$ and $n \geq 3/\epsilon$. This is a contradiction. $\square$.

Theorem 8.3 *Let $k \geq 2$ be an integer and let $\epsilon > 0$. Then there is $n_0 = n_0(k, \epsilon)$ such that if F is a graph of order $n \geq n_0$ with*

$$e(F) \geq \left(\frac{k-2}{k-1} + \epsilon \right) \binom{n}{2},$$

then F contains $K_k(t)$ for some $t \geq \frac{\epsilon \log n}{2^k (k-2)!}$.

Proof. If $n > 3/\epsilon$, by Lemma 8.2, F has a subgraph H of order $n' \geq \epsilon^{1/2}n$ and $\delta(H) \geq \left(\frac{k-2}{k-1} + \frac{\epsilon}{2} \right) n'$. By Lemma 8.1, if n is sufficiently large, then F contains a $K_k(t)$ for some t satisfying

$$t \geq \frac{(\epsilon/2) \log n'}{2^{k-2} (k-2)!} \geq \frac{\epsilon \log n}{2^k (k-2)!},$$

as desired. $\square$

The following corollary is the original Erdős-Stone theorem.

Corollary 8.2 *Let H be a fixed graph with $\chi(H) = k \geq 2$. Then*

$$ex(n; H) = \left(\frac{k-2}{k-1} + o(1) \right) \binom{n}{2}$$

as $n \rightarrow \infty$.

The different proofs for the above corollary can be found in Erdős and Stone(1946), and Erdős and Siminovits(1966), or *Extremal Graph Theory* by Bollobás (1978).

8.2 Bipartite Cases–Upper Bounds

This asymptotical formula of $ex(n; H)$ in the last section is also valid for a graph H with $\chi(H) = 2$. But it does not tell too much, by which we just know that $ex(n; H) = o(n^2)$. The story for bipartite graphs is totally different. We shall discuss the upper bounds of bipartite Tur'an numbers in this section and next section, and we will discuss their lower bounds in next chapter. The following result of Kővári, Sós and Turán (1954) used a powerful double counting argument in the proof.

Theorem 8.4 *Let $t \leq s$ be positive integers. Then*

$$ex(n; K_{t,s}) \leq \frac{1}{2}[(s-1)^{1/t}n^{2-1/t} + (t-1)n]. \quad (8.3)$$

Proof. Let G be a graph of order n that contains no $K_{t,s}$. Let us say that a set is covered by a vertex v if v is joined to every vertex of the set. Since G does not contain $K_{t,s}$, every t -set (i.e., a set with t elements) is covered by at most $s-1$ vertices. Therefore, if G has degree sequence $d_1, d_2, \dots, d_n$, then

$$\sum_{i=1}^n \binom{d_i}{t} \leq (s-1) \binom{n}{t}.$$

Without loss of generality we assume $d_i \geq t$ since otherwise $\binom{d_i}{t} = 0$. Let

$$g(x) = \frac{x(x-1)\dots(x-t+1)}{t!},$$

then $g(x)$ is convex for $x \geq t$. Therefore we have

$$n \frac{d(d-1)\dots(d-t+1)}{t!} \leq (s-1) \frac{n(n-1)\dots(n-t+1)}{t!}$$

where d is the average degree of G . Thus $(d-t+1)^t \leq (s-1)n^{t-1}$. Therefore we have

$$e(G) = \frac{nd}{2} \leq \frac{1}{2}[(s-1)^{1/t}n^{2-1/t} + (t-1)n]$$

as claimed. $\square$

Remark: The upper bound in above theorem can be improved slightly by using

$$\left(\frac{d-t+1}{n-t+1} \right)^t \leq \frac{d(d-1)\dots(d-t+1)}{n(n-1)\dots(n-t+1)},$$

which implies

$$ex(n; K_{t,s}) \leq \frac{1}{2}[(s-1)^{1/t}n(n-t+1) + (t-1)n].$$

We shall improve the upper bound for $ex(n; C_4)$ in above theorem slightly, so that it almost agrees with a lower bound in Chapter 9. The following result is due to Kővári, Sós, and Turán (1954), and due to Reiman (1958).

Theorem 8.5 *Let n be a positive integer, then*

$$ex(n; C_4) \leq \frac{n}{4}(1 + \sqrt{4n-3}).$$

Proof. Let G a graph of order n containing no C_4 , from what we have

$$\sum_{v \in V(G)} \binom{d(v)}{2} \leq \binom{n}{2}$$

or

$$\sum_{v \in V(G)} d^2(v) \leq n(n-1) + \sum_{v \in V(G)} d(v).$$

Applying the Cauchy-Schwarz inequality, we have

$$\left(\sum_{v \in V(G)} d(v) \right)^2 \leq n \sum_{v \in V(G)} d^2(v)$$

hence

$$\left(\sum_{v \in V(G)} d(v) \right)^2 \leq n^2(n-1) + n \sum_{v \in V(G)} d(v).$$

Denote by $x = \frac{1}{2} \sum_{v \in V(G)} d(v)$ for the number of edges of G , then

$$4x^2 \leq n^2(n-1) + 2nx.$$

Solving the corresponding quadratic equation in x we obtain

$$x \leq \frac{n}{4}(1 + \sqrt{4n-3}),$$

yielding the inequality as required. □

Let us turn to the *the problem of Zarankiewicz*. Denote by $G(m, n)$ for a bipartite graph with m vertices in the first class and n vertices in the second. Define $z(m, n; s, t)$ to be the maximum number of edges in a $G(m, n)$ which does not contain $K_{s,t}$ with s vertices in the first class and t in the second. The argument in the proof of the above theorem can apply to obtain the following upper bound.

$$z(m, n; s, t) \leq (s-1)^{1/t} n m^{1-1/t} + (t-1)m. \quad (8.4)$$

Lemma 8.3 *For any positive integers n , s and t ,*

$$ex(n; K_{t,s}) \leq \frac{1}{2} z(n, n; s, t).$$

Proof. Let $G = (V, E)$ be a graph with $|V| = n$ and $|E| = e(G) = ex(n; K_{t,s})$ edges that does not contain $K_{t,s}$. Construct a bipartite graph H as follows. Take two disjoint copies of $V(G)$, say

$$V' = \{v'_1, \dots, v'_n\} \quad \text{and} \quad V'' = \{v''_1, \dots, v''_n\}.$$

The graph H has bipartition (V', V'') , and $x'_i x''_j$ is an edge of H if and only if $x_i x_j$ is an edge of G . Note v'_i and v''_i are not connected. Clearly $d_G(x_i) = d_H(x'_i) = d_H(x''_i)$, thus $e(H) = 2e(G)$, and since G does not contain $K_{t,s}$ hence H does not contain $K_{t,s}$. So $Z(n, n; s, t) \geq e(H) = 2e(G)$. $\square$

For fixed $s \geq t \geq 2$ and large n , the asymptotic form of upper bound (8.3) is

$$ex(n; K_{t,s}) \leq \left(\frac{1}{2} + o(1) \right) (s-1)^{1/t} n^{2-1/t}.$$

This result has not been improved for more than a half century. Füredi did it for fixed $s \geq t \geq 3$ in 1996.

Theorem 8.6 *Let $m \geq s \geq 1$ and $n \geq t \geq 1$ be integers. Then*

$$z(m, n; s, t) \leq (s-t+1)^{1/t} n m^{1-1/t} + tn + tm^{2-2/t},$$

and hence

$$ex(n; K_{t,s}) \leq \frac{1}{2} \left((s-t+1)^{1/t} n^{2-1/t} + tn + tn^{2-2/t} \right).$$

Proof. The case $t = 1$ is trivial, and $t = 2$ is known in (8.4) so we assume that $s \geq t \geq 3$. Let G be a bipartite graph with bipartition

$$V = \{1, 2, \dots, m\} \quad \text{and} \quad V' = \{1', 2', \dots, n'\},$$

which does not contain $K_{t,s}$ with s vertices in V and t vertices in V' . In case that there are some $i \in V$ such that $d(i) < t$ or some $j' \in V'$ such that $d(j') < t$, the validity of the theorem can be verified by the induction on $m + n$. By deleting such vertex i or j' , we know that the number of edges of G is at most

$$z(m-1, n; s, t) + t - 1 \quad \text{or} \quad z(m, n-1; s, t) + t - 1$$

which are less than that desired. We now suppose that $d(i) \geq t$ for any $i \in V$ and $d(j') \geq t$ for any $j' \in V'$. Fix $t-2$ vertices $1 \leq i_1 < i_2 < \dots < i_{t-2} \leq m$ in V . Consider all t -subsets of $N(i_1) \cap \dots \cap N(i_{t-2})$ in V' . Any such set T in V' is covered by at most $s - t + 1$ further more $N(k)$ (with $k \in V$) by the property of G . We obtain

$$\begin{aligned} \sum_{k \neq i_1, \dots, i_{t-2}} \binom{|N(i_1) \cap \dots \cap N(i_{t-2}) \cap N(k)|}{t} \\ \leq (s - t + 1) \binom{|N(i_1) \cap \dots \cap N(i_{t-2})|}{t}, \end{aligned}$$

where the first summation is over $k \neq i_1, \dots, i_{t-2}$. We now need following lemma in which the function $\binom{x}{t}$ on $x \geq 0$ is defined as

$$\binom{x}{t} = \begin{cases} x(x-1) \cdots (x-t+1)/t! & x \geq t-1 \\ 0 & 0 \leq x < t-1. \end{cases}$$

Lemma 8.4 *Let $p, t \geq 1$ be integers, and $c, x_0, x_1, \dots, x_t \geq 0$ be real numbers. If*

$$\sum_{i=1}^p \binom{x_i}{t} \leq c \binom{x_0}{t},$$

then

$$\sum_{i=1}^p x_i \leq x_0 c^{1/t} p^{1-1/t} + (t-1)p.$$

Proof of the lemma. Let $\sigma = \sum_{i=1}^p x_i$. We suppose $\sigma > (t-1)p$. From the convexity of the function $\binom{x}{t}$, we have $p \binom{\sigma/p}{t} \leq \sum_{i=1}^p \binom{x_i}{t}$. Hence

$$\frac{p}{c} \leq \frac{x_0(x_0-1) \cdots (x_0-t+1)}{(\sigma/p)(\sigma/p-1) \cdots (\sigma/p-t+1)} \leq \left(\frac{x_0}{\sigma/p-t+1} \right)^t,$$

the desired inequality follows immediately. $\square$

Now, using the lemma with $p = m - t + 2, c = s - t + 1$, and $x_0 = |N(i_1) \cap \cdots \cap N(i_{t-2})|$, we have

$$\begin{aligned} & \sum_{k \neq i_1, \dots, i_{t-2}} |N(i_1) \cap \cdots \cap N(i_{t-2}) \cap N(k)| \\ & \leq (s - t + 1)^{1/t} (m - t + 2)^{1-1/t} |N(i_1) \cap \cdots \cap N(i_{t-2})| \\ & \quad + (t - 1)(m - t + 2). \end{aligned}$$

Summing up both sides over $i_1, \dots, i_{t-2}$, the left hand side is

$$\begin{aligned} & \sum_{i_1 < \cdots < i_{t-2}} \sum_{k \neq i_1, \dots, i_{t-2}} |N(i_1) \cap \cdots \cap N(i_{t-2}) \cap N(k)| \\ & = (t - 1) \sum_{j=1}^n \binom{d(j')}{t-1}. \end{aligned}$$

And for the right hand side,

$$\sum_{i_1 < \cdots < i_{t-2}} |N(i_1) \cap \cdots \cap N(i_{t-2})| = \sum_{j=1}^n \binom{d(j')}{t-2}.$$

We thus obtain

$$\begin{aligned} & (t - 1) \sum_{j=1}^n \binom{d(j')}{t-1} \\ & \leq (s - t + 1)^{1/t} (m - t + 2)^{1-1/t} \sum_{j=1}^n \binom{d(j')}{t-2} \\ & \quad + (t - 1)(m - t + 2) \binom{m}{t-2}. \end{aligned}$$

We will derive a lower bound for the left hand side in the new inequality. Denote by y_j for $d(j')$. One has

$$\begin{aligned} & [y_i(y_i - 1) \cdots (y_i - (t - 3)) - y_j(y_j - 1) \cdots (y_j - (t - 3))] \\ & \quad \times [(y_i - (t - 2)) - (y_j - (t - 2))] \geq 0, \end{aligned}$$

which yields

$$\left[\binom{y_i}{t-2} - \binom{y_j}{t-2} \right] [(y_i - (t - 2)) - (y_j - (t - 2))] \geq 0.$$

Therefore

$$\begin{aligned} & \binom{y_i}{t-2} (y_j - (t - 2)) + \binom{y_j}{t-2} (y_i - (t - 2)) \\ & \leq \binom{y_i}{t-2} (y_i - (t - 2)) + \binom{y_j}{t-2} (y_j - (t - 2)) \\ & = (t - 1) \left[\binom{y_i}{t-1} + \binom{y_j}{t-1} \right]. \end{aligned}$$

Adding up over pairs of $(y_i, y_j) = (d(i'), d(j'))$ for all $1 \leq i, j \leq n$, we have

$$\left(\sum_{j=1}^n \binom{d(j')}{t-2} \right) \left(\sum_{j=1}^n (d(j') - (t-2)) \right) \leq n(t-1) \sum_{j=1}^n \binom{d(j')}{t-1}.$$

Combining this with what obtained, we have

$$\begin{aligned} & \frac{1}{n} \left(\sum_{j=1}^n \binom{d(j')}{t-2} \right) \left(\sum_{j=1}^n (d(j') - (t-2)) \right) \\ & \leq (s-t+1)^{1/t} (m-t+2)^{1-1/t} \sum_{j=1}^n \binom{d(j')}{t-2} \\ & \quad + (t-1)(m-t+2) \binom{m}{t-2}. \end{aligned}$$

Thus

$$\begin{aligned} & \sum_{j=1}^n d(j') - n(t-2) \\ & \leq (s-t+1)^{1/t} (m-t+2)^{1-1/t} n \\ & \quad + (t-1)(m-t+2) \frac{n \binom{m}{t-2}}{\sum_{j=1}^n \binom{d(j')}{t-2}}. \end{aligned}$$

Since $e(G) = \sum_{j=1}^n d(j')$, we have

$$e(G) \leq (s-t+1)^{1/t} n m^{1-1/t} + tn + tm \frac{n \binom{m}{t-2}}{\sum_{j=1}^n \binom{d(j')}{t-2}}.$$

If the last fraction is at most $m^{1-2/t}$, the the above inequality implies what required for $e(G)$. Otherwise, we have

$$\sum_{j=1}^n \binom{d(j')}{t-2} < \frac{n}{m^{1-2/t}} \binom{m}{t-2}.$$

Apply Lemma 8.4 again with $p = n$, $c = n/m^{1-2/t}$, $x_0 = m$, and $x_i = d(i')$, and replacing t by $t-2$, we then have

$$\begin{aligned} \sum_{j=1}^n d(j') & < m \left(\frac{n}{m^{1-2/t}} \right)^{1/(t-2)} n^{1-1/(t-2)} + (t-3)n \\ & = nm^{1-1/t} + (t-3)n. \end{aligned}$$

The result follows immediately. $\square$

Corollary 8.3 *Let integers $s \geq t \geq 1$ be fixed and $n \rightarrow \infty$. Then*

$$ex(n; K_{t,s}) \leq \left(\frac{1}{2} + o(1)\right) (s - t + 1)^{1/t} n^{2-1/t}.$$

We now discuss a relation between $ex(n; G)$ and $r_k(G)$, where G is a bipartite graph. Note if $ex(n; G) \leq (c + o(1))n^{2-1/t}$ for some constants $c > 0$ and $t \geq 1$, then G must be a bipartite graph.

Theorem 8.7 *If $ex(n; G) \leq (c + o(1))\frac{1}{2}n^{2-1/t}$ as $n \rightarrow \infty$, where $c > 0$ and $t \geq 1$ are constants, then*

$$r_k(G) \leq (1 + o(1))(ck)^t$$

as $k \rightarrow \infty$.

Proof. Setting $n = r_k(G) - 1$, then there is k -coloring of edges of K_n such that there is no monochromatic G . By consider each subgraph induced by edges in a single color and the definition of Turán number, we have

$$\binom{n}{2} \leq k ex(n; G) \leq (c + o(1))\frac{k}{2}n^{2-1/t},$$

which yields $n \leq (1 + o(1))(ck)^t$ hence the required inequality. $\square$

Corollary 8.4 *Let $s \geq t \geq 2$ be fixed integers and let $k \rightarrow \infty$, then*

$$r_k(K_{t,s}) \leq (1 + o(1))(s - t + 1)k^t.$$

The idea in proof of the following result is due to N. Alon, see Chung and Graham(1999).

Theorem 8.8 *If for large n , $ex(n; G) \geq c_1 n^{2-1/t}$, where $c_1 > 0$ is a constant and $t \geq 1$ is a fixed integer, then there exists a constant $c_2 > 0$ such that*

$$r_k(G) \geq c_2 \left(\frac{k}{\log k}\right)^t,$$

as $k \rightarrow \infty$.

Proof. Set $c_2 = (c_1/t)^t$, and set $n = \lfloor c_2(k/\log k)^t \rfloor$. Let H denote a graph on n vertices with $e(H) = ex(n; G)$ edges containing no G . Let $H_1, H_2, \dots, H_k$ be k copies of H placed randomly and independently in

the complete graph K_n on n vertices. For each edge $\{u, v\}$ of K_n , the probability that the edge is not covered by any graph H_i is precisely

$$p = \left(1 - e(H)/\binom{n}{2}\right)^k.$$

If $\binom{n}{2}p < 1$, then there is a choice of placing these H_i so that their union covers all edges of K_n . By referring each edge in H_i as one in color i , then any edge of K_n is colored in at least one color. Keep one color for each edge and delete other colors if the edge got more than one colors, then the edges of K_n is colored in k colors, and there is no monochromatic G . Thus $r_k(G) > n$. Now using $1 - x < e^{-x}$ for $x > 0$, we have

$$\left(1 - e(H)/\binom{n}{2}\right)^k \leq \left(1 - 2t \frac{\log k}{k}\right)^k \leq \left(\frac{1}{k}\right)^{2t},$$

which yields

$$\binom{n}{2}p \leq \frac{c_2^2}{2} \left(\frac{k}{\log k}\right)^{2t} \left(\frac{1}{k}\right)^{2t} \rightarrow 0,$$

the required bound follows. $\square$

It is a widespread belief that the order $n^{2-1/t}$ in the upper bound of $ex(n; K_{t,s})$ is sharp as $s \geq t \geq 2$ fixed and $n \rightarrow \infty$. If so, then the order of $r_k(K_{t,s})$ is between $\left(\frac{k}{\log k}\right)^t$ and k^t from the above results. However, when we have a construction to give a lower bound of form $ex(n; K_{s,t}) \geq c_1 n^{2-1/t}$, we often get a lower bound of form $r_k(K_{s,t}) \geq c_2 k^t$. This situation happens repeatedly in the next chapter.

We conclude this section by remarking that if $\chi(G) \geq 3$, then $r_k(G)$ is much greater than any polynomial function of k as $k \rightarrow \infty$.

Theorem 8.9 *Let G be a graph with $\chi = \chi(G) \geq 3$, then there exists a constant $c > 0$ such that*

$$r_k(G) \geq c(\chi - 1)^k$$

for any integer $k \geq 2$.

Proof. We in fact will prove

$$r_k(G) \geq c(\chi - 1)^k + 1$$

with $c = (r_2(G) - 1)/(\chi - 1)^2$ by induction on k . For $k = 2$, we have nothing to do. For $k \geq 3$, set $N = r_{k-1}(G) - 1$. Suppose all edges of K_N is

colored in $k - 1$ colors so that contains no monochromatic G . Take $\chi - 1$ disjoint copies of the colored K_N , and connect these copies completely by edges in a new color. Then we have a complete graph of order $(\chi - 1)N$ whose edges are colored in k colors, in which there is no monochromatic G , thus by induction

$$\begin{aligned} r_k(G) - 1 &\geq (\chi - 1)(r_{k-1}(G) - 1) \\ &\geq c(\chi - 1)^k, \end{aligned}$$

as desired. $\square$

Therefore $r_k(G)$ can be bounded above by a polynomial of k if and only if G is a bipartite graph.

8.3 Upper Bound of $ex(n; C_{2m})$

In this section, we shall give the following result due to Erdős (1965), who published it without proof, and due to Bondy and Simonovits (1974), who published a proof.

Theorem 8.10 *Let $m \geq 2$ be a fixed integer. Then*

$$ex(n; C_{2m}) \leq 10mn^{1+1/m}$$

for $n \geq 10^{m^2}$.

Let us first recall a result in Chapter 3 as follows.

Lemma 8.5 *Let G be a graph with average degree d . Then there is a subgraph H of G such that $\delta(H) \geq d/2$.*

The next result is similar, in addition to be used here, which has its own interest.

Lemma 8.6 *For any graph G , there is a subgraph H of G such that H is bipartite and $e(H) \geq e(G)/2$.*

Proof. For a subset S of $V(G)$, write $\bar{S} = V(G) \setminus S$. Let $e(S, \bar{S})$ be the number of edges between S and $\bar{S}$. Maximizing $e(S, \bar{S})$ over all proper subsets S of $V(G)$, we obtain a spanning bipartite subgraph H on vertex classes S and $\bar{S}$ with $e(H) = e(S, \bar{S})$. Then for any vertex v , say $v \in S$, at least half of neighbors of v are in $\bar{S}$ since otherwise removing v from S to

$\bar{S}$ would increase $e(S, \bar{S})$, contradicting to the maximality of $e(S, \bar{S})$. This follows by $e(H) \geq e(G)/2$.

The Lemma has a simple proof by probabilistic method. Let S be a random set of $V(G)$ defined by $\Pr(v \in S) = 1/2$, independently. Then it is easy to know that the probability of any edge is an edge between S and $\bar{S}$ is $1/2$. Thus the expectation of $e(S, \bar{S})$ is $e(G)/2$, implying that there is a S such that $e(S, \bar{S}) \geq e(G)/2$. $\square$

Now we shall prove the bipartite version of Theorem 8.10 first. To do this, We shall have more preparative lemmas. Let $t \geq 1$ be an integer. A coloring of vertices of a graph G , not necessarily proper, is called t -periodic if the pair of end vertices of any path of length t have the same color. So in any 1-periodic coloring, all vertices in a connected graph G have the same color. For $t \geq 2$, if G has enough edges, the number of colors used in a t -periodic coloring is small.

Lemma 8.7 *Let $t \geq 1$ be a integer, and let G be a connected graph of order n . If $e(G) \geq 2tn$, then the number of colors in any t -periodic coloring of G is at most two.*

Proof. We separate the proof into several short steps. We first prove that G contains two adjacent vertices joined by two internal vertex-disjoint paths, each of length at least t . We shall call this subgraph as a θ -graph intuitively.

Step 1: $\delta(G) \geq 2t$. We can find such a θ -graph in the following way. Let $x_1x_2 \cdots x_m$ be a longest path. Then x_1 is adjacent only to vertices of this path, $x_{i_1}, x_{i_2}, \dots, x_{i_r}$, say, where

$$2 = i_1 < i_2 < \cdots < i_r, \quad r \geq 2t.$$

The cycle $x_1x_2 \cdots x_{i_{2t}}x_1$ and the edge $x_1x_{i_t}$ form the desired θ -graph.

Step 2: $\delta(G) < 2t$. Since $e(G) \geq 2tn$, so the average degree of G is at least $4t$. From Lemma 8.5, G contains a subgraph H with $\delta(H) \geq 2t$. Then from step 1, H contains desired θ -graph.

Step 3: Any θ -graph has three cycles. Denote by C_1, C_2, C_3 for these cycles with lengths $\ell_1 \leq \ell_2 \leq \ell_3$, respectively, then

$$\ell_1 + \ell_2 - \ell_3 = 2.$$

The restrictions of the t -periodic coloring of G to the θ -graph and to each cycle C_i are also t -periodic. Let $t_i \geq 2$ be the smallest integer such that the coloring of C_i is t_i -periodic for $i = 1, 2, 3$. Clearly $t_i | \ell_i$ and any period on one cycle induces the same period on the other two cycles and hence

$t_1 = t_2 = t_3$. Let $\bar{t}$ be the same value of t_i , then $\bar{t}|\ell_i$ hence $\bar{t}|2$ so $\bar{t} = 2$, implying that the number of colors in the θ -graph is at most two.

Step 4: Since G is connected, any vertex of G is jointed to some vertex in the θ -graph by a path of length kt , probably using some vertices in the θ -graph. Thus both of the end vertices of the path have the same color. Hence the number of colors in G is at most two from step 3. $\square$

Lemma 8.8 *Let G be a bipartite graph of order $n \geq 10^m$ with $\delta(G) \geq 5mn^{1/m}$. Then G contains C_{2m} .*

Proof. Fix a vertex x of G and let

$$V_i = \{v \in V(G) : d(v, x) = i\},$$

which is the set of vertices at distant i from x . Since G is bipartite, each V_i is an independent set.

Suppose that G contains no C_{2m} . We show that

$$|V_i| \geq n^{1/m} |V_{i-1}| \quad (8.5)$$

for $1 \leq i \leq m$, which leads to the contradiction that $|V(G)| > n$ since $|V_m| \geq n$.

We shall prove (8.5) by induction on i . This is trivial for $i = 1$ since $\delta(G) \geq 5mn^{1/m}$. Suppose (8.5) holds for smaller value of i .

Let H be the subset of G induced by $V_{i-1} \cup V_i$ and let $H_1, H_2, \dots, H_q$ be the components of H . Write $W_j = V(H_j) \cap V_{i-1}$.

A path $x_1 x_2 \dots x_k$ in G is called *monotonic* if $d(x_i, x)$ is monotonic. This means that a monotonic path passes through each of some consecutive sets $V_j, V_{j+1}, \dots, V_{j+k}$ exactly once.

We shall show that $e(H_1) < 4m|V(H_1)|$. This is trivial if W_1 has just one vertex hence H_1 is a star. We thus assume that W_1 has at least two vertices. Let $p < i - 1$ be the smallest index such that there is a vertex $a \in V_p$ and there are two monotonic paths P_1, P_2 joining a to W_1 which have just the vertex a in common.

We then show that each vertex of W_1 is jointed to a by a monotonic path. This is clear if $a = x$. Otherwise, for $y \in W_1$, there is a monotonic path P_3 joining y to x . By the minimality of p , P_3 must intersect P_1 or P_2 , say P_1 , at some vertex z . The path consisting of the section of P_3 between y and z and the section of P_1 between z and a is a monotonic path from y to a .

We now assign colors red and blue to the vertices of W_1 in such a way that if two vertices have different colors, then they are joined to a by internal

disjoint monotonic paths. This can be done as follows. Each vertex of W_1 that can be joined to a by a monotonic path disjoint from P_2 is colored red; all other vertices of W_1 are colored blue. To see this is the required coloring, let x_1 and x_2 be vertices of W_1 colored red and blue, respectively. Let P'_1 be a monotonic path from x_1 to a disjoint from P_2 , and let P'_2 be a monotonic path from x_2 to a . Moving along P'_2 from x_2 towards a , let v be first vertex of $(P'_1 \cup P_2) \setminus \{a\}$ encountered. Such a vertex v exists since x_2 is colored blue. Also we see that v cannot belong to P'_1 for otherwise the section of P'_2 between x_2 and v together with the section of P'_1 between v and a would constitute a monotonic path from x_2 to a disjoint from P_2 , contradicting the assumption that x_2 is colored blue. But then $v \in P_2$ and we have a monotonic path $x_2 P'_2 v P_2 a$ disjoint from P'_1 .

We now color the vertices of H_1 in V_i green and show that this coloring of H_1 is t -periodic with $t = 2(m - i + p + 1)$. For, since t is even, if one end vertex of a path of length t in H_1 is green, the so is the other. Also, there can be no path of length t joining a red and a blue vertex, because, if a red x_1 were joined to a blue x_2 by such a path, this path together with vertex-disjoint monotonic paths from x_1 to a and x_2 to a would form a C_{2m} . Therefore, the coloring of H_1 is indeed t -periodic. Since there colors are used in the coloring, Lemma 8.7 implies that

$$e(H_1) < 2t|V(H_1)| < 4m|V(H_1)|.$$

Similarly we have that $e(H_j) < 4m|V(H_j)|$ for $j = 1, 2, \dots, q$ hence

$$e(H) < 4m|V(H)|.$$

Let H' be the subgraph of G induced by $V_{i-2} \cup V_{i-1}$. The same argument gives

$$e(H') < 4m|V(H')|.$$

Clearly, since $\delta(G) \geq 5mn^{1/m}$,

$$e(H) + e(H') \geq 5mn^{1/m}|V_{i-1}|.$$

Combining these inequalities, we get

$$\begin{aligned} & 4m(|V_{i-1}| + |V_i| + |V_{i-1}| + |V_{i-2}|) \\ = & 4m(|V(H)| + |V(H')|) > e(H) + e(H') \geq 5mn^{1/m}, \end{aligned}$$

which implies that

$$|V_i| > \frac{1}{4m}[(5mn^{1/m} - 8m)|V_{i-1}| - 4m|V_{i-2}|].$$

Using the induction hypothesis,

$$|V_{i-1}| \geq n^{1/m} |V_{i-2}|.$$

we obtain

$$\begin{aligned} |V_i| &> \frac{1}{4m} \left(5mn^{1/m} - 8m - \frac{20m^2}{5mn^{1/m}} \right) |V_{i-1}| \\ &> \frac{1}{4m} (5mn^{1/m} - 9m) |V_{i-1}| \\ &> n^{1/m} |V_{i-1}| \end{aligned}$$

as desired. $\square$

Proof of Theorem 8.10. Let G be a graph of order $n \geq 10^{m^2}$ with $e(G) \geq 10mn^{1+1/m}$. The Lemma 8.6 implies that there is a spanning subgraph H of G such that H is bipartite and $e(H) \geq 5mn^{1+1/m}$. The average degree of H is at least $10mn^{1/m}$. From Lemma 8.5, there is a subgraph F of H satisfying that $\delta(F) \geq 5mn^{1/m}$. Let n_0 be the order of F , then

$$n_0 > \delta(F) \geq 5mn^{1/m} > n^{1/m} \geq 10^m,$$

and $\delta(F) \geq 5mn_0^{1/m}$. Thus by Lemma 8.8, the bipartite graph F contains C_{2m} , so does G . This proves the theorem. $\square$

8.4 $r(K_{t,s}, K_n)$ and $r(C_{2m}, K_n)$ with n Large

We have seen that for a fixed graph G , $r(G, K_n)$ is linear on n if and only if G is a forest. We prove an asymptotic upper bound of $r(K_{t,s}, K_n)$ as $n \rightarrow \infty$ in this section, see Li and Zang (2003).

Recall the main result in Chapter 3 that for any graph G of order N with average degree d ,

$$\alpha(G) \geq N f_m(d), \quad (8.6)$$

if the subgraph of G induced by any neighborhood contains no star or path of m edges, and $f(x) = f_m(x) > (\log(x/m) - 1)/x$ for $x > 0$.

Theorem 8.11 *For any fixed integers $s \geq t \geq 2$, as $n \rightarrow \infty$,*

$$r(K_{t,s}, K_n) \leq (1 + o(1))(s - t + 1) \left(\frac{n}{\log n} \right)^t.$$

Proof. Let $N = r(K_{t,s}, K_n) - 1$ and let G be a graph on N vertices with no $K_{t,s}$ whose independence number is at most $n - 1$. By the upper bound of Turán numbers in Section 2, we have

$$d(G) \leq (1 + o(1))(s - t + 1)^{1/t} N^{1-1/t}, \quad (8.7)$$

where $d(G)$ is the average degree of G .

Let v be an arbitrary vertex of G and let G_v be the subgraph of G induced by the neighborhood of v and let $\Delta(G_v)$ be the maximum degree of G_v . Then

$$\Delta(G_v) \leq s - 1 \text{ if } t = 2 \quad (8.8)$$

and

$$\Delta(G_v) \leq r(K_{t-2,s}, K_n) - 1 \text{ if } t \geq 3. \quad (8.9)$$

Indeed, for $t = 2$, (8.8) follows instantly from the fact that G contains no $K_{2,s}$. For $t \geq 3$, suppose to the contrary that the degree of some vertex u in G_v is at least $r(K_{t-2,s}, K_n)$. Since G contains no K_n , $G_u \cap G_v$ must contain $K_{t-2,s}$, which together with u and v yield a $K_{t,s}$ in G , a contradiction.

Now let us apply induction on $t \geq 2$.

For $t = 2$, in view of (8.7) and (8.8), from (8.6) we have,

$$\begin{aligned} n &> N f_s(d(G)) \\ &\geq (1 - o(1)) \frac{N \log(\sqrt{(s-1)N}/s)}{\sqrt{(s-1)N}} \\ &= (1 - o(1)) \sqrt{\frac{N}{s-1}} \frac{\log N}{2}. \end{aligned}$$

It follows that $N \leq (s - 1 + o(1))(n/\log n)^2$ since otherwise, there exists a constant $\delta > 0$ such that $N \geq (s - 1 + \delta)(n/\log n)^2$ for infinitely many n , which will lead a contradiction.

Let us proceed to the induction step. Recall (8.7) and (8.9), by the main theorem with $a = r(K_{t-2,s}, K_n)$ and $d = d(G)$ we have $n > N f_a(d)$. It follows that

$$n \geq (1 - o(1)) \left(\frac{N}{s - t + 1} \right)^{1/t} \log \frac{N^{1-1/t}}{r(K_{t-2,s}, K_n)}. \quad (8.10)$$

Now we distinguish between two cases according to the value of t .

Case 1. $t = 3$. Our proof begins at Chvátal's discovery on $r(T, K_n)$ mentioned in Chapter 2,

$$r(K_{1,s}, K_n) = s(n - 1) + 1.$$

So the inequality (8.10) becomes

$$\begin{aligned} n &\geq (1 - o(1)) \left(\frac{N}{s-2} \right)^{1/3} \log \frac{N^{2/3}}{s(n-1)+1} \\ &= (1 - o(1)) \left(\frac{N}{s-2} \right)^{1/3} \log \frac{N^{2/3}}{n}. \end{aligned} \quad (8.11)$$

For any fixed $0 < \epsilon < 1$, divide the natural number set into two sets $S = \{n'\}$ and $T = \{n''\}$ such that

$$\frac{N^{2/3}}{n} \geq (n')^{1-\epsilon} \quad \text{and} \quad \frac{N^{2/3}}{n''} < (n'')^{1-\epsilon}.$$

We suppose both sizes of S and T are infinite without loss of generality. For large $n \in S$, from (8.11), we have

$$n \geq (1 - o(1))(1 - \epsilon) \left(\frac{N}{s-2} \right)^{1/3} \log n,$$

which follows by $N \leq \frac{1}{(1-2\epsilon)^3} (s-2)(n/\log n)^3$. On the other hand, for large $n \in T$, we have directly

$$N \leq n^{3-3\epsilon/2} = o\left(\frac{n}{\log n}\right)^3.$$

Since $\epsilon > 0$ can be arbitrarily small, we get the desired statement.

Case 2. $t \geq 4$. Our proof goes as follows. For any fixed $0 < \epsilon < 1$, divide the natural number set into two sets $S = \{n'\}$ and $T = \{n''\}$ such that

$$\frac{N^{1-1/t}}{r(K_{t-2,s}, K_{n'})} \geq (n')^{1-\epsilon} \quad \text{and} \quad \frac{N^{1-1/t}}{r(K_{t-2,s}, K_{n''})} < (n'')^{1-\epsilon}.$$

Let n be any sufficiently large integer. If $n \in S$ then from (8.10) we have

$$n \geq (1 - \epsilon - o(1)) \left(\frac{N}{s-t+1} \right)^{1/t} \log n.$$

By letting $\epsilon \rightarrow 0$, we get the desired statement. If $n \in T$ then, by definition of T and the induction hypothesis on $r(K_{t-2,s}, K_n)$, we have $N^{1-1/t} < n^{1-\epsilon} c (n/\log n)^{t-1}$ with some constant $c > 0$, which yields

$$N \leq c' \left(\frac{n}{\log n} \right)^{t-\epsilon t/(t-1)} = o\left(\frac{n}{\log n}\right)^t.$$

So the desired statement follows. $\square$

Combining the above theorem and the lower bounds obtained in Chapter 5, we have

$$\begin{aligned} c \left(\frac{n}{\log n} \right)^{(st-1)/(s+t-2)} &\leq r(K_{t,s}, K_n) \\ &\leq (1 + o(1))(s - t + 1) \left(\frac{n}{\log n} \right)^t. \end{aligned}$$

Note that the exponent $(st-1)/(s+t-2)$ in the lower bound can be arbitrarily close to the exponent t in the upper bound when s is large but fixed. So it is natural to ask if the order of upper bound is sharp. Assuming yes, we shall be able to establish a somehow unexpected result: asymptotically, all the extremal graphs for $r(K_{t,s}, K_n)$ come from those for $ex(K_{t,s}; N)$. Thus it is very interesting to estimate the independence numbers of known extremal graphs for $ex(n; K_{t,s})$. No doubt, this assumption is a bold adventure.

Proposition 1 *For any fixed integers $s \geq t \geq 2$, if there exists a constant $c > 0$ such that*

$$r(K_{t,s}, K_n) \geq (c - o(1)) \left(\frac{n}{\log n} \right)^t \quad (8.12)$$

as $n \rightarrow \infty$, then

$$ex(K_{t,s}; N) \geq \frac{1}{2}(c - o(1))^{1/t} N^{2-1/t} \quad (8.13)$$

for all sufficiently large N of the form $N = r(K_{t,s}, K_n) - 1$. Furthermore, the extremal graphs yielding (8.12) also yield (8.13).

Proof. To prove it, assume the contrary: there exists $\delta > 0$ such that $ex(K_{t,s}; N) \leq \frac{1}{2}(c - \delta)^{1/t} N^{2-1/t}$ for infinitely many n , where $N = r(K_{t,s}, K_n) - 1$. Imitating the proof of (8.10), we have

$$n \geq (1 - o(1)) \left(\frac{N}{c - \delta} \right)^{1/t} \log \frac{N^{1-1/t}}{r(K_{t-2,k}, K_n)},$$

which implies $N \leq (c - \delta + o(1)) \left(\frac{n}{\log n} \right)^t$, contradicting (8.12). $\square$

The main idea for the proof of the above theorem is simple: if the average degree of a graph is small, its independence number must be big.

The main result in last section $ex(C_{2m}; N) \leq c(m)N^{1+1/m}$ will be used to improve the result of Erdős, Faudree, Rousseau and Schelp (1978) that $r(C_{2m}, K_n) \leq cn^{m/(m-1)}$ with a factor $\log n$.

Theorem 8.12 *For any fixed $m \geq 2$, there exists constant $c = c(m) > 0$ such that for all sufficiently large n ,*

$$r(C_{2m}, K_n) \leq c \left(\frac{n}{\log n} \right)^{m/(m-1)}.$$

Proof. Let G be a graph of order $N = r(C_{2m}, K_n) - 1$ which contains no C_{2m} and $\alpha(G) < n$. For any vertex v of G , consider the subgraph G_v induced by the neighborhood of v . The subgraph G_v does not contain path P_{2m-2} since G does not contain C_{2m} , where P_{2m-2} is a path of $2m-2$ edges. Also since $ex(C_{2m}; N) \leq c_1 N^{1+1/m}$ for some constant $c_1 = c_1(m) > 0$ thus the average degree of G is at most $2c_1 N^{1/m}$. Note that for fixed a , the function $f_a(x)$ is asymptotic equal to $\log x/x$ as $x \rightarrow \infty$. Apply (8.6) once again,

$$\begin{aligned} n > \alpha(G) &\geq (1 - o(1))N \frac{\log 2c_1 N^{1/m}}{2c_1 N^{1/m}} \\ &> c_2 N^{1-1/m} \log N \end{aligned}$$

for some constant $c_2 > 0$. Now if for any large $c > 0$, there are infinitely many n such that $N \geq c(n/\log n)^{m/(m-1)}$, then $\log N \geq c_3 \log n$, where $c_3 > 0$ is a constant increasing as c increasing, and

$$n \geq c_2 c^{(m-1)/m} \frac{n}{\log n} c_3 \log n = c_2 c_3 c^{(m-1)/m} n$$

which would lead a contradiction if c is large and $n \rightarrow \infty$.

Using the similar proof we can have that for any fixed integers $k \geq 2$, and $s \geq t \geq 2$,

$$r_k(K_{t,s}, \dots, K_{t,s}, K_n) \leq c \left(\frac{n}{\log n} \right)^t,$$

where $c = c(k, t, s) > 0$ is a constant. We conclude this section with the following problems.

Problem 8.1 *For fixed $s \geq t \geq 2$, determine if the order of $r(K_{t,s}, K_n)$ is $(n/\log n)^t$. If yes, does it grow linear on s ?*

8.5 Involving $K_{t,n}$ with Large n

We cannot solve the problem proposed at the end of the last section. The following result seems to support for a positive solution.

Theorem 8.13 *Let $t \geq 2$ be fixed. Then for all large n ,*

$$c \frac{n^{t+1}}{(\log n)^t} \leq r(K_{t,n}, K_n) \leq (1 + o(1)) \frac{n^{t+1}}{(\log n)^t},$$

where $c = c(t) > 0$ is a constant. In particular, the order of $r(K_{t,n}, K_n)$ is $n^{t+1}/(\log n)^t$.

Proof. The lower bound was proved in Chapter 5. The proof for the upper bound is the similar to that for Theorem 8.11, we thus omit it. $\square$

The above result can be generalized as follows with proof by induction on $t = 2$ starting at $r(K_{2,n}, K_n)$.

Theorem 8.14 *Let H be a graph on $t \geq 2$ vertices that is not complete. Then the order of $r(H + \overline{K_n}, K_n)$ is $n^{t+1}/(\log n)^t$ as $n \rightarrow \infty$.*

$\square$

We conjecture the above result may hold for any graph H of order $t \geq 2$. To confirm this, one needs only to verify it holds for $H = K_2$ by the same proof.

We now discuss asymptotic formula of k -color Ramsey number $r_k(K_{m,n})$ as k and m fixed and $n \rightarrow \infty$.

Theorem 8.15 *Let integers $k \geq 2$ and $m \geq 2$, and real $\epsilon > 0$ be fixed. Then*

$$(1 - \epsilon)k^m n \leq r_k(K_{m,n}) \leq k^m(n - 1) + m(km - k + 1) + 1,$$

for all sufficiently large n .

Proof. The lower bound appeared in Chapter 5. For the upper bound we first show that $r_k(K_{m,n}) \leq (1 + o(1))k^m(n - m + 1)$. Set $N = r_k(K_{m,n}) - 1$. By definition, there is a k -coloring of the edges of K_N which contains no monochromatic $K_{m,n}$. So there are at most $ex(N; K_{m,n})$ edges in each single color, using Theorem 8.6, we thus have

$$\begin{aligned} \binom{N}{2} &\leq k \, ex(N; K_{m,n}) \\ &\leq \frac{k}{2} z(N, N; m, n) \\ &\leq \frac{k}{2} [(n - m + 1)^{1/m} N^{2-1/m} + mN + mN^{2-2/m}]. \end{aligned}$$

Suppose m and k are fixed. Clearly $N \rightarrow \infty$ as $n \rightarrow \infty$. Now take $\delta > 0$ such that $(1 + \delta\epsilon)^m < 1 + \epsilon/2$. Then for all large N , we deduce that $N^2 \leq (1 + \delta\epsilon)k(n - m + 1)^{1/m}N^{2-1/m}$, which yields $r_k(K_{m,n}) = N + 1 \leq (1 + \epsilon/2)k^m(n - m + 1) + 1 < (1 + \epsilon)k^m(n - m + 1)$, as desired. Combining the lower bound and the upper bound, we know

$$N \sim k^m n \sim k^m(n - 1) \text{ as } n \rightarrow \infty. \quad (8.14)$$

To get a slightly better upper bound, note that (by upper bound of Turán number in Section 2)

$$\binom{N}{2} \leq ex(K_{m,n}; N) \leq \frac{k}{2}[(n - 1)^{1/m}N^{2-1/m} + (m - 1)N],$$

So $N^2 \leq k(n - 1)^{1/m}N^{2-1/m} + (km - k + 1)N$, and hence

$$\begin{aligned} N^{1/m} &\leq k(n - 1)^{1/m} + \frac{km - k + 1}{N^{1-1/m}} \\ &= k(n - 1)^{1/m} \left(1 + \frac{km - k + 1}{kN} \left(\frac{N}{n - 1} \right)^{1/m} \right). \end{aligned}$$

Using the fact $(1 + x)^m = 1 + mx + O(x^2)$ as $x \rightarrow 0$, it follows that

$$\begin{aligned} N &\leq k^m(n - 1) \left(1 + \frac{km - k + 1}{kN} \left(\frac{N}{n - 1} \right)^{1/m} \right)^m \\ &\leq k^m(n - 1) \left(1 + \frac{m(km - k + 1)}{kN} \left(\frac{N}{n - 1} \right)^{1/m} + \frac{O(1)}{N^2} \right) \\ &= k^m(n - 1) + k^{m-1}m(km - k + 1) \left(\frac{N}{n - 1} \right)^{1/m-1} + o(1). \end{aligned}$$

By (8.14), the second term in the last line tends to $m(km - k + 1)$. Thus $r_k(K_{m,n}) = N + 1 \leq k^m(n - 1) + m(km - k + 1) + 1 + o(1)$. Clearly the term $o(1)$ can be omitted for large n , this completes the proof. $\square$

For $m = k = 2$, we have $(1 - o(1))4n \leq r(K_{2,n}) \leq 4n + 3$. This should be improved. The following result is due to Exoo, Harborth and Mengersent (1991).

Theorem 8.16 *For $n \geq 2$, $r(K_{2,n}) \leq 4n - 2$. If $4n - 3$ is a prime power, $r(K_{2,n}) = 4n - 2$.*

Proof. We prove $r(K_{2,n}) \leq 4n - 2$ first. Suppose, to the contrary, this is not valid. Then for $N = 4n - 2$, there is a red-blue edge coloring of K_N on vertex V that contains no monochromatic $K_{2,n}$. We now estimate the number of red edges between $N_R(v)$ and $N_B(v)$ for $v \in V$. Set

$$A(v) = \sum_{x \in N_R(v)} |N_R(x) \cap N_B(v)|.$$

Since for $x \in N_R(v)$

$$d_B(v) = |N_R(x) \cap N_B(v)| + |N_B(x) \cap N_B(v)|,$$

and $|N_B(x) \cap N_B(v)| \leq n - 1$, so $|N_R(x) \cap N_B(v)| \geq d_B(v) - n + 1$. Hence

$$A(v) \geq d_R(v)(d_B(v) - n + 1).$$

On the other hand,

$$A(v) = \sum_{x \in N_B(v)} |N_R(x) \cap N_R(v)| \leq d_B(v)(n - 1).$$

Therefore, $d_R(v)(d_B(v) - n + 1) \leq d_B(v)(n - 1)$, which yields

$$d_R(v)d_B(v) \leq (n - 1)(4n - 3). \quad (8.15)$$

For any vertex set $U \subseteq V$, consider the sum $\sum_{x,y \in U, x \neq y} |N_B(x) \cap N_B(y)|$, in which each vertex $z \in V$ is counted $\binom{|N_B(z) \cap U|}{2}$ times, thus we have

$$\sum_{z \in V} \binom{|N_B(z) \cap U|}{2} \leq \binom{|U|}{2}(n - 1). \quad (8.16)$$

Since either $d_R(v) \geq 2n - 1$ or $d_B(v) \geq 2n - 1$, we suppose that $d_R(v) \geq 2n - 1$ without loss of generality.

Case 1. $d_R(v) = 2n - 1$. Then $d_B(v) = 2n - 2$ so that (8.15) is violated.

Case 2. $d_R(v) \geq 2n$. Take a subset $U \subseteq N_R(v)$ with $|U| = 2n$. Then for any $z \in U$, $|N_B(z) \cap U| \geq n$ since otherwise $|N_R(z) \cap U| \geq n$ hence z and v , together with their common red neighbors, would form a red $K_{2,n}$. Similarly, for any $z \in V \setminus U$, $|N_B(z) \cap U| \geq n + 1$. So the left hand side of (8.16) is at least $n(n - 1)(2n + 1)$. But the right hand side is just $n(n - 1)(2n - 1)$, which is a contradiction, proving the upper bound.

If $q = 4n - 3 = 4(n - 1) + 1$ is a prime power, then the Paley graph of order q constructed in Chapter 2 is a strongly regular graph with parameters $(4n - 3, 2n - 2, n - 2, n - 1)$. This graph and its complement do not contain

$K_{2,n}$ since it is self-complementary. Thus $r(K_{2,n}) \geq 2n - 2$, completing the proof. $\square$

It would be interesting to look back the process of studying the behavior of $r(K_{m,n})$ for fixed m , which may offer an epitome of solving a class of Ramsey problems.

For $m = 1$, the proof in Chapter 2 is easy, and we get a formula for all $r(K_{1,n})$, which is $2n$ for odd n and $2n - 1$ for even n .

For $m = 2$, the proof is less easy. We use some technique to obtain an upper bound and use Paley graph to obtain a lower bound for some n . We get a formula $4n - 2$ just for infinitely many n .

For general m , the proof is not trivial. We apply a result on Turán number of $K_{m,n}$ to obtain an upper bound and use probabilistic method to obtain lower bound which is non-constructive. We get only an asymptotic formula $2^m n$ as $n \rightarrow \infty$.

A natural problem on $r_k(B_{m,n})$ related to results in this chapter is as follows, which is easier than the conjecture of Thomason mentioned in Chapter 2.

Problem 8.2 *Prove or disprove $r_k(B_{m,n}) \sim k^m n$ as $n \rightarrow \infty$ for fixed m and k . What can we say for the case $k \rightarrow \infty$ if m and n are fixed?*

The second problem may be very hard. We will see how far one is away from determining the order of $r_k(K_3)$ as $k \rightarrow \infty$.

Exercises

8.1 Let G and H be graphs of order n . Prove that G contains a subgraph with at least $e(G)e(H)/\binom{n}{2}$ edges that is isomorphic to a subgraph of H . (See proof of Theorem 8.8.)

8.2 Let G be a graph with n vertices and m edges. Considering random bipartition of $V(G)$ of sizes $\lfloor n/2 \rfloor$ and $\lceil n/2 \rceil$, show G has a bipartite graph with at least $2m \lfloor n^2/4 \rfloor / n(n-1)$ edges.

8.3 Complete the proof of Theorem 8.14. Where is the incompleteness of graph H used?

8.4 Show that if G is a bipartite d -regular graph of order n containing neither C_4 nor C_6 , then $d \leq n^{1/3} + 1$.

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Chapter 9

Constructions Forbidding Bipartite Graph

9.1 Constructions Forbidding C_4

The starting point of problem involving complete bipartite graph is usually $K_{2,2} = C_4$. We begin with a construction of Erdős-Rényi (1962) and Brown(1966), which contains no C_4 .

Let $F = F(q)$ be the Galois field with q elements. Define an equivalence relation $\equiv$ on $F^3 \setminus \{(0, 0, 0)\}$ by letting $(a_1, a_2, a_3) \equiv (b_1, b_2, b_3)$ if there is $\lambda \in F^* = F \setminus \{0\}$ such that $(a_1, a_2, a_3) = \lambda(b_1, b_2, b_3)$. Let $\langle a_1, a_2, a_3 \rangle$ denote the equivalence class of (a_1, a_2, a_3) , and let V be the set of all equivalence classes.

Define a graph G_q with vertex set V by letting distinct vertices $\langle a_1, a_2, a_3 \rangle$ and $\langle x_1, x_2, x_3 \rangle$ be adjacent if

$$a_1x_1 + a_2x_2 + a_3x_3 = 0.$$

Clearly,

$$|V| = \frac{q^3 - 1}{q - 1} = q^2 + q + 1.$$

For a vertex $\langle a_1, a_2, a_3 \rangle$, since $a_1x_1 + a_2x_2 + a_3x_3 = 0$ has $q^2 - 1$ solutions forming $q + 1$ vertices, it has degree q if $a_1^2 + a_2^2 + a_3^2 = 0$, or degree $q + 1$ otherwise.

Lemma 9.1 *Let $q = p^m$ with prime p and odd m . Then there are precisely*

q^2 solutions (x_1, x_2, x_3) of the equation

$$x_1^2 + x_2^2 + x_3^2 = 0$$

in $F(q)$, and hence precisely $q+1$ vertices in G_q of degree q and q^2 vertices of degree $q+1$.

Proof. Label the vertex set of G_q as

$$V(G_q) = \{A, B, \dots, X, \dots, Y, \dots, Z\}$$

in some order. We write $X \perp Y$ if and only if

$$x_1y_1 + x_2y_2 + x_3y_3 = 0,$$

where $X = \langle x_1, x_2, x_3 \rangle$ and $Y = \langle y_1, y_2, y_3 \rangle$. Let $n = q^2 + q + 1$ and define an $n \times n$ real matrix $M = (m_{ij})$ by

$$m_{ij} = \begin{cases} 1 & \text{if } X \perp Y, \\ 0 & \text{otherwise,} \end{cases}$$

where X and Y represent the i th vertex and the j th vertex, and X and Y are associated with the i th row and j th row in M , respectively. Note the diagonal elements of M are different from that in the adjacency matrix of G_q . We have $m_{ii} = 1$ if $X \perp X$, that is, X lies on the conic $x_1^2 + x_2^2 + x_3^2 = 0$. All that remains to show is that

$$\text{tr}(M) = q + 1,$$

where $\text{tr}(M) = \sum_{i=1}^n m_{ii}$ is the trace of M . We know that the trace equals the sum of eigenvalues.

Fact 1. Any row of M contains precisely $q+1$ ones hence $q+1$ is an eigenvalue of M .

This is because $ML = (q+1)L$, where $L = (1, 1, \dots, 1)^T$.

Fact 2. For $i \neq j$, there is exactly one column with 1 in the both i th row and j th row.

Suppose that X and Y represent i th row and j th row, respectively. Then there is a unique row, say k th row, corresponding to the solution (w_1, w_2, w_3) of the equation system

$$\begin{cases} x_1w_1 + x_2w_2 + x_3w_3 & = & 0, \\ y_1w_1 + y_2w_2 + y_3w_3 & = & 0. \end{cases}$$

That is to say, $m_{ik} = m_{jk} = 1$. Note that M is symmetric, so we see that only in k th column, the elements in both i th row and j th row are 1.

Using these two facts, we have

$$M^2 = \begin{pmatrix} q+1 & 1 & \cdots & 1 & 1 \\ 1 & q+1 & \cdots & 1 & 1 \\ \vdots & & & & \vdots \\ 1 & 1 & \cdots & 1 & q+1 \end{pmatrix} = qI + J,$$

where I is the identity matrix and J is the all-ones-matrix. It is easy to see J has the eigenvalues $n = q^2 + q + 1$ (of multiplicity 1) and 0 (of multiplicity $n - 1 = q^2 + q$). This follows by that M^2 has the eigenvalues $q + n = (q + 1)^2$ (of multiplicity 1) and q (of multiplicity $n - 1 = q^2 + q$).

Since M is symmetric and hence diagonalizable into

$$\begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix},$$

where $\lambda_1, \dots, \lambda_n$ are eigenvalues of M . This implies that M^2 is diagonalizable into

$$\begin{pmatrix} \lambda_1^2 & & & \\ & \lambda_2^2 & & \\ & & \ddots & \\ & & & \lambda_n^2 \end{pmatrix},$$

implying that $\lambda_1^2 = (q + 1)^2$ and $\lambda_2^2 = \dots = \lambda_n^2 = q$, in which the eigenvalues may be relabelled. So $\lambda_1 = q + 1$ as $q + 1$ is an eigenvalue of M , and $\lambda_i = \pm\sqrt{q}$ for $i = 2, \dots, n$. Let s and t be the numbers of eigenvalues of M equal to $\sqrt{q}$ and $-\sqrt{q}$, respectively. Then $s + t = n - 1$ and

$$\text{tr}(M) = (q + 1) + (s - t)\sqrt{q}.$$

Since the trace is an integer and $q = p^m$ with m odd, we must have $s = t$ hence $\text{tr}(M) = q + 1$, completing the proof. $\square$

Remark. In fact for any prime power q , the assertion in above lemma holds.

Lemma 9.2 *The order of the graph G_q is $q^2 + q + 1$ and*

$$e(G_q) = \frac{1}{2}q(q + 1)^2$$

for odd prime power q .

We now come to the point to see that G_q contains no C_4 .

Theorem 9.1 *The graph G_q contains no C_4 .*

Proof. Suppose to the contrary that G_q contains C_4 , then there are two distinct vertices $\langle a_1, a_2, a_3 \rangle$ and $\langle b_1, b_2, b_3 \rangle$ such that

$$\begin{cases} a_1x_1 + a_2x_2 + a_3x_3 &= 0 \\ b_1x_1 + b_2x_2 + b_3x_3 &= 0 \end{cases}$$

has nonzero solutions that form two distinct vertices. This is impossible as (a_1, a_2, a_3) and (b_1, b_2, b_3) are linearly independent that the above equation system has exactly $q - 1$ solutions forming only one vertex. $\square$

Denote by $n = q^2 + q + 1$. Then $q = (\sqrt{4n - 3} - 1)/2$. Thus for $n = q^2 + q + 1$,

$$ex(n; C_4) \geq \frac{1}{4}(n - 1)(1 + \sqrt{4n - 3}).$$

This is very close to the upper bound $ex(n; C_4) \leq \frac{1}{4}n(1 + \sqrt{4n - 3})$ obtained in last chapter. We will show the lower bound gives the quality for infinite many $n = q^2 + q + 1$ in next section. Before this, we shall obtain an asymptotic formula of $ex(n; C_4)$.

Theorem 9.2 *As $n \rightarrow \infty$,*

$$ex(n; C_4) \sim \frac{1}{2}n^{3/2}.$$

Proof. The desired upper bound has been proved in the last chapter. For any large n , let p_k be the maximum prime satisfying that

$$p_k^2 + p_k + 1 \leq n < p_{k+1}^2 + p_{k+1} + 1.$$

Then $p_k \sim \sqrt{n}$ since $p_k^2 \sim p_{k+1}^2$ from a result in Chapter 2. So

$$ex(n; C_4) \geq ex(p_k^2 + p_k + 1; C_4) \geq \frac{1}{2}p_k^3 \sim \frac{1}{2}n^{3/2}$$

by the construction mentioned. The required inequality follows. $\square$

Lemma 9.3 *All vertices of degree q in graph G_q are independent.*

Proof. Suppose that $\langle a_1, a_2, a_3 \rangle$ and $\langle b_1, b_2, b_3 \rangle$ are distinct vertices of degree q in G . Then the vectors (a_1, a_2, a_3) and (b_1, b_2, b_3) are linearly independent in F^3 . Therefore, the dimension of the subspace S of F^3 consisting of all solutions (x_1, x_2, x_3) of the equation system

$$\begin{cases} a_1x_1 + a_2x_2 + a_3x_3 &= 0 \\ b_1x_1 + b_2x_2 + b_3x_3 &= 0 \end{cases}$$

is one. If $\langle a_1, a_2, a_3 \rangle$ and $\langle b_1, b_2, b_3 \rangle$ are adjacent, then both (a_1, a_2, a_3) and (b_1, b_2, b_3) would be in S , a contradiction. $\square$

Since there are exactly $q + 1$ vertices of degree q in G_q , these vertices form an independent set, so $\alpha(G_q) \geq q + 1$. Using the main result in Chapter 3, one has $\alpha(G_q) \geq (1 - o(1))q \log q$ as $q \rightarrow \infty$. It is interesting to determine whether this is an asymptotical equality.

Among all graphs of the same order that contain no C_4 , the extremal graphs for $ex(n; C_4)$ have the largest average degree. So they are expected to give good lower bounds for $r(C_4, K_{1,n})$ if they are near regular. We now apply the above constructions to give such Ramsey lower bounds. The following results are due to Parsons (1975).

Lemma 9.4 *Let q be a prime power, then*

$$\begin{aligned} r(C_4, K_{1,q^2}) &\geq q^2 + q + 1, \\ r(C_4, K_{1,q^2+1}) &\geq q^2 + q + 2. \end{aligned}$$

Proof. Denote by G_q the graph we just constructed. Then G_q has $q^2 + q + 1$ vertices. It contains no C_4 and its minimum degree is q , so the maximum degree of $\overline{G}_q$ is q^2 . This proves the second lower bound in the lemma.

For the first lower bound, let v be a vertex of degree q in the graph G_q , by Lemma 9.3, each neighbor of v has degree $q + 1$. Deleting the vertex v , we obtain a graph with $q^2 + q$ vertices and minimum degree q , so its complement has a maximum degree $q^2 - 1$. This proves the first lower bound. $\square$

Lemma 9.5 *Let $n \geq 2$ be an integer. Then*

$$r(C_4, K_{1,n}) \leq n + \sqrt{n-1} + 2.$$

Proof. Let $N = r(C_4, K_{1,n}) - 1$ and let G be a graph of order N that contains no C_4 and its complement $\overline{G}$ has the maximum degree at most $n - 1$. Since it is easy to see that $r(C_4, K_{1,2}) = 4$ and $r(C_4, K_{1,3}) = 6$, so we suppose that $n \geq 3$ and $N \geq 5$. The fact $\Delta(\overline{G}) \leq n - 1$ implies that for

any vertex v of G , $\deg(v) \geq N - n$. Since the fact $G \not\supseteq C_4$ implies that for any distinct vertices u and v of G , $|N(u) \cap N(v)| \leq 1$, we thus have

$$\sum_{w \in V(G)} \binom{\deg(w)}{2} \leq \binom{N}{2}. \quad (9.1)$$

This and $\deg(w) \geq N - n$ give

$$N^2 - 2(n+1)N + n^2 + n + 1 \leq 0. \quad (9.2)$$

The above inequality becomes an equality if and only if G is regular with degree $N - n$ and any pair of vertices have exactly one common neighbor. We now need the following result known as “the Friendship Theorem” of Erdős, Rényi and Sós (1966). We shall attach a proof as the last section of this chapter. Call the graph $F_k = K_1 + kK_2$ as *Friendship graph* or a k -*fan*, which consists of k triangles with a vertex in common.

Lemma 9.6 (Friendship Theorem) *Let G be a graph with N vertices such that any pair of vertices is joined by exactly one path of length two in G . Then $N = 2k + 1$ and $G = F_k$.*

Return to our current proof. If the inequality (9.1) is an equality, then by Lemma 9.6, $G = F_n$, but G is regular, contradicting to $n \geq 3$. So this is not the case hence the inequality (9.2) is strict, thus

$$N^2 - 2(n+1)N + n^2 + n + 2 \leq 0,$$

yielding $r(C_4, K_{1,n}) = N + 1 \leq n + \sqrt{n-1} + 2$. $\square$

Theorem 9.3 *Let q be a prime power, then*

$$r(C_4, K_{1,q^2}) = q^2 + q + 1.$$

$\square$

Remark. A careful analysis shows $r(C_4, K_{1,q^2+1}) = q^2 + q + 2$, so both lower bounds in Lemma 9.4 are equalities.

The following construction of Lazebnik and Woldar (2000) yields a lower bound for $r_k(C_4)$.

Theorem 9.4 *Let k be a prime power. Then*

$$r_k(C_4) \geq k^2 + 2.$$

Proof. Let $F(k)$ be the field of k elements and let $V = F(k) \times F(k)$. We shall color the edges of K_{k^2+1} on vertex set $V \cup \{u\}$ with k colors so that there is no monochromatic C_4 . We will do so for all edges in V first. Let e be an edge with end vertices (a_1, a_2) and (b_1, b_2) in V . We assign e with color

$$\alpha = a_1b_1 + a_2 + b_2.$$

We claim that there is no monochromatic C_4 . Since otherwise, suppose that (a_1, a_2) , (b_1, b_2) , (c_1, c_2) , and (d_1, d_2) are four consecutive (distinct) vertices of a C_4 in some color α . then

$$\begin{aligned} \alpha &= a_1b_1 + a_2 + b_2 = b_1c_1 + b_2 + c_2 \\ &= c_1d_1 + c_2 + d_2 = d_1a_1 + d_2 + a_2, \end{aligned}$$

yielding $(a_1 - c_1)(b_1 - d_1) = 0$. So either $a_1 = c_1$ or $b_1 = d_1$, which imply either $(a_1, a_2) = (c_1, c_2)$ or $(b_1, b_2) = (d_1, d_2)$, a contradiction.

It remains to color the edges of form $\{u, v\}$ with $v = (a_1, a_2) \in V$. For such an edge, we assign it with color a_1 , the first coordinate of v . Suppose that there is a monochromatic C_4 in color α , which must contain the vertex u . Let (α, a_2) , (b_1, b_2) and (α, c_2) be the other three vertices in the C_4 , so

$$\alpha = \alpha b_1 + a_2 + b_2 = \alpha b_1 + b_2 + c_2.$$

Thus $a_2 = c_2$, which implies that the vertices (α, a_2) and (α, c_2) are identical, a contradiction. This completes the proof. $\square$

Combining the upper bound on $r_k(K_{t,s})$ in the last chapter and what in above theorem, we obtain the asymptotic formula of $r_k(C_4)$ as follows.

Theorem 9.5 As $k \rightarrow \infty$, $r_k(C_4) \sim k^2$.

9.2 Exact Values of $ex(n; C_4)$

We have obtained that $ex(n; C_4) \sim n^{3/2}/2$. It is hopeless to find all exact values of $ex(n; C_4)$ because of the difficulty of constructions for the lower bounds and estimating for exact upper bounds. However, it is possible to be lucky to find such values for some special values of n .

Recall the graph G_q constructed by Erdős and Rényi in last section that has $n = q^2 + q + 1$ vertices and $q(q+1)^2/2$ edges. Each vertex v of G_q has

degree q or $q + 1$. There are exactly $q + 1$ vertices of degree q , and exactly q^2 vertices of degree $q + 1$.

The graph G_q and the upper bound obtained in last chapter give

$$\frac{1}{4}(n-1)(1+\sqrt{4n-3}) \leq ex(n; C_4) \leq \frac{1}{4}n(1+\sqrt{4n-3})$$

with $n = q^2 + q + 1$. That is

$$\frac{1}{2}q(q+1)^2 \leq ex(n; C_4) \leq \frac{1}{2}(q^2 + q + 1)(q + 1).$$

Erdős, Rényi and Sós (1966) proved that the second inequality is strict, and Erdős (1966) conjectured that the first inequality is an equality. This conjecture was confirmed by Füredi in 1996, who obtained a partial answer for $q = 2^k$ in 1983.

Theorem 9.6 *Let $q > 13$ be a prime power, and let $n = q^2 + q + 1$, then*

$$ex(n; C_4) = \frac{1}{2}q(q+1)^2.$$

To avoid some burden with extra background for the proof, we only introduce the proof of Füredi for the special case where q is a power of 2.

Lemma 9.7 *Let $q \geq 2$ be an even integer and let G be a C_4 -free graph of order $q^2 + q + 1$ with $\Delta(G) \leq q + 1$. Then any vertex of degree $q + 1$ is adjacent to at least one vertex of degree q or less. Consequently, G has at least $q + 1$ vertices of degree q or less.*

Proof. Let $v \in V(G)$ be a vertex with $\deg(v) = q + 1$. In the subgraph of G induced by $N(v)$, there is no vertex with degree two or more, thus

$$|E(N(v))| \leq \left\lfloor \frac{q+1}{2} \right\rfloor = \frac{q}{2}$$

since q is even. Noticing that any pair of vertices in $N(v)$ have the unique neighbor in common, namely $\{v\}$, and $V(G) \setminus (N(v) \cup \{v\})$ contains $q^2 - 1$ vertices, so we have

$$\begin{aligned} \sum_{x \in N(v)} \deg(x) &\leq \deg(v) + 2|E(N(v))| + |V(G) \setminus (N(v) \cup \{v\})| \\ &\leq (q+1) + q + (q^2 - 1) = (q+1)^2 - 1. \end{aligned}$$

Thus there exists at least one vertex $x \in N(v)$ with $\deg(x) \leq q$ as claimed. Set $S = \{x \in V(G) : \deg(x) \leq q\}$. Then any vertex $x \in S$ can be a neighbor of at most q vertices. From what has just proved, we get

$$\bigcup_{x \in S} (N(x) \cup \{x\}) = V(G),$$

thus $(q+1)|S| \geq q^2 + q + 1$, which implies that $|S| \geq q+1$. $\square$

Lemma 9.8 *Let G be a C_4 -free graph on $q^2 + q + 1$ vertices with $\Delta(G) \geq q+2$, where q is an even integer. Then $e(G) \leq q(q+1)^2/2$.*

Proof. Set

$$V = V(G) = \{v_1, v_2, \dots, v_n\},$$

where $n = q^2 + q + 1$, and $\deg(v_1) = \Delta = \Delta(G) \geq q+2$. Since G contains no C_4 , so

$$|N(v_i) \setminus N(v_1)| \geq |N(v_i)| - 1 = \deg(v_i) - 1$$

for $2 \leq i \leq n$. Then

$$\begin{aligned} \binom{n-\Delta}{2} &\geq (\text{number of pairs of } V \setminus N(v_1)) \\ &\geq \sum_{2 \leq i \leq n} (\text{number of pairs of } N(v_i) \cap (V \setminus N(v_1))) \\ &\geq \sum_{2 \leq i \leq n} \binom{\deg(v_i) - 1}{2}. \end{aligned}$$

Suppose to the contrary, $e(G) > q(q+1)^2/2$. Since $q(q+1)^2$ is an integer thus $e(G) \geq q(q+1)^2/2 + 1$, that is, $2e(G) \geq (n-1)(q+1) + 2$. Using the Jensen inequality, we have

$$\begin{aligned} \binom{n-\Delta}{2} &\geq (n-1) \binom{\sum_{2 \leq i \leq n} (\deg(v_i) - 1) / (n-1)}{2} \\ &= (n-1) \binom{[2e(G) - (n-1) - \Delta] / (n-1)}{2} \\ &\geq (n-1) \binom{[(n-1)q + 2 - \Delta] / (n-1)}{2}. \end{aligned}$$

This yields

$$(n-1)(n-\Delta)(n-\Delta-1) \geq [(n-1)q + 2 - \Delta][(n-1)(q-1) + 2 - \Delta]. \quad (9.3)$$

However, the fact $\Delta q > (q+2)q - 1 = n + q - 2$ for $\Delta \geq q+2$ implies

$$(n-1)q + 2 - \Delta > (q+1)(n-\Delta), \quad (9.4)$$

and the fact $\Delta(q-1) \geq (q+2)(q-1) = n-3$ implies

$$(n-1)(q-1) + 2 - \Delta \geq q(n-\Delta-1). \quad (9.5)$$

Multiplying the expressions on the left-hand sides and on the right-hand sides, respectively, we have an inequality contradicting to (9.3). $\square$

Proof of Theorem 9.6 for case $q = 2^k$. The graph G_q gives that $ex(n; C_4) \geq q(q+1)^2/2$. On the other hand, if G is a C_4 -free graph of order $n = q^2 + q + 1$ with q a power of 2, by Lemma 9.8, we may assume that $\Delta(G) \leq q+1$. By Lemma 9.7, there are at least $q+1$ vertices of degree q or less. Hence

$$2e(G) \leq (q+1)n - (q+1) = q(q+1)^2,$$

the desired equality follows immediately. $\square$

9.3 Constructions with Forbidden $K_{2,s+1}$

The following construction is due to Füredi (1996), and Axenovich, Füredi and Mubayi (2000), which gives a lower bound for $r_k(K_{2,s+1})$ hence that for $ex(n, K_{2,s+1})$. Recall the Siegel-Walfisz theorem in Chapter 2, which ensures that about $1/\phi(s)$ primes have the form $ks+1$ for $k \geq 0$, where $\phi(s)$ is the number of integers in $\{1, 2, \dots, s\}$ that are prime with s .

Theorem 9.7 *For any fixed integer $s \geq 1$, $r_k(K_{2,s+1}) \sim s k^2$ as $k \rightarrow \infty$.*

Proof. The upper bound is in the last Chapter. We need to show that

$$r_k(K_{2,s+1}) \geq (1 - o(1))s k^2$$

as $k \rightarrow \infty$. Let q be a prime power such that $k = (q-1)/s$ is an integer. Set $n = (q-1)^2/s = s k^2$. We will color all edges of K_n with slightly more than k colors such that there is no monochromatic $K_{2,s+1}$.

Let $F = F(q)$ be the q -element finite field, and let $h \in F$ be an element of order s , and

$$H = \{1, h, \dots, h^{s-1}\}.$$

Denote the cosets of H by

$$H_1, H_2, \dots, H_k,$$

which partition $F^* = F \setminus \{0\}$. We introduce an equivalence relation " $\equiv$ " in $F^* \times F^*$ as $(a_1, a_2) \equiv (x_1, x_2)$ if $(a_1, a_2) = h^t(x_1, x_2)$ for some $0 \leq t \leq s-1$. It is easy to verify that the relation is an equivalence relation. The equivalence class represented by (a_1, a_2) is denoted by $\langle a_1, a_2 \rangle$. Let V be the set of all equivalence classes and $n = |V| = sk^2$. Consider the complete graph K_n with vertex set V . Color the edge joining two vertices $\langle a_1, a_2 \rangle$ and $\langle x_1, x_2 \rangle$ with color i if $a_1x_1 + a_2x_2 \neq 0$ and $a_1x_1 + a_2x_2 \in H_i$. Clearly the definition for the coloring is compatible with the equivalence class, that is to say, $a_1x_1 + a_2x_2 \in H_i$, $(a_1, a_2) \equiv (b_1, b_2)$ and $(x_1, x_2) \equiv (y_1, y_2)$ imply $b_1y_1 + b_2y_2 \in H_i$. Note that the edges of form $\{\langle a_1, a_2 \rangle, \langle x_1, x_2 \rangle\}$ with $a_1x_1 + a_2x_2 = 0$ are still uncolored.

Let G_i denote the graph induced by all edges in color i . We shall show that G_i contains no $K_{2,s+1}$. Let $\langle a_1, a_2 \rangle$ and $\langle b_1, b_2 \rangle$ be a pair of distinct vertices. Consider the equation system

$$\begin{cases} a_1x_1 + a_2x_2 &= u \\ b_1x_1 + b_2x_2 &= v. \end{cases} \quad (9.6)$$

We claim the system has at most one solution (x_1, x_2) for every $u, v \in H_i$. Indeed, the solution is unique if the determinant of the system is not 0. Otherwise, there exists $\lambda \in F^*$ such that $(b_1, b_2) = \lambda(a_1, a_2)$. If the system (9.6) has solution (x_1, x_2) , then $\lambda u = v$ hence $\lambda = v/u \in H$, contradicting to the fact that (a_1, a_2) and (b_1, b_2) are not equivalent. Finally, there are s^2 possibilities for $u, v \in H_i$ in (9.6). The set of solutions form s equivalence classes hence s vertices. So there are at most s vertices $\langle x_1, x_2 \rangle$ joined simultaneously to $\langle a_1, a_2 \rangle$ and $\langle b_1, b_2 \rangle$.

Now turn to the uncolored edges $\{\langle a_1, a_2 \rangle, \langle b_1, b_2 \rangle\}$ with $a_1b_1 + a_2b_2 = 0$. Let G_0 be the graph induced by these edges. We are going to color the edges of G_0 by an additional $O(\sqrt{k})$ colors. Since the equation

$$a_1x_1 + a_2x_2 = 0$$

has $q-1$ solutions forming $k = (q-1)/s$ equivalence classes hence k vertices, thus vertex $\langle a_1, a_2 \rangle$ of G_0 has degree k if $a_1^2 + a_2^2 \neq 0$ and degree $k-1$ if $a_1^2 + a_2^2 = 0$. Let $\{\langle a_1, a_2 \rangle, \langle b_1, b_2 \rangle\}$ be an edge of G_0 , then $a_1b_1 + a_2b_2 = 0$. Set

$$V_1 = \{\langle x_1, x_2 \rangle : a_1x_1 + a_2x_2 = 0\}$$

and

$$V_2 = \{\langle y_1, y_2 \rangle : b_1y_1 + b_2y_2 = 0\}.$$

Then $\langle a_1, a_2 \rangle \in V_2$ and $\langle b_1, b_2 \rangle \in V_1$, and $|V_1| = |V_2| = k$. For any $\langle x_1, x_2 \rangle \in V_1$ and $\langle y_1, y_2 \rangle \in V_2$, we have

$$\begin{aligned} x_1 y_1 + x_2 y_2 &= x_1 y_2 \left(\frac{y_1}{y_2} + \frac{x_2}{x_1} \right) \\ &= x_1 y_2 \left(-\frac{b_2}{b_1} - \frac{a_1}{a_2} \right) \\ &= -\frac{x_1 y_2}{a_2 b_2} (a_1 b_1 + a_2 b_2) \\ &= 0. \end{aligned}$$

So V_1 and V_2 are completely connected in G_0 . If $V_1 \cap V_2 = \emptyset$, they induce a complete bipartite graph $K_{k,k}$ that is not connecting to any other vertex of G_0 since the maximum degree of G_0 is k . If $V_1 \cap V_2 \neq \emptyset$, $\langle z_1, z_2 \rangle \in V_1 \cap V_2$, then we see that $z_1^2 + z_2^2 = 0$ and the degree of $\langle z_1, z_2 \rangle$ in G_0 is $k - 1$. Since it is adjacent to all other vertices in V_1 and V_2 , so $V_1 = V_2$ and it induces a complete graph K_k . As any vertex in $V_1 = V_2$ has degree $k - 1$, so any vertex in this complete graph is not connecting to any other vertex of G_0 . Therefore, G_0 is a union of $K_{k,k}$ and K_k .

From Theorem 9.5, $r_k(C_4) \sim k^2$, we can color the edges of K_{2k} with at most $(1 + o(1))\sqrt{2k} < \sqrt{3k}$ colors such that there is no monochromatic C_4 hence no monochromatic $K_{2,s+1}$. Let p_k be the k th prime such that $(p_k - 1)/s$ is an integer, and let $n_k = (p_k - 1)/s + \lfloor 3\sqrt{(p_k - 1)/s} \rfloor$, then by what has been proved, $r_{n_k}(K_{2,s+1}) \geq (1 - o(1))s n_k^2$. For any n with $n_k \leq n < n_{k+1}$, then $n \sim n_k$ as $n \rightarrow \infty$, thus

$$r_n(K_{2,s+1}) \geq r_{n_k}(K_{2,s+1}) \geq (1 - o(1))s n_k^2 = (1 - o(1))s n^2,$$

as desired. $\square$

By counting the edges in a single color, say, that in G_1 , we obtain the asymptotic formula of $ex(n; K_{2,s+1})$ as follows.

Corollary 9.1 *For any fixed integer $s \geq 1$,*

$$ex(n; K_{2,s+1}) \sim \frac{1}{2}\sqrt{s}n^{3/2}$$

as $n \rightarrow \infty$.

9.4 Orders of $ex(n; C_{2m})$ for $m = 2, 3, 5$

In the last chapter, we proved that for $m \geq 2$

$$ex(n; C_{2m}) \leq 10mn^{1+1/m}.$$

This upper bound has the right order for $m = 2$. We shall prove that its order is also right for $m = 3$ and $m = 5$.

Theorem 9.8 *There exist some constant $c = c(m) > 0$ such that*

$$ex(n; C_{2m}) \geq cn^{1+1/m}$$

for $m = 2, 3, 5$.

The following constructions for the desired lower bounds is due to Wenger (1991). The same lower bound for $ex(n; C_6)$ has obtained by Benson (1966). Let p be a prime number. Construct a bipartite graph $H_m(p)$ as follows on vertex classes X and Y , where both X and Y are copies of $F^m(p)$. Thus $|X| = |Y| = p^m$. For a vertex $A \in X$ and a vertex $B \in Y$ with

$$A = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix},$$

they are adjacent if

$$B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_{m-1} \\ b_m \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_{m-1} \\ 0 \end{pmatrix} + b_m \begin{pmatrix} a_2 \\ a_3 \\ \vdots \\ a_m \\ 1 \end{pmatrix}.$$

For every $A \in X$, the value b_m uniquely determines a neighbor of A , so each vertex in X has degree p . Hence $H_m(p)$ has $2p^m$ vertices and at least p^{m+1} edges.

Remark The construction of Lazebnik and Woldar in Section 1 has some similarity with the above construction for $m = 2$.

Before giving some properties of $H_m(p)$, let define generalized *Vandermonde matrix* on $F(p)$ as follows. For $m \geq k \geq 2$ and $a_i \in F(p)$, set

$$M_m = M_m(a_1, a_2, \dots, a_k) = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ a_1 & a_2 & \cdots & a_k \\ \vdots & \vdots & \ddots & \vdots \\ a_1^{m-1} & a_2^{m-1} & \cdots & a_k^{m-1} \end{pmatrix}$$

as an $m \times k$ matrix on $F(p)$. Note that M_k is the commonly defined the Vandermonde matrix and

$$\det(M_k) = \prod_{1 \leq i < j \leq k} (a_j - a_i).$$

So M_k is singular if and only if there are some $i \neq j$ such that $a_i = a_j$. However, we need a more specific property of the Vandermonde matrix.

Lemma 9.9 *If $p \geq m \geq k \geq 2$ and if the i th column of the Vandermonde matrix M_m is a linear combination of the others, then there exists $j \neq i$ such that $a_i = a_j$.*

Proof. Without loss of generality, we assume that the last column is a linear combination of the other columns.

Case 1. $a_1, a_2, \dots, a_{k-1}$ are pairwise distinct. Define a polynomial $P(x)$ of degree $k-1$ as

$$P(x) = \det(M_k(a_1, \dots, a_{k-1}, x)) = c(x - a_1)(x - a_2) \cdots (x - a_{k-1}),$$

where $c = \prod_{1 \leq i < j \leq k-1} (a_j - a_i) \neq 0$. Since k th column is a combination of the first $k-1$ columns, so $M(a_1, \dots, a_{k-1}, a_k)$ is singular hence $P(a_k) = 0$. Thus there is some a_i with $1 \leq i \leq k-1$ such that $a_k = a_i$ as claimed.

Case 2. There exist a_i and a_j with $1 \leq i \neq j \leq k-1$ such that $a_i = a_j$. Without loss of generality, we assume that a_i and a_j are the unique pair which are equal. Thus we can find at least two $a'_j \in F(p)$ such that $a_1, \dots, a_{j-1}, a'_j, a_{j+1}, \dots, a_{k-1}$ are pairwise distinct. It is easy to see that the last column in the following Vandermonde matrix

$$M_m(a_1, \dots, a_{j-1}, a'_j, a_{j+1}, \dots, a_k)$$

is still a linear combination of the other columns. From that proved in Case 1, either $a_k = a_i$ for some $1 \leq i \leq k-1$ or $a_k = a'_j$. The latter is not true since there are different choices for a'_j . $\square$

Lemma 9.10 *Let p be a prime and m and k be integers with $p \geq m \geq k \geq 2$. If $H_m(p)$ contains a cycle $C_{2k} = (A_1, B_1, A_2, B_2, \dots, A_k, B_k)$ with $A_i \in X$ and $B_i \in Y$, then for each B_i , there exists a B_j with $j \neq i$ such that $b_{im} = b_{jm}$, where b_{im} and b_{jm} are the m th (last) coordinates of B_i and B_j , respectively.*

Proof. Let A, B and A' be three consecutive vertices in the cycle C_{2k} with $B \in Y$. By the definition of adjacency we have

$$\begin{pmatrix} a_1 - a'_1 \\ \vdots \\ a_{m-1} - a'_{m-1} \\ 0 \end{pmatrix} = -b_m \begin{pmatrix} a_2 - a'_2 \\ \vdots \\ a_m - a'_m \\ 0 \end{pmatrix},$$

which gives $a_i - a'_i = -b_m(a_{i+1} - a'_{i+1}) = (-b_m)^{m-i}(a_m - a'_m)$ hence

$$\begin{pmatrix} a_1 - a'_1 \\ a_2 - a'_2 \\ \vdots \\ a_{m-1} - a'_{m-1} \\ a_m - a'_m \end{pmatrix} = (a_m - a'_m) \begin{pmatrix} (-b_m)^{m-1} \\ (-b_m)^{m-2} \\ \vdots \\ -b_m \\ 1 \end{pmatrix}.$$

Clearly $a_m \neq a'_m$ since otherwise A and A' are the same vertex. By taking A, B and A' as A_i, B_i and A_{i+1} , respectively, and by writing $x_i = a_{im} - a_{i+1m}$, and $c_i = -b_{im}$, we obtain

$$A_i - A_{i+1} = x_i \begin{pmatrix} c_i^{m-1} \\ c_i^{m-2} \\ \vdots \\ c_i \\ 1 \end{pmatrix},$$

and $x_i \neq 0$. From the trivial fact that $\sum_{i=1}^k (A_i - A_{i+1})$ is a zero vector, where A_{k+1} is A_1 , we have

$$\begin{pmatrix} c_1^{m-1} & c_2^{m-1} & \cdots & c_k^{m-1} \\ c_1^{m-2} & c_2^{m-2} & \cdots & c_k^{m-2} \\ \vdots & \vdots & \ddots & \vdots \\ c_1 & c_2 & \cdots & c_k \\ 1 & 1 & \cdots & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Note that the left-hand side matrix is a Vandermonde matrix, and the i th column is a linear combination of the others since $x_i \neq 0$. Thus $c_i = c_j$ hence $b_{im} = b_{jm}$ for some $j \neq i$ by Lemma 9.9. $\square$

Lemma 9.11 *If two distinct vertices in the same vertex class of $H_m(p)$ have a neighbor in common, then they have different m th (last) coordinates.*

Proof. If two distinct vertices $A, A' \in X$ have a neighbor $B \in Y$ in common, the proof of the Lemma 9.10 gives the assertion. If two distinct vertices $B, B' \in Y$ have a common neighbor $A \in X$, then the assertion can be seen easily by the definition of adjacency, since otherwise B and B' would be the same vertex in Y of $H_m(p)$. $\square$

Corollary 9.2 *Let $p \geq m$ be a prime number. Then $H_m(p)$ contains no C_4 for $m \geq 2$, and $H_m(p)$ contains neither C_4 , nor C_6 for $m \geq 3$; and $H_m(p)$ contains neither C_4 , nor C_6 , nor C_{10} for $m \geq 5$.*

Proof. Suppose $m \geq 2$. If $H_m(p)$ contains a cycle $C_4 = (A_1, B_1, A_2, B_2)$ with $A_i \in X$ and $B_i \in Y$. By Lemma 9.10, $b_{12} = b_{22}$. However, both B_1 and B_2 are adjacent to A_1 , which contradicts to Lemma 9.11.

Suppose $m \geq 3$. If $H_m(p)$ contains a cycle $C_6 = (A_1, B_1, A_2, B_2, A_3, B_3)$ with $A_i \in X$ and $B_i \in Y$. By Lemma 9.10, $b_{13} = b_{23}$ or $b_{13} = b_{33}$. However, B_1 and B_2 have a neighbor A_2 in common, and B_1 and B_3 have a neighbor A_1 in common, which contradict to Lemma 9.11.

Suppose $m \geq 5$. If $H_m(p)$ contains a cycle $C_{10} = (A_1, B_1, \dots, A_5, B_5)$ with $A_i \in X$ and $B_i \in Y$. By Lemma 9.10, there exist three distinct vertices B_i, B_j and B_k such that $b_{i5} = b_{j5} = b_{k5}$. Two of these vertices must be consecutive in the cyclic sequence $B_1, B_2, \dots, B_5, B_1$, so they have a neighbor in common, which contradicts to Lemma 9.11. $\square$

Proof of Theorem 9.8. Note that $H_m(p)$ has $n = 2p^m$ vertices and $(1 - o(1))p^{1+1/m} = (1 - o(1))(n/2)^{1+1/m}$ edges. The proof of Theorem 9.8 is then immediate from the above corollary and a result in last section on the gaps of consecutive prime numbers. $\square$

It is not difficult to find that $H_4(p)$ contains C_8 , and $H_6(p)$ contains C_{12} . However, We believe the answer of the following problem is affirmative.

Problem 9.1 *Prove or disprove that $ex(n; C_{2m}) = \Theta(n^{1+1/m})$ as $m \geq 2$ fixed and $n \rightarrow \infty$.*

9.5 Constructions with Forbidden $K_{3,3}$

We now turn to discuss $ex(n; K_{3,3})$ and $r_k(K_{3,3})$. The following construction is due to Brown (1966).

Let p be an odd prime, and let $\alpha \in F^*(p) = F(p) \setminus \{0\}$ be a fixed quadratic non-residue if $p \equiv 1 \pmod{4}$, and a quadratic residue if $p \equiv 3 \pmod{4}$ in $F(p)$. Set $V = F^3(p)$. We shall call $x, a \in V$ as points or vectors and write them as $x = (x_1, x_2, x_3)$ and $a = (a_1, a_2, a_3)$. For $x \in V$, define

$$N(x) = \{y \in V : \sum_{i=1}^3 (x_i - y_i)^2 = \alpha\}.$$

Note that $y \in N(x)$ if and only if $x \in N(y)$. We construct a graph G on vertex set V by connecting a pair of vectors x and y if and only if $y \in N(x)$. The degree of each vertex x is $p^2 - p$ by admitting without proof a theorem of Lebesgue that $N(x)$ contains $p^2 - p$ vertices. So the order of G is p^3 and the number of edges of G is $(p^5 - p^4)/2$.

We shall prove that G contains no $K_{3,3}$. Suppose to the contrary, G contain a $K_{3,3}$ with bipartition a, a', a'' and b, b', b'' . Thus $w = b, b', b''$ are three solutions of the equations

$$\sum_{i=1}^3 (a_i - w_i)^2 = \sum_{i=1}^3 (a'_i - w_i)^2 = \sum_{i=1}^3 (a''_i - w_i)^2 = \alpha,$$

which yields

$$\begin{cases} 2(a_1 - a'_1)w_1 & +2(a_2 - a'_2)w_2 & +2(a_3 - a'_3)w_3 & = \sum (a_i^2 - a_i'^2) \\ 2(a'_1 - a''_1)w_1 & +2(a'_2 - a''_2)w_2 & +2(a'_3 - a''_3)w_3 & = \sum (a_i'^2 - a_i''^2) \\ 2(a''_1 - a_1)w_1 & +2(a''_2 - a_2)w_2 & +2(a''_3 - a_3)w_3 & = \sum (a_i''^2 - a_i^2) \end{cases}$$

Set a matrix A as

$$A = \begin{pmatrix} a_1 - a'_1 & a_2 - a'_2 & a_3 - a'_3 \\ a'_1 - a''_1 & a'_2 - a''_2 & a'_3 - a''_3 \\ a''_1 - a_1 & a''_2 - a_2 & a''_3 - a_3 \end{pmatrix} = \begin{pmatrix} a - a' \\ a' - a'' \\ a'' - a \end{pmatrix},$$

then we have

$$2A \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} \sum (a_i^2 - a_i'^2) \\ \sum (a_i'^2 - a_i''^2) \\ \sum (a_i''^2 - a_i^2) \end{pmatrix}. \quad (9.7)$$

Clearly, the rank of the matrix A is one or two.

For fixed vectors $c, d \in V = F^3(p)$ with d is not a zero vector, we shall call the set of points

$$\{c + td : t \in F(p)\}$$

as a line of V . In particular, when c is a zero vector, the line passes through the origin.

Lemma 9.12 *Let S be a subset of V with at least two points. Then the points in S are collinear if and only if there is some $c \in V$, such that the rank of $\{x - c : x \in S\}$ is one; that is to say, $\{x - c : x \in S\}$ are collinear passing through the origin.*

Claim 1 *If the rank of A is one, then vectors a, a', a'' are collinear, and if the rank of A is two, then b, b', b'' are collinear.*

If fact, if the rank of A is one,

$$\text{rank} \begin{pmatrix} a - a' \\ a' - a'' \\ a'' - a \end{pmatrix} = \text{rank} \begin{pmatrix} a - a' \\ a' - a'' \\ 0 \end{pmatrix} = \text{rank} \begin{pmatrix} a - a'' \\ a' - a'' \\ 0 \end{pmatrix} = 1.$$

So the points a, a', a'' are collinear from Lemma 9.12 (with $c = a''$). Suppose that the rank of A is two, and suppose that c is a solution of equation (9.7). Then $b - c, b' - c, b'' - c$ are solutions of equation $2Aw = 0$, thus the rank of them must be one since the rank of $2A$ is two. So b, b', b'' are collinear from the lemma 9.12.

The fact that G contains no $K_{3,3}$ will follows the following claim.

Claim 2 *No three points in $N(x + c)$ are collinear for any $x, c \in V$.*

In fact, if a line $\{c + td : t \in F(p)\}$ meets $N(x + c)$ at least three points, then the line $\{td : t \in F(p)\}$ meets $N(x)$ at least three points. Thus

$$\sum_{i=1}^3 (x_i - td_i)^2 = \alpha,$$

which yields a quadratic equation in t as

$$\left(\sum d_i^2\right)t^2 - 2\left(\sum d_i x_i\right)t + \sum x_i^2 = \alpha.$$

This equation has three solutions in t only if

$$\sum d_i^2 = 0, \sum d_i x_i = 0, \sum x_i^2 = \alpha.$$

Since $d \neq (0, 0, 0)$, we assume that $d_1 \neq 0$ without loss of generality. Then

$$\begin{aligned} d_1^2 \alpha &= d_1^2 \sum x_i^2 \\ &= (-d_2 x_2 - d_3 x_3)^2 + d_1^2 (x_2^2 + x_3^2) \\ &= (d_2 x_2 + d_3 x_3)^2 - (d_2^2 + d_3^2)(x_2^2 + x_3^2) \\ &= -(d_3 x_2 - d_2 x_3)^2. \end{aligned}$$

This contradicts the choice of α since -1 is a quadratic residue as $p \equiv 1 \pmod{4}$, and a quadratic non-residue otherwise.

Brown did not know that his construction is in fact asymptotically sharp when the paper appeared in 1966. Combining his construction and Füredi's upper bound in 1996 proved in the last chapter, we have an asymptotic formula of $ex(n; K_{3,3})$.

Theorem 9.9 *As $n \rightarrow \infty$, $ex(n; K_{3,3}) \sim \frac{1}{2}n^{5/3}$.*

However, Brown's construction cannot give a good lower bound for $r_k(K_{3,3})$. The following construction is due to Kollár, Rónyai and Szabó (1996), and Alon, Rónyai and Szabó (1999), which yields both asymptotic formulas of $ex(n; K_{3,3})$ and $r_k(K_{3,3})$.

Let q be a prime power. For any $X \in F(q^2)$, set $N(X) = X^{1+q}$. Note that the roots of the polynomial $x^q - x$ are precisely the elements of $F(q)$. Since $X^{q^2} = X$ for $X \in F(q^2)$ and

$$N(X)^q = (X^{1+q})^q = X^q X^{q^2} = X^q X = N(X),$$

we infer that $N(X) \in F(q)$ for any $X \in F(q^2)$. Clearly, N is a multiplicative function: if $A, B \in F(q^2)$, then $N(AB) = N(A)N(B)$. We then define a graph H as follows. The vertex set $V(H)$ is $F(q^2) \times F^*(q)$. Two distinct vertices (A, a) and (B, b) in $V(H)$ are connected if and only if $N(A + B) = ab$. The order of H is $q^3 - q^2$. If (A, a) and (B, b) are adjacent, then (A, a) and $B \neq -A$ determine b . Thus a vertex (A, a) has degree $q^2 - 1$ if $N(2A) \neq a^2$, and degree $q^2 - 2$ otherwise.

Lemma 9.13 *The graph does not contain $K_{3,3}$ as a subgraph.*

Proof. The lemma is a direct consequence of the following statement: if $(D_1, d_1), (D_2, d_2), (D_3, d_3)$ are distinct vertices in $V(H)$, then the system of equations

$$\begin{cases} N(X + D_1) &= x d_1, \\ N(X + D_2) &= x d_2, \\ N(X + D_3) &= x d_3 \end{cases} \quad (9.8)$$

has at most two solutions $(X, x) \in F(q^2) \times F^*(q)$.

If the system (9.8) has one solution (X, x) , then

- $X \neq -D_i$ for any $i = 1, 2, 3$,
- $D_i \neq D_j$ for $i \neq j$.

The former is true, because $xd_i \neq 0$; and so is the latter since if $D_i = D_j$, then $d_i = d_j$ hence $(D_i, d_i) = (D_j, d_j)$. From the system (9.8) and the property that N is multiplicative, we have

$$\begin{cases} N\left(\frac{X+D_1}{X+D_3}\right) &= \frac{d_1}{d_3}, \\ N\left(\frac{X+D_2}{X+D_3}\right) &= \frac{d_2}{d_3}, \end{cases}$$

which yields

$$\begin{cases} N\left(\frac{X+D_1}{(X+D_3)(D_1-D_3)}\right) &= \frac{d_1}{d_3N(D_1-D_3)}, \\ N\left(\frac{X+D_2}{(X+D_3)(D_2-D_3)}\right) &= \frac{d_2}{d_3N(D_2-D_3)}. \end{cases}$$

Denote by b_i for $d_i/(d_3N(D_i-D_3))$, A_i for $1/(D_i-D_3)$, and Y for $1/(X+D_3)$, then

$$\begin{cases} N(Y+A_1) &= b_1, \\ N(Y+A_2) &= b_2. \end{cases}$$

Note that $(A+B)^q = A^q + B^q$, so $N(Y+A_i) = (Y+A_i)(Y^q+A_i^q)$ hence

$$\begin{cases} (Y+A_1)(Y^q+A_1^q) &= b_1, \\ (Y+A_2)(Y^q+A_2^q) &= b_2. \end{cases} \quad (9.9)$$

We now refer unknown Y and A_i and b_i as elements of $F(q^2)$. Consider the system of equations

$$\begin{cases} (x_1 - a_{11})(x_2 - a_{12}) &= b_1, \\ (x_1 - a_{21})(x_2 - a_{22}) &= b_2 \end{cases} \quad (9.10)$$

with $a_{11} \neq a_{21}$ and $a_{12} \neq a_{22}$, where $a_{ij}, b_i \in F(q^2)$. We claim that the system has at most two solutions $(x_1, x_2) \in F^2(q^2)$. In fact, from the system we get

$$(a_{11} - a_{21})x_2 + (a_{12} - a_{22})x_1 + a_{21}a_{22} - a_{11}a_{12} = b_2 - b_1.$$

By expressing x_1 in terms of x_2 and substituting it into one equation in system (9.10), we obtain a quadratic equation in x_2 with a nonzero leading coefficient. This has at most two solutions in x_2 and each determines a unique x_1 . The claim follows.

Setting $x_1 = Y$, $x_2 = Y^q$, we see the system (9.9) has at most two solutions in unknown Y . These solutions are corresponding with the solutions (X, x) of system (9.8), so the proof is finished. $\square$

It is just one more step to obtain an asymptotic formula of $r_k(K_3, 3)$.

Theorem 9.10 As $k \rightarrow \infty$,

$$r_k(K_{3,3}) \sim k^3.$$

Proof. The upper bound $r_k(K_{3,3}) \leq (1 + o(1))k^3$ comes from last chapter. For a lower bound, let q be a prime power. For a complete graph on vertex set $V = F(q^2) \times F^*(q)$, color the edge connecting (A, a) and (B, b) by color $N(A + B)/ab$ if $A + B \neq 0$. It can be verified as the proof for Lemma 9.13 that there is no monochromatic $K_{3,3}$ in edges that have been colored. The uncolored edges form $(q^2 - 1)/2$ pairwise disjoint complete bipartite graphs. In fact, for any fixed $A \in F(q^2)$,

$$V_1 = \{(A, x) : x \in F^*(q)\} \quad \text{and} \quad V_2 = \{(-A, y) : y \in F^*(q)\}$$

form such a complete bipartite graph on $2(q - 1)$ vertices for $A \neq 0$ or a complete graph on $q - 1$ vertices for $A = 0$. Using the same technique recursively, we can color the edges of each such bipartite graph (in fact the complete graph the same vertex set) with at most $(1 + o(1))(2q)^{1/3}$ additional colors. Since the uncolored copies of graph are pairwise disjoint, we can use the same set of new colors for each of them. The total number of colors is $q + o(q)$, implying the required lower bound, in view of known results about the density of primes, mentioned in Chapter 2. $\square$

The above construction can be generalized as follows. Let $V(H) = F(q^{t-1}) \times F^*(q)$ for $t \geq 3$. Two distinct vertices (A, a) and (B, b) are adjacent if and only if $N(A + B) = ab$, where

$$N(X) = X^{1+q+\dots+q^{t-2}}.$$

In this way, the same authors obtained that for any fixed $t \geq 3$ and $s \geq (t - 1)! + 1$, the order of $ex(n; K_{t,s})$ is $n^{1-1/t}$, and the order of $r_k(K_{t,s})$ is k^t . In particular, for $s \geq t = 3$, the generalization yields

$$\begin{aligned} (1 - o(1)) \frac{1}{2} \left(\frac{s-1}{2} \right)^{1/3} n^{5/3} &\leq ex(n; K_{3,s}) \\ &\leq (1 + o(1)) \frac{1}{2} (s-2)^{1/3} n^{5/3}, \end{aligned}$$

and

$$(1 - o(1)) \frac{s-1}{2} k^3 \leq r_k(K_{3,s}) \leq (1 + o(1))(s-2)k^3.$$

A natural problem is as follows.

Problem 9.2 Determine the orders of $ex(n; K_{t,t})$ and $r_k(K_{t,t})$ for fixed t . Is the former $n^{2-1/t}$? Is the latter k^t ?

Another problem is related to the constructions in this chapter.

Problem 9.3 Estimate the upper bounds of the independence numbers of the graphs constructed in this chapter. How far away are they from the lower bounds given by the main theorem in Chapter 3?

9.6 Friendship Theorem

We now prove Friendship Theorem that we have used. The theorem states that if in a group of people any pair of persons have precisely one common friend, then there is always a person (the “politician”) who is everyone’s friend.

Theorem 9.11 Let G be a graph of order n in which any two vertices have precisely one common neighbor. Then there $n = 2m + 1$ for some integer m and $G = K_1 + mK_2$ is a fan.

Proof. If $n \leq 3$, then G is K_1 or K_3 , so we assume that $n \geq 4$. It suffices to show that G has a vertex adjacent to all other vertices. Suppose to the contrary that the assertion is false and G is a counterexample. To arrive at a contradiction, we separate the proof into several parts.

Claim 1. The graph G contains no C_4 as a subgraph.

This is clear since otherwise a pair of vertices in a C_4 have two common neighbors.

Claim 2. The graph G is k -regular with $k \geq 3$.

We first show that any two non-adjacent vertices have the same degree. Let u and v be a pair of non-adjacent vertices of G with $d(u) = k$. Clearly $k \geq 2$ since any vertex must be a vertex of a triangle. Suppose $N(u) = \{w_1, w_2, \dots, w_k\}$. Exactly one of w_i , say w_2 , is the common neighbor of u and v . The vertices u and w_2 have the unique common neighbor, say w_1 . By Claim 1, w_1 and v are not adjacent, and neither w_1 nor w_2 is adjacent to any vertex in $\{w_3, \dots, w_k\}$. However, for $i \geq 2$, v and w_i have a common neighbor, say $z_i \notin \{w_1, w_2, \dots, w_k\}$. By Claim 1 again, all $z_2, \dots, z_k$ are distinct so $d(v) \geq k = d(u)$ since $N(v) \supseteq \{w_2, z_2, \dots, z_k\}$. Furthermore $d(u) \geq d(v)$ by symmetry hence $d(u) = d(v) = k$.

If $k = 2$, then $d(u) = d(v) = d(w_1) = d(w_2) = 2$ and $V(G) = \{u, v, w_1, w_2, w_3\}$, with w_2 is adjacent to any other vertices, which is not the case. Thus $k \geq 3$. To finish the proof of Claim 2, we need to show that any pair of adjacent vertices have the same degree. It suffices to show that $d(w_1) = d(w_2) = k$. Since w_i and v are not adjacent with $i = 1, 3$, so $d(w_i) = d(v) = k$ by what just proved. Also, w_2 and w_3 are not adjacent, so $d(w_2) = d(w_3) = k$. This proves Claim 2.

Claim 3. The order n of G satisfies that $n = k^2 - k + 1$.

Observing that in the sum $\sum_{w_i \in N(u)} d(w_i) = k^2$, each vertex x other than u is counted exactly once since x and u have precisely one common neighbor, and u is counted k times. This proves Claim 3.

To derive a contradiction needed, we consider the adjacency matrix $A = (d_{ij})$ of G . By Claim 1 and Claim 2, any row of A has exactly k 1's, and any two rows have exactly one common coordinate, so

$$A^2 = \begin{pmatrix} k & 1 & \cdots & 1 & 1 \\ 1 & k & \cdots & 1 & 1 \\ \vdots & & & & \vdots \\ 1 & 1 & \cdots & 1 & k \end{pmatrix} = (k-1)I + J,$$

where I is the identity matrix and J is the matrix of all 1's. It is easy to see J has the eigenvalues n (of multiplicity 1) and 0 (of multiplicity $n-1$). This follows by that A^2 has the eigenvalues $k-1+n = k^2$ (of multiplicity 1) and $k-1$ (of multiplicity $n-1$).

Since A is a real symmetric and hence diagonalizable into

$$\begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix},$$

where $\lambda_1, \dots, \lambda_n$ are eigenvalues of A . This implies that A^2 is diagonalizable into

$$\begin{pmatrix} \lambda_1^2 & & & \\ & \lambda_2^2 & & \\ & & \ddots & \\ & & & \lambda_n^2 \end{pmatrix},$$

implying that $\lambda_1^2 = k^2$ and $\lambda_2^2 = \dots = \lambda_n^2 = k-1$, in which the eigenvalues may be relabeled. So $\lambda_1 = \pm k$, and $\lambda_i = \pm\sqrt{k-1}$ for $i = 2, \dots, n$. More

precisely speaking, $\lambda_1 = k$ as k is an eigenvalue of A since $AL = kL$ with $L = (1, 1, \dots, 1)^T$. Let s and t be the numbers of eigenvalues of A equal to $\sqrt{k-1}$ and $-\sqrt{k-1}$, respectively. Then $s + t = n - 1$. Denote by $\text{tr}(A) = \sum_{i=1}^n a_{ii}$ for the trace of A , which is zero, then

$$\lambda_1 + \lambda_2 + \dots + \lambda_n = \text{tr}(A) = 0.$$

Thus

$$k + s\sqrt{k-1} - t\sqrt{k-1} = 0,$$

which implies $s \neq t$ and $\sqrt{k-1} = k/(t-s)$ is rational, so it is an integer. Write $h = \sqrt{k-1}$, then

$$h(t-s) = k = h^2 + 1.$$

So h divides $h^2 + 1$, hence $h = 1$ so $k = 2$, contradicting to Claim 2. This completes the proof. $\square$

Exercises

9.1 Using the constructions in the proofs of Theorem 9.4 and Theorem 9.7, respectively, give lower bounds of $ex(n; C_4)$ for infinite many n .

9.2 Give a lower bound and an upper bound for $r(K_{3,3}, K_{1,n})$.

9.3 Prove $r_k(K_{3,s}) \geq (1/2 - o(1))(s-1)k^3$ for fixed $s \geq 3$.

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Chapter 10

Constructions from Hypergraphs

10.1 Oddtown-Theorem

To illustrate the methods, we introduce some algebraic constructions for two-coloring Ramsey numbers $r(m, n)$. We select these topics from a planned book by Babai and Frankl: *Linear Algebra Methods in Combinatorics with Applications to Geometry and Computer Science*, re-originated in Lecture Notes by Gyárfas. These results on Ramsey numbers are much weaker than that obtained from probabilistic methods. We still do so with two reasons. The first reason is that these methods are developing gradually and they are promising. The second is that some constructions may be good enough but we cannot estimate their parameters currently. For example, the graph G_q of order $n = q^2 + q + 1$ constructed by Erdős and Rényi that contains no C_4 , which is discussed in last chapter, may have given a better lower bound of $r(C_4, K_n)$ than $c(n/\log n)^{3/2}$ by probabilistic method, but we cannot estimate its independence number $\alpha(G_q)$ with a good upper bound. The current best upper bound due to Alon (1986) is $\alpha(G_q) \leq cq^{3/2} \sim cn^{3/4}$ hence the constructive lower bound of $r(C_4, K_n)$ is $\Omega(n^{4/3})$ from G_q . The best lower bound of $\alpha(G_q)$ is $(1 - o(1))q \log q$, which is perhaps much closer to the truth.

Recall a *hypergraph* $\mathcal{H}$ on vertex set $V \neq \emptyset$ is a pair $(V, \mathcal{E})$, where the edge set $\mathcal{E}$ is a family of subsets of V . All hypergraphs are *simple* in this book, that is to say, any pair of edges are distinct as subsets of V (no multiedges). We assume that $e \neq \emptyset$ for all $e \in \mathcal{E}$. Let $V^{(r)}$ be the set of all r -subsets of V . If $\mathcal{E} \subseteq V^{(r)}$, then the hypergraph is called r -uniform.

Most of selected proofs use the following simple fact: if $x_1, x_2, \dots, x_m$ are linearly independent in a linear space, then m is at most the dimension of the space.

In the following result, the vertex set consists of all residents in a town, named Oddtown, and edge set consists of all clubs in the town. The citizens form these clubs by some rules, from which the name of the town came. These rules are a bit odd.

Theorem 10.1 (Oddtown-theorem) *Assume a hypergraph $\mathcal{H} = (V, \mathcal{E})$ has the following properties:*

1. $|e|$ is odd for all $e \in \mathcal{E}$;
2. $|e \cap f|$ is even for all $e, f \in \mathcal{E}$ with $e \neq f$.

Then $|\mathcal{E}| \leq |V|$.

Proof. Assume $|V| = n$ and $\mathcal{E} = \{e_1, e_2, \dots, e_m\}$. View e_i as a 0–1 vector, namely as a row vector with n components,

$$e_i = (e_{i1}, e_{i2}, \dots, e_{in}),$$

where

$$e_{ij} = \begin{cases} 1 & \text{if } v_j \in e_i, \\ 0 & \text{if } v_j \notin e_i. \end{cases}$$

Then all these row vectors form an $m \times n$ matrix M , which is the incidence matrix of $\mathcal{H}$. Since all e_i are elements of the linear space of n –component vectors over the field $F_2 = \{0, 1\}$, the inner product in the space for $x = (x_1, x_2, \dots, x_n)$ and $y = (y_1, y_2, \dots, y_n)$ is

$$x \cdot y = x_1y_1 + x_2y_2 + \dots + x_ny_n.$$

Then the two conditions can be nicely expressed by

$$e_i \cdot e_j = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Claim. $e_1, e_2, \dots, e_m$ are linear independent.

The proof of the claim is simple. Assume

$$\lambda_1 e_1 + \lambda_2 e_2 + \dots + \lambda_m e_m = 0,$$

where $\lambda_i \in F_2$ for $1 \leq i \leq m$. Multiplying both sides of the above equation by e_i yields $\lambda_i = 0$ and the claim is proved.

Thus the conclusion $m \leq n$ follows from the claim immediately. $\square$

Theorem 10.2 (Fisher Inequality, 1940) Assume $\mathcal{H} = (V, \mathcal{E})$ is a hypergraph for which there is a constant $\lambda \geq 0$ such that any pair of distinct edges e and f satisfy $|e \cap f| = \lambda$. Then

$$|\mathcal{E}| \leq |V|.$$

Proof. Let $|V| = n$ and $\mathcal{E} = \{e_1, e_2, \dots, e_m\}$. There are several cases to consider.

Case 1. $\lambda = 0$. It is trivial to see $m \leq n$.

Case 2. $\lambda \geq 1$ and there is an edge, say e_m , with $|e_m| = \lambda$. Then $e_m \subseteq e_i$ and $e_i \cap e_j = e_m$ for any $i \neq j$. Set $e'_i = e_i \setminus e_m$, then $\mathcal{E}' = \{e'_1, e'_2, \dots, e'_{m-1}\}$ is an edge set and the hypergraph $\mathcal{H}' = (V \setminus e_m, \mathcal{E}')$ satisfies the condition in case 1. Thus $m - 1 \leq n - \lambda$ thus $m \leq n$.

Case 3. $\lambda \geq 1$ and any edge contains at least $\lambda + 1$ vertices. As in the proof of Oddtown-theorem, e_i is viewed as a row vector of the incidence matrix of $\mathcal{H}$. But now we consider e_i as an element of the linear space of n -component vectors over R (real numbers). In this space, the inner product will be

$$e_i \cdot e_j = \begin{cases} \lambda + \mu_i & \text{if } i = j \\ \lambda & \text{if } i \neq j, \end{cases}$$

where $\mu_i = |e_i| - \lambda \geq 1$ is an integer.

Claim. $e_1, e_2, \dots, e_m$ are linearly independent.

The proof is as follows. Assume

$$\alpha_1 e_1 + \alpha_2 e_2 + \dots + \alpha_m e_m = 0,$$

where α_i are real numbers. Taking the inner product of both sides by e_i , we have

$$\lambda(\alpha_1 + \alpha_2 + \dots + \alpha_m) + \alpha_i \mu_i = 0.$$

Setting $\beta = \sum_{j=1}^m \alpha_j$, thus $\lambda\beta + \alpha_i \mu_i = 0$, which yields

$$\alpha_i = -\frac{\lambda}{\mu_i} \beta.$$

Summing the both sides of the above equation over $1 \leq i \leq m$, we get

$$\beta = -\lambda \left(\sum_{j=1}^m \frac{1}{\mu_j} \right) \beta,$$

implying that $\beta = 0$ since otherwise the signs of both sides would be different. Therefore $\alpha_1 = \alpha_2 = \dots = \alpha_m = 0$ and the claim is proved. Then the desired inequality $m \leq n$ follows from the claim immediately. $\square$

The following is a constructive lower bound of diagonal Ramsey numbers $r_2(n)$ due to Zs. Nagy (1972), which is much weaker than given by probabilistic method as mentioned. In proof we shall write $K_n^{(r)}$ for complete r -uniform hypergraph that has n vertices and all r -uniform edges. Thus the number of edges of $K_n^{(r)}$ is $\binom{n}{r}$. We write $K_n^{(2)}$ for K_n as usual.

Corollary 10.1 *Let $n \geq 4$ be an integer. Then $r_2(n) > \binom{n-1}{3}$.*

Proof. Associate each vertex of complete graph $K_{\binom{n-1}{3}}$ with an edge of $K_{n-1}^{(3)}$. Color an edge xy of $K_{\binom{n-1}{3}}$ with red if the corresponding edges (triples) of $K_{n-1}^{(3)}$ intersect in one element, otherwise (they intersect in zero or two elements) color xy with blue. The Fisher inequality implies that there is no red K_n , and the Oddtown-theorem implies that there is no blue K_n , as required. $\square$

In the next section, we shall try to obtain a super-polynomial lower bound for $r_2(n)$.

10.2 Cross Intersecting Hypergraphs

In the proofs of theorems in last section, the linear spaces were spaces of vectors. For some cases the linear spaces are spaces of functions. For example, the set of functions $\Omega \rightarrow F$, where Ω is an arbitrary set and F is a field, forms such a space. The dimension of the space is $|\Omega|$. Another example is that consisting of all homogeneous polynomials of degree k in n variables over a field. The dimension of this space is $\binom{n+k-1}{k}$. Specifically, if $n = k = 3$,

$$xyz, xy^2, xz^2, x^2y, x^2z, x^3, yz^2, y^2z, y^3, z^3$$

form a basis, which consists of 10 polynomials of degree 3.

To check the linear independence in spaces of functions, we may need the following propositions.

Proposition 10.1 (Diagonal Criterion) *For $1 \leq i \leq m$, let $f_i : \Omega \rightarrow F$ be functions and let $a_i \in \Omega$ be elements such that*

$$f_i(a_j) \begin{cases} \neq 0 & \text{if } i = j \\ = 0 & \text{if } i \neq j. \end{cases}$$

Then $f_1, f_2, \dots, f_m$ are linearly independent.

Proposition 10.2 (Triangular Criterion) For $1 \leq i \leq m$, let $f_i : \Omega \rightarrow F$ be functions and let $a_i \in \Omega$ be elements such that

$$f_i(a_j) \begin{cases} \neq 0 & \text{if } i = j \\ = 0 & \text{if } i < j. \end{cases}$$

Then $f_1, f_2, \dots, f_m$ are linearly independent.

An s -uniform hypergraph $(V, \mathcal{E})$ and a t -uniform hypergraph $(V, \mathcal{F})$ are called *cross-intersecting* if the numbers of their edges are same with $\mathcal{E} = \{e_1, e_2, \dots, e_m\}$ and $\mathcal{F} = \{f_1, f_2, \dots, f_m\}$, and they satisfy $e_i \cap f_j = \emptyset$ if and only if $i = j$.

Theorem 10.3 (Bollobás, 1965) If a hypergraph $(V, \mathcal{E})$ is s -uniform and $(V, \mathcal{F})$ is t -uniform, and they are cross-intersecting, then

$$m \leq \binom{s+t}{s},$$

where m is the common value of the numbers of edges.

An s -uniform hypergraph $(V, \mathcal{E})$ and a t -uniform hypergraph $(V, \mathcal{F})$ are called *skew cross-intersecting* if the numbers of their edges are same with $\mathcal{E} = \{e_1, e_2, \dots, e_m\}$ and $\mathcal{F} = \{f_1, f_2, \dots, f_m\}$, and they satisfy $e_i \cap f_i = \emptyset$ for $i = 1, 2, \dots, m$, and $e_i \cap f_j \neq \emptyset$ for $1 \leq i < j \leq m$.

The next result generalizes the Bollobás theorem, and the proof is due to Lovász (1977).

Theorem 10.4 If $(V, \mathcal{E})$ is s -uniform, and $(V, \mathcal{F})$ is t -uniform, and they are skew cross-intersecting, then

$$m \leq \binom{s+t}{s},$$

where m is the common value of the numbers of edges.

Proof. Associate each vertex $v \in V$ with a vector in R^{s+1} as

$$Y(v) = (y_0(v), y_1(v), \dots, y_s(v)) \in R^{s+1}$$

so that the set of vectors so obtained will be in general position, namely, any s of them will be linearly independent.

Define a polynomial g_i in $s+1$ variables $x_0, x_1, \dots, x_s$ as

$$\begin{aligned} g_i(X) &= \prod_{v \in f_i} (X \cdot Y(v)) \\ &= \prod_{v \in f_i} (x_0 y_0(v) + x_1 y_1(v) + \dots + x_s y_s(v)). \end{aligned}$$

Then $g_i(X)$ is a homogeneous polynomial of degree t since $|f_i| = t$ for $i = 1, 2, \dots, m$, and $g_i(X) = 0$ if and only if there is some $v \in f_i$ such that $Y(v) \perp X$.

The vectors corresponding to the vertices in e_j generate a subspace of dimension s since $|e_j| = s$. Denote by A_j for this space and let a_j be a nonzero vector orthogonal to this subspace. From the fact that the vectors $Y(v)$ are in the general position, the subspace A_j contains $Y(v)$ if and only if $v \in e_j$. This means

$$g_i(a_j) \begin{cases} \neq 0 & \text{if } i = j \\ = 0 & \text{if } i < j, \end{cases}$$

from skew interesting condition, implying (by Triangular Criterion) that g_i are linearly independent. Therefore the number m of the functions g_i is at most the dimension of the space of homogenous polynomials of degree t in $s + 1$ variables, so

$$m \leq \binom{s+1+t-1}{t} = \binom{s+t}{t} = \binom{s+t}{s},$$

completing the proof. $\square$

For a set L of integers and integers r, t , we shall write

$$t \in L \pmod{r}$$

if $t = \ell \pmod{r}$ for some $\ell \in L$. The negation of this statement will be written as $t \notin L \pmod{r}$.

The following result is nonuniform modular version of a theorem of Ray-Chaudhuri and Wilson (1969), which is due to Deza, Frankl and Singhi (1983).

Theorem 10.5 *Let p be a prime and let L be a set of integers. Assume that $(V, \mathcal{E})$ is a hypergraph with $|V| = n$ and $\mathcal{E} = \{e_1, e_2, \dots, e_m\}$ satisfying*

$$\begin{aligned} |e_i| &\notin L \pmod{p} \quad \text{for } 1 \leq i \leq m \\ |e_i \cap e_j| &\in L \pmod{p} \quad \text{for } 1 \leq i \neq j \leq m. \end{aligned}$$

Then

$$m \leq \binom{n}{0} + \binom{n}{1} + \dots + \binom{n}{|L|} = \sum_{i=0}^{|L|} \binom{n}{i}.$$

Proof. The proof is due to Alon, Babai, Suzuki (1991). Define a polynomial

$$F(x, y) = \prod_{\ell \in L} (x \cdot y - \ell)$$

in $2n$ variables: $x, y \in F_p^n$, which are n -component vectors with coordinates from the p -element field F_p , where $x \cdot y = \sum_{i=1}^n x_i y_i$ is the inner product in F_p^n . Set $f_i : F_p^n \rightarrow F_p$ as

$$f_i(x) = F(x, v_i)$$

for $v_i \in F_p^n$, which is the incidence vector of e_i ($1 \leq i \leq m$). The condition of the theorem ensures

$$f_i(v_j) \begin{cases} \neq 0 & \text{if } i = j \\ = 0 & \text{if } i \neq j. \end{cases}$$

These equations remains true if the domain of f_i is restricted to $\Omega = \{0, 1\}^n \subseteq F_p^n$. In Ω , $x_i^2 = x_i$ for each variable thus every polynomial $f_i : \Omega \rightarrow F_p$ is multilinear. This shows that the restricted $f_1, f_2, \dots, f_m$ are linearly independent from the diagonal criterion. The degree of such a polynomial is at most $|L|$. The dimension of the space consisting of all multilinear polynomials is

$$\binom{n}{0} + \binom{n}{1} + \dots + \binom{n}{|L|} = \sum_{i=0}^{|L|} \binom{n}{i},$$

which is an upper bound of m . □

Remark. If $L = \{0\}$ and $p = 2$, the above theorem implies that the number of clubs in Oddtown is at most $1 + n$, slightly weaker than that from Oddtown-theorem.

Corollary 10.2 *Let L be a set of integers and let $(V, \mathcal{E})$ be a uniform hypergraph with $|V| = n$ and $\mathcal{E} = \{e_1, e_2, \dots, e_m\}$ such that*

$$|e_i \cap e_j| \in L \quad \text{for all } 1 \leq i \neq j \leq m.$$

Then

$$m \leq \binom{n}{0} + \binom{n}{1} + \dots + \binom{n}{|L|} = \sum_{i=0}^{|L|} \binom{n}{i}.$$

Proof. Assume that the hypergraph is r -uniform. Select a prime $p > k$ and set $L' = L \setminus \{k\}$ if $k \in L$ and $L' = L$ otherwise. Then the Theorem 10.5 can be applied. □

Let $V^{(r)}$ be the set of all r -subsets of V . Then a hypergraph $\mathcal{H} = (V, \mathcal{E})$ is r -uniform if and only if $\mathcal{E} \subseteq V^{(r)}$, denoted by $\mathcal{H} \subseteq K_n^{(r)}$ in short, where $n = |V|$. Note that complete $(2p-1)$ -uniform hypergraph $K_{4p-1}^{(2p-1)}$ has about 2^{4p-1} edges. The following result says that forbidding a single interesting size, the number of edges must decrease significantly.

Corollary 10.3 (Omitted Intersection Theorem) *Let p be a prime and let $\mathcal{H} \subseteq K_{4p-1}^{(2p-1)}$ such that no two edges of $\mathcal{H}$ intersect in exactly $p-1$ elements. Then the number of edges of $\mathcal{H}$ is at most*

$$2 \binom{4p-1}{p-1} < 1.7548^{4p-1}.$$

Proof. Set $L = \{0, 1, \dots, p-2\}$. The $\mathcal{H}$ satisfies the conditions of Theorem 10.5 giving

$$|\mathcal{E}| \leq \sum_{i=0}^{p-1} \binom{4p-1}{i} \leq 2 \binom{4p-1}{p-1} < (1.7548)^{4p-1},$$

where the second inequality is an exercise and the last follows from Stirling formula. $\square$

We now have a constructive super-polynomial lower bound for diagonal Ramsey number.

Let p be a prime and let $n > 2p^2$. Define a complete graph of order $\binom{n}{p^2-1}$ whose vertex set is $V^{(p^2-1)}$, where V is an n -set. Color an edge ef of the graph red if $|e \cap f| \not\equiv p-1 \pmod{p}$, where $e, f \in V^{(p^2-1)}$ are subsets, otherwise color it blue. Assume that K_m is a monochromatic red clique. Setting $L = \{0, 1, \dots, p-2\}$, the Corollary 10.2 gives $m \leq 2 \binom{n}{p-1}$. If K_m is a monochromatic blue clique. Setting $L = \{p-1, 2p-1, \dots, p^2-p-1\}$, the nonuniform modular version of Ray-Chaudhuri and Wilson theorem gives

$$m \leq \binom{n}{p-1} + \binom{n}{p-2} + \dots + \binom{n}{0} \leq 2 \binom{n}{p-1}.$$

For case $p = 3$, this construction is exactly that in Corollary 10.1. Let $k = 2 \binom{n}{p-1} + 1$, then $\binom{n}{p^2-1} > ck^{p+1}$ for all large n , where c is a constant depending only on p . This gives $r_2(k) > ck^{p+1}$. Basing on the above construction, Frankl (1977) and Frankl-Wilson (1981) obtained the following super-polynomial lower bound for $r_2(k)$.

Theorem 10.6 *Let $\omega(k) = \log k / (4 \log \log k)$. For any $\epsilon > 0$, if k is sufficiently large, then*

$$r_2(k) > k^{(1-\epsilon)\omega(k)}.$$

Proof. Let us select p and n appropriately. Let $n = p^3$ and let p be the largest prime such that $\binom{p^3}{p-1} < k$. So there are no monochromatic K_k . For any $\epsilon > 0$ and large enough k , using the Prime Number Theorem and elementary estimate of $\binom{p^3}{p-1}$, we have

$$\frac{(1-\epsilon)\log k}{2\log\log k} \leq p \leq \frac{(1+\epsilon)\log k}{2\log\log k}.$$

Easy calculation confirms that number of vertices of the complete graph satisfies

$$\binom{p^3}{p^2-1} \geq k^{(1-\epsilon)\omega(k)}$$

as desired. $\square$

Using Omitted Intersection Theorem, one can construct a graph G_p for a prime p by associating its vertices with edges of $K_{4p-1}^{(2p-1)}$, in which two of vertices are adjacent if and only if the corresponding edges of $K_{4p-1}^{(2p-1)}$ intersect in exactly $p-1$ vertices. The above corollary gives an upper bound for independence number of G_p as

$$\alpha(G_p) < 1.7548^{4p-1},$$

then

$$\chi(G_p) \geq \frac{|V(G_p)|}{\alpha(G_p)} = \frac{\binom{4p-1}{2p-1}}{\alpha(G_p)} > 1.1397^{4p-1}.$$

We have seen the algebraic methods have much success for Turán numbers and multi-color Ramsey number of bipartite graphs. In this section we have seen that results on classic Ramsey numbers by algebraic methods are much weaker than given by probabilistic methods. Also, the latter has a much wider applications than the former. However, we introduce two jokes given by Spencer (1987) to say that for many topics the probabilistic method cannot provide the best results. The problem that he was joking is serious. In order to have verisimilitude, we write the joked results as usual but without indices.

The following results is due to Joker. In the proof he uses the basic probabilistic method.

Theorem (Joker) *Let S and T be nonempty sets. If $|T| > \binom{|S|}{2}$, then there exists an injection $f: S \rightarrow T$.*

Proof. Consider the probability space consisting of all maps from S to T . Let A_{xy} signify the event $f(x) = f(y)$. Then $\Pr(A_{xy}) = 1/|T|$ and

$$\Pr(\cup_{x,y \in S} A_{xy}) \leq \sum_{x,y \in S} \frac{1}{|T|} = \frac{\binom{|S|}{2}}{|T|} < 1,$$

which implies that $\Pr(\cap_{x,y \in S} \overline{A_{xy}}) > 0$ and the desired injection exists. $\square$

Later Joker himself improved the above result by using the local lemma. The new result is tight up to a constant factor.

Theorem (Joker) *Let S and T be nonempty sets. If $|T| \geq 2e|S|$, then there exists an injection $f : S \rightarrow T$.*

Proof. The same as that for Joke 1 but apply the local lemma. In the dependence graph, the vertex A_{xy} is adjacent to $A_{xy'}$ with $y' \in S \setminus \{y\}$ and $A_{x'y}$ with $x' \in S \setminus \{x\}$. Let $d = 2(|S| - 1)$, then the independence graph is d regular, in which the event A_{xy} is mutually independent to all non-neighbors. As $p = 1/|T|$, the condition ensures $e(d+1)p < 1$, so the symmetric form of the Local Lemma gives that $\Pr(\cap_{x,y \in S} \overline{A_{xy}}) > 0$, implying the existence of the desired injection. $\square$

10.3 Borsuk's Conjecture $\star$

The essential concept in Ramsey theory is that exceptions occur in small size of the structures. Sometimes the size can be the dimension of the linear space. This is the case we discuss in this section. These small size cases may have bigger effect since they are more concrete, which is possible to lead to an error. We have seen one such case in Chapter 6 that almost all large (random) graphs disprove a conjecture. We shall see another example from geometry.

In 1933, Borsuk conjectured that every set of diameter one in real space R^d can be partitioned into $d+1$ sets of *smaller* diameter.

The conjecture has been proved for $d = 1, 2, 3$ and also for all d if the set is special, like centrally symmetric, having smooth surface.

Let $f(d)$ be the minimum integer such that every set of diameter one in R^d can be partitioned into $f(d)$ sets of smaller diameter. Borsuk's conjecture was $f(d) \leq d+1$. This conjecture fails badly for large d .

The following surprising result is due to Kahn and Kalai (1992), who proved that $f(d) > 1.2^{\sqrt{d}}$. An upper bound as $f(d) \leq (\sqrt{3/2} + o(1))^d$ was obtained by Schramm (1988).

Theorem 10.7

$$f(d) > 1.1397^{\sqrt{2d}} > 1.2^{\sqrt{d}}.$$

Let p be a prime and $d = \binom{4p-1}{2}$. The Kahn-Kalai hypergraph $K(d)$ is defined as follows. Let $\mathcal{H}_p = K_{4p-1}^{(2p-1)}$ be the complete $(2p-1)$ -uniform hypergraph on $4p-1$ vertices. Let each pair of distinct vertices $x, y \in V(\mathcal{H}_p)$ associate with a vertex a_{xy} of $K(d)$. That is to say, the vertex set of hypergraph $K(d)$ is

$$\{a_{xy} : x, y \in V(\mathcal{H}_p)\}.$$

Notice that $K(d)$ has $d = \binom{4p-1}{2}$ vertices since $4p-1$ vertices.

The edges of $K(d)$ are associated with edges of $\mathcal{H}_p$ as follows. If $e \in \mathcal{E}(\mathcal{H})$ then the edge A_e of $K(d)$ is

$$A_e = \{a_{xy} : x \in e, y \in V(\mathcal{H}_p) \setminus e\}.$$

That is to say, A_e is defined by the pairs of $V(\mathcal{H}_p)$ “split by e ”.

Notice that $K(d)$ is a $2p(2p-1)$ -uniform hypergraph and it has $d = \binom{4p-1}{2}$ vertices and $\binom{4p-1}{2p-1}$ edges.

Let $\mathcal{K}(d)$ be the representation of $K(d)$ in R^d (row vectors of incidence matrix). Since $K(d)$ is a $2p(2p-1)$ -uniform hypergraph, the distance of the points representing A_{e_1} and A_{e_2} is

$$\sqrt{2[2p(2p-1) - |A_{e_1} \cap A_{e_2}|]}$$

Therefore the maximum distance, that is the diameter of $\mathcal{K}(d)$, is realized between two points of $\mathcal{K}(d)$ if and only if

$$|A_{e_1} \cap A_{e_2}| = \min\{|A_e \cap A_{e'}| : e, e' \in \mathcal{E}(\mathcal{H}_p)\} = \mu.$$

Claim. $|A_{e_1} \cap A_{e_2}| = \mu$ if and only if $|e_1 \cap e_2| = p-1$.

In fact, $|A_{e_1} \cap A_{e_2}|$ is the number of pairs of $V(\mathcal{H})$ split by *both* e_1 and e_2 . It depends only on $|e_1 \cap e_2|$. Assume $|e_1 \cap e_2| = r$. Then

$$\mu = \min\{r(r-1) + (2p-1-r)^2\}.$$

It is easy to check that this expression has minimum at $r = p-1$, which proves the claim.

Using the claim, the partition of $\mathcal{K}(d)$ into sets of smaller diameter is equivalent to partition the edges of $\mathcal{H}_p$ into classes without intersection of

size $p-1$, and that is clearly at least $\chi(G_p)$, where G_p is the graph defined in last section. From the properties of G_p , we have

$$f(d) \geq \chi(G_p) > (1.1397)^{4p-1} > (1.1397)^{\sqrt{2d}} > (1.2)^{\sqrt{d}},$$

the third inequality follows from the fact that $4p-1 > \sqrt{(4p-1)(4p-2)} = \sqrt{2d}$, completing the proof. $\square$

Exercises

10.1 Assume that $\mathcal{H} = (V, \mathcal{E})$ is a simple hypergraph such that each pair of edges $e, f \in \mathcal{E}$ satisfies $|e \cap f| = 0$ or $|e \cap f| = \lambda$ edges, then $|\mathcal{E}| \leq |V|$.

10.2 Prove the dual of Fisher Inequality as follows. Assume that $\mathcal{H} = (V, \mathcal{E})$ is a simple hypergraph such that each pair of vertices is contained in exactly λ edges, then $|\mathcal{E}| \geq |V|$.

10.3 Generalize Odd-town Theorem for “mod p town” or even “mod p^k town”.

10.4 Prove that the dimension of homogeneous polynomials of degree k in n variables is $\binom{n+k-1}{k}$.

10.5 Prove the inequality $\sum_{i=0}^{p-1} \binom{4p-1}{i} \leq 2 \binom{4p-1}{p-1}$, which has been used in the proof of Omitted Intersection Theorem.

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Chapter 11

Ramsey Numbers and Communication

11.1 Communication Channels

A communication channel consists of a finite input set X , an output set Y . For each input $x \in X$, there is a nonempty fan-shaped output $S_x \subseteq Y$, which is the set of outputs that may be received for the input x by the receiver. In each use of the channel, a sender transmits an input $x \in X$, and receiver receives an arbitrary output $y \in S_x$. For distinct inputs u and v , they can be received as the same output if and only if $S_u \cap S_v \neq \emptyset$. Suppose that the sender and receiver agree in advance on an input set $I \subseteq X$. In order to avoid error, the outputs of any two distinct inputs in I cannot intersect. In a noiseless channel, there is no intersect between two outputs. Shannon (1956) first studied the amount that an information channel can communicate without error. He formulated the problem to a problem of graph theory.

Let X be the input set of a channel and let G be a graph with vertex set X in which two distinct vertices are adjacent if and only if their outputs intersect. The graph G is called the *characteristic graph* of the channel. The characteristic graph of a completely noisy channel is K_k ; and that of a noiseless channel is an empty graph (a graph with edge set empty).

In most situations, a channel is in repeated uses. When the channel is used n times, the sender transmits a sequence $x = (x_1, \dots, x_n)$, where $x_i \in X$, and receiver receives a sequence $y = (y_1, \dots, y_n)$, where $y_i \in S_{x_i} \subseteq Y$. The repeated use of the channel can be viewed as a single use of a larger channel. The large channel has an input set X^n , the Cartesian product of

X . For $x = (x_1, \dots, x_n) \in X^n$, its output set is

$$S_x = S_{x_1} \times \dots \times S_{x_n} = \{(y_1, \dots, y_n) : y_i \in S_{x_i}, 1 \leq i \leq n\}.$$

Denote by $\mathcal{G}$ as the characteristic graph of the large channel, then its vertex set is X^n . Two distinct vertices $x = (x_1, \dots, x_n)$ and $x' = (x'_1, \dots, x'_n)$ can be received as the same output if and only if for each $1 \leq i \leq n$, $S_{x_i} \cap S_{x'_i} \neq \emptyset$, when $x_i \neq x'_i$. Thus x and x' are adjacent in $\mathcal{G}$ if and only if for each $1 \leq i \leq n$, x_i and x'_i are adjacent in G , when $x_i \neq x'_i$. Hence the edge set of $\mathcal{G}$ is defined.

We now define the n th AND product of graph $G = (X, E)$, denoted by $\wedge^n G$. Its vertex set is X^n , two vertices $x, x' \in X^n$ are adjacent in $\wedge^n G$ if and only if for each $1 \leq i \leq n$, x_i and x'_i are adjacent in G , when $x_i \neq x'_i$; namely, either x_i and x'_i are adjacent in G , or $x_i = x'_i$. (It is slightly more convenient to give the definition if we admit any vertex in G is adjacent to itself.) Clearly, if the characteristic graph of a channel is G , and when its repeated use is viewed as a single use of a large channel, the characteristic graph of the large channel is $\wedge^n G$.

When we consider to transmit the sequence $x = (x_1, \dots, x_n)$ of length n , where x_i may come from different input X_i , the characteristic graph of the channel is the AND product of the graphs G_i defined as follows. Let $G_1, \dots, G_n$ be graphs, and let $V_1, \dots, V_n$ be their vertex sets, respectively. Define their AND product $G_1 \wedge \dots \wedge G_n$ as a graph with vertex set $V_1 \times \dots \times V_n$, two distinct vertices $x = (x_1, \dots, x_n)$ and $x' = (x'_1, \dots, x'_n)$ are adjacent if and only if for each $1 \leq i \leq n$, either x_i and x'_i are adjacent in G_i or $x_i = x'_i$. In the following proposition, the closed neighborhood of a vertex v , denoted by $N[v]$, consists of v and its neighbors, that is to say, $N[v] = \{v\} \cup N(v)$.

Proposition 11.1 *Let $G_1, \dots, G_n$ be graphs and $\mathcal{G} = G_1 \wedge \dots \wedge G_n$. Then for any vertex $x = (x_1, \dots, x_n)$ of $\mathcal{G}$, its closed neighborhood $N_{\mathcal{G}}[x]$ satisfies*

$$N_{\mathcal{G}}[x] = N_{G_1}[x_1] \times \dots \times N_{G_n}[x_n].$$

□

Note the above equality does not hold for neighborhoods in general. Let $G = K_2 \cup K_1$ on vertex set $V = u, v, w$ with only one edge uv . Consider $\mathcal{G} = \wedge^2 G$. Take vertex $x = (u, w)$ of $\mathcal{G}$, its neighborhood $N_{\mathcal{G}}(x)$ is singleton $\{(v, w)\}$, but $N_G(u) \times N_G(w) = \emptyset$ since $N_G(w) = \emptyset$.

Proposition 11.2 *Let I_i be an independent set of G_i . Then $I_1 \times \dots \times I_n$ is an independent set of $G_1 \wedge \dots \wedge G_n$. Consequently,*

$$\alpha(G_1 \wedge \dots \wedge G_n) \geq \alpha(G_1) \cdots \alpha(G_n),$$

and

$$\alpha(\wedge^{m+n}G) \geq \alpha(\wedge^m G)\alpha(\wedge^n G).$$

Proof. For two distinct vertices $x = (x_1, \dots, x_n)$ and $x' = (x'_1, \dots, x'_n)$ of $I_1 \times \dots \times I_n$, there is some $1 \leq i \leq n$, $x_i \neq x'_i$. Since I_i is independent in G_i , so x_i and x'_i are non-adjacent in G_i . So x and x' are non-adjacent in $G_1 \wedge \dots \wedge G_n$, hence $I_1 \times \dots \times I_n$ is an independent set. $\square$

Let $\{f(n) : n \geq 1\}$ be a series of positive real numbers. It is said to be super-multiplicative if $f(m+n) \geq f(m)f(n)$. The following result is elementary in calculus.

Lemma 11.1 *Suppose that $f(n)$ is positive and super-multiplicative. Then the limit of $f(n)^{1/n}$ exists as $n \rightarrow \infty$, and*

$$\lim_{n \rightarrow \infty} f(n)^{1/n} = \sup_{n \geq 1} f(n)^{1/n}.$$

Proof. Set $\ell = \sup_{n \geq 1} f(n)^{1/n}$. Then $0 < \ell \leq \infty$ and $\overline{\lim}_{n \rightarrow \infty} f(n)^{1/n} \leq \ell$. We shall show that $\underline{\lim}_{n \rightarrow \infty} f(n)^{1/n} \geq \ell$.

Case 1. $\ell < \infty$. For any $\epsilon > 0$, there is some $N \geq 1$ such that $f(N)^{1/N} > \ell - \epsilon$. For any $n \geq N$, let $n = q_n N + r_n$ with $0 \leq r_n < N$. Thus

$$f(n)^{1/n} \geq f(q_n N)^{1/n} f(r_n)^{1/n} \geq f(N)^{q_n/n} f(r_n)^{1/n}.$$

Since $q_n/n \rightarrow 1/N$ and $f(r_n)^{1/n} \rightarrow 1$, we have

$$\underline{\lim}_{n \rightarrow \infty} f(n)^{1/n} \geq f(N)^{1/N} > \ell - \epsilon.$$

Thus $\underline{\lim}_{n \rightarrow \infty} f(n)^{1/n} \geq \ell$ since $\epsilon > 0$ is arbitrary.

Case 2. $\ell = \infty$. For any $M > 0$, there is $N \geq 1$ such that $f(N)^{1/N} > M$, we can similarly show that $\underline{\lim}_{n \rightarrow \infty} f(n)^{1/n} \geq M$, so $\underline{\lim}_{n \rightarrow \infty} f(n)^{1/n} = \infty$ since $M > 0$ is arbitrary. $\square$

From Proposition 11.2, we know that $\alpha(\wedge^n G)$ is super-multiplicative, hence the limit

$$\lim_n (\alpha(\wedge^n G))^{1/n} = \sup_n (\alpha(\wedge^n G))^{1/n}$$

exists, which is denoted by $\Theta(G)$, called the *Shannon capacity* of G (or of the corresponding channel). Then we have an easy lower bound for $\Theta(G)$ as follows.

Theorem 11.1 *For any graph G , $\Theta(G) \geq \alpha(G)$.*

□

The inequality can be strict. The first such graph is C_5 , which we will encounter in the next section.

We now define OR product of graphs. Let $V_1, \dots, V_n$ be vertex sets of graphs $G_1, \dots, G_n$, respectively. The OR product of $G_1, \dots, G_n$, denoted by $G_1 \vee \dots \vee G_n$, is defined a graph on vertex set $V_1 \times \dots \times V_n$, in which two distinct vertices $x = (x_1, \dots, x_n)$ and $x' = (x'_1, \dots, x'_n)$ are adjacent if and only if for *some* $1 \leq i \leq n$, $x_i \neq x'_i$, and they are adjacent in G_i . For $G_1 = \dots = G_n = G$, this OR product is denoted by $\vee^n G$. By the definition, we have

$$\overline{G_1 \vee \dots \vee G_n} = \overline{G_1} \wedge \dots \wedge \overline{G_n},$$

and hence

$$\overline{\overline{G_1} \wedge \dots \wedge \overline{G_n}} = \overline{G_1} \vee \dots \vee \overline{G_n}.$$

Combining these with the fact that $\alpha(G) = \omega(\overline{G})$, we obtain

$$\alpha(G_1 \wedge \dots \wedge G_n) = \omega(\overline{G_1} \vee \dots \vee \overline{G_n}). \quad (11.1)$$

11.2 Shannon Capacities of Cycles

This section is devoted to compute the Shannon capacities of cycles.

Lemma 11.2 *The independence number of $\wedge^2 C_5$ is 5.*

Proof. Setting the vertex set of C_5 as $\{0, 1, 2, 3, 4\}$, we arrange all vertices of $\wedge^2 C_5$ in a matrix as follows.

$$\begin{array}{ccccc} (0, 0)^* & (0, 1) & (0, 2) & (0, 3) & (0, 4) \\ (1, 0) & (1, 1) & (1, 2)^* & (1, 3) & (1, 4) \\ (2, 0) & (2, 1) & (2, 2) & (2, 3) & (2, 4)^* \\ (3, 0) & (3, 1)^* & (3, 2) & (3, 3) & (3, 4) \\ (4, 0) & (4, 1) & (4, 2) & (4, 3)^* & (4, 4) \end{array}$$

It is easy to check that the set $\{(0, 0), (1, 2), (2, 4), (3, 1), (4, 3)\}$, each element of which is marked by a star in above matrix, forms an independent set in $\wedge^2 C_5$, so $\alpha(\wedge^2 C_5) \geq 5$. For each $i = 0, 1, 2, 3, 4$, the consecutive two rows

$$\begin{array}{ccccc} (i, 0) & (i, 1) & (i, 2) & (i, 3) & (i, 4) \\ (i+1, 0) & (i+1, 1) & (i+1, 2) & (i+1, 3) & (i+1, 4) \end{array}$$

with $i+1 \pmod{4}$, contain only two non-adjacent vertices. Thus $\alpha(\wedge^2 C_5) \leq 5$, proving the lemma. $\square$

Corollary 11.1 $\Theta(C_5) \geq \sqrt{5}$.

We know that there is graph G with the chromatic number $\chi(G)$ is considerably larger than its clique number $\omega(G)$, see an exercise of Chapter 6. A graph G is called *perfect* if any induced subgraph H of G satisfies that $\chi(H) = \omega(H)$. Any even cycle is a perfect graph and any odd cycle with length at least 5 is not a perfect graph. Shannon (1956) proved that when G is a perfect graph, then $\Theta(G) = \alpha(G)$. The equality may not hold in general as $\Theta(C_5) \geq \sqrt{5}$. However, Lovász (1979) proved that $\Theta(C_5) = \sqrt{5}$. The knowledge on the Shannon capacity of imperfect graphs is very limited.

We adopt a simpler way from *Proofs from THE BOOK* by Aigner and Ziegler to obtain the Shannon capacities of even cycles and a general upper bound for odd cycles. We shall introduce the Lovász theta function later.

Call a real vector $X = \{x_v : v \in V\}$ as a probability distribution or simply a *distribution* on the set V if $x_v \geq 0$ and $\sum_{v \in V} x_v = 1$. Denote by $\mathcal{T}$ for the set of all cliques of G . For a fixed distribution X , we write

$$\lambda(X) = \max_{T \in \mathcal{T}} \sum_{v \in T} x_v,$$

and $\lambda(G) = \inf_X \lambda(X)$. Since the inf is achievable as $\lambda(X)$ is continuous on the compact set consisting of all distributions, so

$$\lambda(G) = \min_X \lambda(X) = \min_X \max_{T \in \mathcal{T}} \sum_{v \in T} x_v, \quad (11.2)$$

where the min runs through all distributions X on vertex set $V(G)$.

In order to get another expression for $\lambda(G)$, we need a basic result in the Game Theory or Linear Programming, called Minimax Theorem.

Theorem 11.2 *Let $A = (a_{ij})$ be a real $n \times m$ matrix, and let $X = (x_1, \dots, x_m)$ and $Y = (y_1, \dots, y_n)$ be probability distributions. Then*

$$\min_X \max_Y YAX^T = \max_Y \min_X YAX^T,$$

where the min runs through all probability distributions X and the max does through all such Y . Furthermore, there exist probability distributions X^* and Y^* such that

$$\min_X Y^*AX^T = \max_Y YAX^{*T}.$$

The proof is based on Duality Theorem, which can be found in most of textbooks on Game Theory or Linear Programming. We thus omit it.

Set $\mathcal{T} = \{T_1, T_2, \dots, T_n\}$ and $V(G) = \{v_1, v_2, \dots, v_m\}$. For any clique T_i and any vertex v_j of G , define

$$a_{ij} = \begin{cases} 1 & \text{if } v_j \in T_i, \\ 0 & \text{otherwise.} \end{cases}$$

We thus have an $n \times m$ real matrix $A = (a_{ij})$. Let $e_i = (0, \dots, 0, 1, 0, \dots, 0)$ be the vector of R^n of all zeros except for a one in the i th position. Then $e_i A X^T = \sum_{v \in T_i} x_v$, so the expression (11.2) is

$$\lambda(G) = \min_X \max_{1 \leq i \leq n} e_i A X^T.$$

We now consider the left-hand side in the first equality in Minimax Theorem, for a given distribution X , suppose in the i th position that the vertex $A X^T$ has the maximum component, then

$$\max_Y Y A X^T = e_i A X^T = \max_{1 \leq i \leq n} e_i A X^T.$$

Denote by f_j for the vector of R^m of all zeros except for a one in the j th position. So

$$\min_X \max_Y Y A X^T = \min_X \max_{1 \leq i \leq n} e_i A X^T,$$

thus

$$\begin{aligned} \min_X \max_{T \in \mathcal{T}} \sum_{v \in T} x_v &= \min_X \max_{1 \leq i \leq n} e_i A X^T \\ &= \max_Y \min_{1 \leq j \leq m} Y A f_j^T = \max_Y \min_{v \in V} \sum_{T \ni v} y_T, \end{aligned}$$

where in the last expression the sum is on T over $\mathcal{T}$, and the max runs through all distributions $Y = \{y_T : T \in \mathcal{T}\}$ on $\mathcal{T}$.

We thus obtain the second expression for $\lambda(G)$ as

$$\lambda(G) = \max_Y \min_{v \in V} \sum_{T \ni v} y_T. \quad (11.3)$$

Let $U \subseteq V(G)$ be an independent set of G with $|U| = \alpha(G) = \alpha$, and define a distribution $X(U) = \{x_v : v \in V\}$ by

$$x_v = \begin{cases} 1/\alpha & \text{if } v \in U, \\ 0 & \text{otherwise.} \end{cases}$$

Since each clique contains at most one vertex of U , so $\lambda(X(U)) = 1/\alpha$ and $\lambda(G) \leq 1/\alpha$. We then have the following lemma.

Lemma 11.3 For any graph G ,

$$\alpha(G) \leq \frac{1}{\lambda(G)}.$$

Lemma 11.4 For any graphs G_1 and G_2 ,

$$\lambda(G_1 \wedge G_2) = \lambda(G_1)\lambda(G_2).$$

Proof. We now have two expressions for $\lambda(G)$ as

$$\lambda(G) = \min_X \max_{T \in \mathcal{T}} \sum_{v \in T} x_v = \max_Y \min_{v \in V} \sum_{T \ni v} y_T.$$

Let X and X' be the distributions which achieve the minima, that is to say,

$$\lambda(X) = \lambda(G_1) \quad \text{and} \quad \lambda(X') = \lambda(G_2).$$

We define a distribution Z on vertex set of $G_1 \wedge G_2$ as $z_{(u,v)} = x_u x'_v$ for a vertex (u, v) of $G_1 \wedge G_2$. The fact that $Z = \{z_{(u,v)} : (u, v) \in V(G_1) \times V(G_2)\}$ is truly a probability distribution can be seen by $\sum z_{(u,v)} = \sum x_u \sum x'_v = 1$.

Claim. A clique of $G_1 \wedge G_2$ is maximal if and only if it has the form $T_1 \times T_2$ with T_i is a maximal clique of G_i for $i = 1, 2$.

The claim is easy. Note the “clique” in the statement cannot be replaced by “independent set”. The dual form of the claim is “an independent set of $G_1 \vee G_2$ is maximal if and only if it has the form $S_1 \times S_2$ with S_i is a maximal independent set of G_i for $i = 1, 2$.”

Using the above claim, we have

$$\begin{aligned} \lambda(G_1 \wedge G_2) &\leq \lambda(Z) = \max_{X \times X'} \sum_{(u,v) \in X \times X'} z_{(u,v)} \\ &= \max_{X \times X'} \sum_{u \in X} x_u \sum_{v \in X'} x'_v \\ &= \lambda(G_1)\lambda(G_2). \end{aligned}$$

On the other hand, denote still by T for clique of $G_1 \wedge G_2$, and Y for distribution on the set of cliques of $G_1 \wedge G_2$, then

$$\begin{aligned} \lambda(G_1 \wedge G_2) &= \max_Y \min_{(u,v) \in V(G_1) \times V(G_2)} \sum_{T \ni (u,v)} y_T \\ &\geq \lambda(Z) = \max_{X \times X'} \sum_{(u,v) \in X \times X'} z_{(u,v)} \end{aligned}$$

$$\begin{aligned}
 &= \max_{X \times X'} \sum_{u \in X} x_u \sum_{v \in X'} x'_v \\
 &= \lambda(G_1) \lambda(G_2),
 \end{aligned}$$

proving the lemma. $\square$

Theorem 11.3 *For any graph G , we have*

$$\alpha(G) \leq \Theta(G) \leq \frac{1}{\lambda(G)}.$$

Proof. From Lemma 11.4, we see $\lambda(\wedge^n G) = \lambda^n(G)$. This and Lemma 11.3 give

$$\alpha(\wedge^n G) \leq \frac{1}{\lambda(\wedge^n G)} = \frac{1}{\lambda^n(G)},$$

hence $\Theta(G) \leq \frac{1}{\lambda(G)}$. $\square$

Lemma 11.5 *Let $k \geq 4$ be an integer and let C_k be a cycle of length k . Then $\lambda(C_k) = \frac{2}{k}$, hence*

$$\Theta(C_k) \leq \frac{2}{k}.$$

Proof. Let $X_0 = (1/k, \dots, 1/k)$ be the uniform distribution on the vertex set. Since any clique T meets C_k at most two vertices, so

$$\lambda(C_k) = \min_X \lambda(X) \leq \lambda(X_0) = \max_{T \in \mathcal{T}} \sum_{v \in T} \frac{1}{k} = \frac{2}{k}.$$

On the other hand, $\mathcal{T}$ consists of all k vertices and all k edges. Defining a distribution Y_0 by choosing a component as $1/k$ for an edge and 0 for a vertex, and using the expression (11.3) for $\lambda(G)$, we similarly have $\lambda(C_k) \geq 2/k$, hence $\lambda(C_k) = 2/k$. $\square$

Theorem 11.4 *Let $m \geq 2$ be an integer. Then $\Theta(C_{2m}) = m$.*

$\square$

The above discuss is not sufficient to obtain the Shannon capacity of odd cycles. The first one, $\Theta(C_5)$, was obtained by Lovász with an elegant solution. His main new idea was to represent the vertices $V(G) = \{v_1, v_2, \dots, v_m\}$ by real vectors of length one such that any two non-adjacent

are orthogonal. Let us call such a representation an *orthogonal representation* of G . Note such a representation always exists: just take unit vectors

$$e_1 = (1, 0, \dots, 0), e_2 = (0, 1, 0, \dots, 0), \dots, e_m = (0, 0, \dots, 1)$$

in R^m .

Let $T = \{v^{(1)}, v^{(2)}, \dots, v^{(m)}\}$ be an orthogonal representation of G in R^s with $v^{(i)}$ corresponding to the vertex v_i . Denote by

$$\bar{v} = \frac{1}{m}(v^{(1)} + v^{(2)} + \dots + v^{(m)})$$

We assume that any vector $v^{(i)}$ has the same angle ($\neq \pi/2$) with $\bar{v}$, or equivalently that the inner product

$$\langle v^{(i)}, \bar{v} \rangle = \sigma_T(G)$$

has the same value $\sigma_T \neq 0$ for each i . We shall call the value $\sigma_T(G)$ the constant of the representation T .

Denote by $|v| = \sqrt{\langle v, v \rangle}$ for the length of vector v . For a probability distribution $X = (x_1, \dots, x_m)$ on vertex set V , set

$$\mu(X) = |x_1 v^{(1)} + x_2 v^{(2)} + \dots + x_m v^{(m)}|^2$$

and

$$\mu(G) = \inf_X \mu(X) = \min_X \mu(X).$$

Lemma 11.6 *Let $T = \{v^{(1)}, v^{(2)}, \dots, v^{(m)}\}$ be an orthogonal representation of G . Then*

$$\sigma_T(G) = \mu_T(G).$$

Proof. From the Cauchy-Schwarz inequality $\langle a, b \rangle^2 \leq |a|^2 |b|^2$, we have

$$\langle x_1 v^{(1)} + x_2 v^{(2)} + \dots + x_m v^{(m)}, \bar{v} \rangle^2 \leq \mu(X) |\bar{v}|^2.$$

However, the left-hand side is

$$\langle x_1 v^{(1)} + x_2 v^{(2)} + \dots + x_m v^{(m)}, \bar{v} \rangle^2 = (x_1 + \dots + x_m)^2 \sigma_T^2(G) = \sigma_T^2,$$

which in fact holds for any distribution X . Let $X_0 = (1/m, \dots, 1/m)$ be the uniform distribution, for which $|\bar{v}|^2 = \sigma_T(G)$, we have $\sigma_T^2(G) \leq \mu(X_0) \sigma_T(G)$, or $\sigma_T(G) \leq \mu(X_0)$. On the other hand, for the uniform distribution X_0 , we obtain

$$\mu_T(G) \leq \mu(X_0) = \left| \frac{1}{m}(v^{(1)} + \dots + v^{(m)}) \right|^2 = |\bar{v}|^2 = \sigma_T(G),$$

so $\mu_T(G) = \sigma_T(G)$. □

Lemma 11.7 *Let $T = \{v^{(1)}, v^{(2)}, \dots, v^{(m)}\}$ be an orthogonal representation of G . Then*

$$\alpha(G) \leq \frac{1}{\mu_T(G)}.$$

Proof. Let U be an independent set of G with $|U| = \alpha(G) = \alpha$. Define a distribution $X(U)$ on $V(G)$ by

$$x_i = \begin{cases} 1/\alpha & \text{if } x_i \in U, \\ 0 & \text{otherwise} \end{cases}$$

Since $\langle v^{(i)}, v^{(j)} \rangle = 0$ for any pair of two non-adjacent vertices and $\langle v^{(i)}, v^{(i)} \rangle = 1$, we have

$$\mu_T(G) \leq \mu(X(U)) = \left| \sum_{i=1}^m x_i v^{(i)} \right|^2 = \sum_{i=1}^m x_i^2 = \frac{1}{\alpha},$$

proving the lemma. $\square$

Lemma 11.8 *Let $T = \{v^{(1)}, v^{(2)}, \dots, v^{(m)}\}$ be an orthogonal representation of G . Then*

$$\alpha(G) \leq \frac{1}{\sigma_T(G)}.$$

Let G_1 and G_2 be graphs with orthogonal representations R and S in R^r and R^s , respectively. We now do not distinguish the vertices and their representations. For $u = (u_1, \dots, u_r) \in R$ and $v = (v_1, \dots, v_s) \in S$, the *tensor product* of u and v is defined as the vector

$$u \circ v = (u_1 v_1, \dots, u_1 v_s, u_2 v_1, \dots, u_2 v_s, \dots, u_r v_1, \dots, u_r v_s)$$

of R^{rs} . Denote by $R \circ S$ for the set $\{u \circ v : u \in R, v \in S\}$.

Lemma 11.9 *Let $u, x \in R^r$ and $v, y \in R^s$ be vectors. Then*

$$\langle u \circ v, x \circ y \rangle = \langle u, x \rangle \langle v, y \rangle.$$

Proof. Directly from the definitions of tensor product and inner product. $\square$

Lemma 11.10 *Let R and S be orthogonal representations of G_1 and G_2 , respectively. Then $R \circ S$ is an orthogonal representation of $G_1 \wedge G_2$ with*

$$\mu_{R \circ S}(G_1 \wedge G_2) = \mu_R(G_1) \mu_S(G_2).$$

Proof. In fact, for any $u \in R$ and $v \in S$,

$$|u \circ v|^2 = \sum_{i,j} (u_i v_j)^2 = \sum_i u_i^2 \sum_j v_j^2 = 1.$$

If (u, v) and (x, y) are two non-adjacent vertices of $G_1 \wedge G_2$, then either u and x are non-adjacent in G_1 or v and y are non-adjacent in G_2 , so $\langle u, x \rangle = 0$ or $\langle v, y \rangle = 0$. Then the inner product

$$\langle u \circ x, v \circ y \rangle = \langle u, x \rangle \langle v, y \rangle = 0.$$

□

Theorem 11.5 Suppose that $T = \{v^{(1)}, \dots, v^{(m)}\}$ is an orthonormal representation of G with constant $\sigma_T(G)$, then

$$\Theta(G) \leq \frac{1}{\sigma_T(G)}.$$

Proof. Repeatedly using Lemma 11.10, we know that T^n is an orthonormal representation of $\wedge^n G$ with constant $\mu_{T^n}(\wedge^n G) = \mu_T(G)^n$. Thus by Lemma 11.8,

$$\alpha(\wedge^n G) \leq \frac{1}{\mu_{T^n}(\wedge^n G)} = \frac{1}{\sigma_T(G)^n},$$

which yields $\sqrt[n]{\alpha(\wedge^n G)} \leq 1/\sigma_T(G)$, hence the inequality as desired. □

An “umbrella” used by Lovász in the proof of the following theorem is called “Lovász umbrella”.

Theorem 11.6 $\Theta(C_5) = \sqrt{5}$.

Proof. For the graph C_5 , Lovász obtained an orthogonal representation T in R^3 by considering an “umbrella” with five ribs $v_1, \dots, v_5$ of unit length. Now open the umbrella (with tip at the origin) to the point where the angles between alternate ribs are $\pi/2$. Let h be the distance from the origin to the plan determined by end-vertices of ribs, then a simple calculation show that $h^2 = \frac{1}{\sqrt{5}}$. So $\bar{v} = (0, 0, h) = (0, 0, 1/\sqrt[4]{5})$ hence

$$\sigma_T(C_5) = \langle v^{(i)}, \bar{v} \rangle = h^2 = \frac{1}{\sqrt{5}},$$

which and Theorem 11.5 prove that $\Theta(C_5) \leq \sqrt{5}$. The inverse inequality has been obtained. □

In order to improve the obtained upper bound $\Theta(C_k) \leq k/2$ for odd k , we are going to find the eigenvalues of the adjacency matrix of C_k first.

Lemma 11.11 *Let $k = 2m + 1 \geq 3$ be an integer. Then*

$$\lambda_\ell = 2 \cos \frac{2\ell\pi}{k} \quad (\ell = 0, 1, \dots, k-1)$$

are all eigenvalues of the adjacency matrix of C_k , in which

$$\max_\ell \lambda_\ell = \lambda_0 = 2 \quad \text{and} \quad \min_\ell \lambda_\ell = \lambda_m = -2 \cos \frac{\pi}{k}.$$

Proof. Let $A = (a_{ij})$ be the adjacency matrix of C_k , then

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 & 1 \\ 1 & 0 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ \cdots & & & & & & \cdots \\ 0 & 0 & 0 & \cdots & 1 & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 1 & 0 \end{pmatrix},$$

in which each row contains two zeros. Let $\zeta = e^{2\pi i/k}$. Note that $1, \zeta, \dots, \zeta^{k-1}$ are all k th roots of unity. Denote by β for ζ^ℓ with $0 \leq \ell \leq k-1$. We shall show that $X^{(\ell)} = (1, \beta, \beta^2, \dots, \beta^{k-1})^T$ is an eigenvector of A corresponding to eigenvalue $\beta + \beta^{-1} = \zeta^\ell + \zeta^{-\ell}$. In fact,

$$AX^{(\ell)} = \begin{pmatrix} \beta + \beta^{k-1} \\ \beta^2 + 1 \\ \beta^3 + \beta \\ \vdots \\ 1 + \beta^{k-2} \end{pmatrix} = (\beta + \beta^{-1})X^{(\ell)}.$$

Since $X^{(0)}, X^{(1)}, \dots, X^{(k-1)}$ are independent (seen from the Vandermonde matrix formed by them), so

$$\lambda_\ell = \zeta^\ell + \zeta^{-\ell} = 2 \cos \frac{2\ell\pi}{k} \quad (\ell = 0, 1, \dots, k-1),$$

are all eigenvalues of A , which are decreasing from $\ell = 0$ to m and the increasing, proving the lemma. $\square$

We need to recall some results in linear algebra:

Facts from Linear Algebra. If $M = (m_{ij})$ is an $m \times m$ real symmetric matrix, then it has m real eigenvalues. Furthermore, if all such eigenvalues are non-negative, then there are vectors $v^{(1)}, \dots, v^{(m)}$ in R^s with $s = \text{rank}(M)$ such that $m_{ij} = \langle v^{(i)}, v^{(j)} \rangle$.

Let $A = (a_{ij})$ be the matrix of graph G of order k with eigenvalues

$$\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k.$$

Since $\sum \lambda_i = \sum d_{ii} = 0$, so $\lambda_k < 0$ (unless G has no edge). Let $p = |\lambda_k| = -\lambda_k$, then the matrix

$$M = I + \frac{1}{p}A,$$

where I is the identity matrix, has k eigenvalues

$$1 + \frac{\lambda_1}{p} \geq 1 + \frac{\lambda_2}{p} \geq \dots \geq 1 + \frac{\lambda_k}{p} = 0.$$

So we obtain a set of vectors $T = \{v^{(1)}, \dots, v^{(k)}\}$ in R^s with $s = \text{rank}(M)$ such that

$$\langle v^{(i)}, v^{(i)} \rangle = m_{ii} = 1 \quad \text{and} \quad \langle v^{(i)}, v^{(j)} \rangle = m_{ij} = \frac{a_{ij}}{p} \quad (i \neq j).$$

If two vertices v_i and v_j are not adjacent, then $a_{ij} = 0$, thus T forms an orthonormal representation of G . Now for an odd cycle C_k , we have

$$\begin{aligned} \sigma_T(C_k) &= \langle v^{(i)}, \bar{v} \rangle = \frac{1}{k} \langle v^{(i)}, v^{(1)} + \dots + v^{(k)} \rangle \\ &= \frac{1}{k} \left(1 + \frac{2}{p} \right) = \frac{1 + \cos(\pi/k)}{k \cos(\pi/k)}. \end{aligned}$$

So we obtain that for an odd $k \geq 3$,

$$\Theta(C_k) \leq \frac{1}{\sigma_T(C_5)} = \frac{k \cos(\pi/k)}{1 + \cos(\pi/k)}.$$

Theorem 11.7 *Let $k \geq 3$ be an odd integer. Then*

$$\Theta(C_k) \leq \frac{k \cos(\pi/k)}{1 + \cos(\pi/k)}.$$

□

Note the equality holds for $k = 3$ in the above theorem with $\Theta(C_3) = 1$, and it does also for $k = 5$ with $\Theta(C_5) = \sqrt{5}$ and $\cos(\pi/5) = (\sqrt{5} + 1)/4$. Whether or not it holds for $k \geq 7$ is unknown.

We now introduce Lovász theta function briefly.

Let $T = \{v^{(1)}, \dots, v^{(m)}\}$ with $v^{(i)} \in R^r$ be an orthonormal representation of graph G of order m . Define

$$val_T(G) = \min_c \max_{1 \leq i \leq m} \frac{1}{\langle c, v^{(i)} \rangle^2},$$

where c runs over all unit vectors in R^r . The vector yielding the minimum is called the *handle* of the representation. The name “handle” comes from the proof for $\Theta(C_5) = \sqrt{5}$. The theta function is defined as

$$\vartheta(G) = \min_T val_T(G),$$

where T runs over all representations of G . Call a representation *optimal* if it achieves the minimum value.

Lemma 11.12 $\vartheta(G_1 \wedge G_2) = \vartheta(G_1)\vartheta(G_2)$.

Proof. Let $R = \{u^{(1)}, \dots, u^{(m)}\}$ in R^r and $S = \{v^{(1)}, \dots, v^{(n)}\}$ in R^s be optimal orthonormal representations of G_1 and G_2 with handles c and d , respectively. Then $R \circ S$ is an orthonormal representation of $G_1 \wedge G_2$ and $c \circ d$ is a unit vector of R^{rs} . Hence

$$\begin{aligned} \vartheta(G_1 \wedge G_2) &\leq \max_{i,j} \frac{1}{\langle c \circ d, u^{(i)} \circ v^{(j)} \rangle^2} \\ &= \max_{i,j} \frac{1}{\langle c, u^{(i)} \rangle^2} \frac{1}{\langle d, v^{(j)} \rangle^2} \\ &= \vartheta(G_1)\vartheta(G_2). \end{aligned}$$

□

In fact, the equality holds in the above lemma.

Lemma 11.13 $\alpha(G) \leq \vartheta(G)$.

Proof. Let $V(G) = \{v_1, \dots, v_m\}$ be the vertex set of G , and let $R = \{v^{(1)}, \dots, v^{(m)}\}$ be an optimal orthonormal representation of G in R^r with handle C . Suppose that $\{v_1, \dots, v_k\}$ be an independent set of G with $k = \alpha(G)$. Then the vectors in $S = \{v^{(1)}, \dots, v^{(k)}\}$ are pairwise orthogonal, thus they can be extended to a base of R^r by adding some unit vectors $w^{(1)}, \dots, w^{(r-k)}$. So

$$\begin{aligned} 1 &= |c|^2 = \sum_{i=1}^k \langle c, v^{(i)} \rangle^2 + \sum_{i=1}^{r-k} \langle c, w^{(i)} \rangle^2 \\ &\geq \sum_{i=1}^k \langle c, v^{(i)} \rangle^2 \geq \frac{k}{\vartheta(G)}, \end{aligned}$$

where the last inequality holds as by definition $\vartheta(G) \geq \frac{1}{\langle c, v^{(i)} \rangle^2}$ for each i .
 $\square$

Theorem 11.8 $\Theta(G) \leq \vartheta(G)$.

Proof. By above two lemmas,

$$\alpha(\wedge^n G) \leq \vartheta(\wedge^n G) \leq \vartheta(G)^n,$$

which follows by the inequality as required immediately. $\square$

Using linear algebra extensively, with a special case in a general upper bound, Lovász obtained that for odd $k \geq 3$,

$$\vartheta(C_k) = \frac{k \cos(\pi/k)}{1 + \cos(\pi/k)},$$

which is an upper bound for $\Theta(C_k)$ as we have seen.

11.3 A Connection

The independence number of a graph product is important for the corresponding communication channel. As the capacity of a channel is described by the independence number of the graph, the following definition comes naturally. For integers $k_1 \geq 2, \dots, k_n \geq 2$, define

$$\begin{aligned} & \rho(k_1, \dots, k_n) \\ = & \max\{\alpha(G_1 \wedge \dots \wedge G_n) : \alpha(G_i) < k_i, i = 1, \dots, n\}. \end{aligned}$$

From the equality (11.1), we have

Proposition 11.3 *Let $k_1 \geq 2, \dots, k_n \geq 2$ be integers. Then*

$$\begin{aligned} & \rho(k_1, \dots, k_n) \\ = & \max\{\omega(G_1 \vee \dots \vee G_n) : \omega(G_i) < k_i, i = 1, \dots, n\}. \end{aligned}$$

$\square$

When $k_1 = \dots = k_n = k \geq 2$,

$$\begin{aligned} & \rho(k, \dots, k) \\ = & \max\{\alpha(G_1 \wedge \dots \wedge G_n) : \alpha(G_i) < k, i = 1, \dots, n\}, \end{aligned}$$

where $G_1, \dots, G_n$ are formally independent in the definition. However, Erdős, McEliece, and Taylor (1971), and later Alon and Orlitsky (1995) proved that $G_1, \dots, G_n$ can be taken to one graph G . That is to say, $\rho(k, \dots, k) = \rho_n(k)$, where

$$\rho_n(k) = \max\{\alpha(\wedge^n G) : \alpha(G) < k\}.$$

Theorem 11.9 *Let $k \geq 2$ be an integer. Then $\rho_n(k) = \rho(k, \dots, k)$. That is to say,*

$$\begin{aligned} & \max\{\alpha(\wedge^n G) : \alpha(G) < k\} \\ = & \max\{\alpha(G_1 \wedge \dots \wedge G_n) : \alpha(G_i) < k, i = 1, \dots, n\}. \end{aligned}$$

Namely,

$$\begin{aligned} & \max\{\omega(\vee^n G) : \omega(G) < k\} \\ = & \max\{\omega(G_1 \vee \dots \vee G_n) : \omega(G_i) < k, i = 1, \dots, n\}. \end{aligned}$$

Proof. We prove the second equality. Denote by ρ for $\rho(k, \dots, k)$, that is,

$$\begin{aligned} \rho &= \rho(k, \dots, k) \\ &= \max\{\omega(G_1 \vee \dots \vee G_n) : \omega(G_i) < k, i = 1, \dots, n\}. \end{aligned}$$

Clearly $\rho_n(k) \leq \rho$. It suffices to show that there exists a graph with $\omega(G) < k$ such that $\omega(\vee^n G) \geq \rho$. By definition, there are graphs $G_1, \dots, G_n$ with $\omega(G_i) < k$ such that $G_1 \vee \dots \vee G_n$ contains a clique

$$\begin{aligned} S &= \{(x_{1,1}, x_{1,2}, \dots, x_{1,n}), (x_{2,1}, x_{2,2}, \dots, x_{2,n}), \\ &\quad \dots, (x_{\rho,1}, x_{\rho,2}, \dots, x_{\rho,n})\} \end{aligned}$$

with size $|S| = \rho$. Let G be the graph on the vertex set

$$\begin{aligned} V &= \{1, 2, \dots, \rho\} \times \{1, 2, \dots, n\} \\ &= \{(1, 1), (1, 2), \dots, (1, n), (2, 1), (2, 2), \dots, (2, n), \\ &\quad \dots, (\rho, 1), (\rho, 2), \dots, (\rho, n)\}. \end{aligned}$$

The size of V is $n\rho$. Two distinct vertices (a, i) and (b, j) are adjacent if and only if $i = j$ and $x_{a,i}$ and $x_{b,i}$ are adjacent in G_i . Now let W be a clique of G with size ℓ ,

$$W = \{(a_1, i_1), (a_2, i_2), \dots, (a_\ell, i_\ell)\}.$$

Then $i_1 = i_2 = \dots = i_\ell$, denoted by i for this common value, and vertices $x_{a_1,i}, x_{a_2,i}, \dots, x_{a_\ell,i}$ form a clique in G_i . Therefore $|W| \leq \omega(G_i) < k$ and hence

$$\omega(G) \leq \max_{1 \leq i \leq n} \{\omega(G_i)\} < k.$$

However, consider the following set T of V^n of the vertex set of $\vee^n G$,

$$T = \{[(1, 1), (1, 2), \dots, (1, n)], \dots, [(\rho, 1), (\rho, 2), \dots, (\rho, n)]\}.$$

The size of T is ρ . Let

$$I = [(i, 1), (i, 2), \dots, (i, n)] \quad \text{and} \quad J = [(j, 1), (j, 2), \dots, (j, n)]$$

be two distinct vertices of T with $i \neq j$. They correspond to two distinct vertices

$$(x_{i,1}, x_{i,2}, \dots, x_{i,n}) \quad \text{and} \quad (x_{j,1}, x_{j,2}, \dots, x_{j,n})$$

of S , which is a clique of $G_1 \vee \dots \vee G_n$. Thus there is some m , $1 \leq m \leq n$, such that $x_{i,m}$ and $x_{j,m}$ are adjacent in G_m , hence (i, m) and (j, m) are adjacent in G . Thus I and J are adjacent in $\vee^n G$, and so T is a clique of $\vee^n G$. It follows by $\omega(\vee^n G) \geq |T| = \rho$, completing the proof. $\square$

More importantly, Erdős *et al.* obtained a relation of function $\rho(k_1, \dots, k_n)$ and Ramsey number $r(k_1, \dots, k_n)$ as follows.

Theorem 11.10 *Let $k_1 \geq 2, \dots, k_n \geq 2$ be integers. Then*

$$\rho(k_1, \dots, k_n) = r(k_1, \dots, k_n) - 1.$$

Proof. Set $\rho = \rho(k_1, \dots, k_n)$, and $r = r(k_1, \dots, k_n) - 1$. We first prove that $\rho \geq r$. Recall the definition that r is the largest integer for which there exists a coloring of edges of K_r with colors $\{1, 2, \dots, n\}$ such that any monochromatic clique in color i has size less than k_i for $1 \leq i \leq n$. Let

$$V = \{1, 2, \dots, r\}$$

be the vertex set of this K_r , and let G_i be the subgraph with vertex set V whose edge set consists of all edges in color i . Then $\omega(G_i) < k_i$. By considering the OR product graph $G = G_1 \vee \dots \vee G_n$, we have $\rho \geq \omega(G)$. On the other hand, G contains a set

$$S = \{(1, 1, \dots, 1), (2, 2, \dots, 2), \dots, (r, r, \dots, r)\}.$$

For any pair of vertices i and j of K_r with $1 \leq i \neq j \leq r$, they are adjacent in some G_ℓ , so the vertices $(i, i, \dots, i)$ and $(j, j, \dots, j)$ of S are adjacent in G . Thus S is a clique and hence $\rho \geq \omega(G) \geq r$.

We then prove that $r \geq \rho$. From the definition, there are graphs $G_1, \dots, G_n$ with $\omega(G_i) < k_i$ such that $G = G_1 \vee \dots \vee G_n$ contains a clique T of size ρ ,

$$T = \{(x_1^1, x_2^1, \dots, x_n^1), (x_1^2, x_2^2, \dots, x_n^2), \dots, (x_1^\rho, x_2^\rho, \dots, x_n^\rho)\}.$$

Define a coloring for the edges of K_ρ on vertex set

$$U = \{1, 2, \dots, \rho\}$$

with colors $\{1, 2, \dots, n\}$ in following way. For an edge ab in this K_ρ , consider two vertices in T

$$(x_1^a, x_2^a, \dots, x_n^a) \quad \text{and} \quad (x_1^b, x_2^b, \dots, x_n^b),$$

which are adjacent in G . There is some i so that $x_i^a \neq x_i^b$ and $x_i^a x_i^b$ is an edge of G_i . We color the edge ab of K_ρ with the such smallest color i , then all edges of K_ρ are colored.

We claim that there is no monochromatic clique of size k_i in any fixed color i . In fact, if

$$A \subseteq U = \{1, 2, \dots, \rho\}$$

is a monochromatic clique in color i , consider

$$X = \{x_i^a : a \in A\} \subseteq V(G_i).$$

For any distinct x_i^a and x_i^b of X , the edge ab of K_ρ is colored i , so $x_i^a x_i^b$ is an edge of G_i , thus X is a clique of G_i . Therefore

$$|A| = |X| \leq \omega(G_i) < k_i,$$

yielding the fact that $r \geq \rho$ and completing the proof. $\square$

11.4 Upper Bounds for $r_n(3)$

From last section, the study of ρ function is exactly that of multicolor Ramsey function. The first nontrivial such function is $r_n(3)$. In case that the number of colors is at least three, the only known exact value is $r_3(3) = 17$ (Greenwood and Gleason, see Chapter 2). For $n = 4$, we have $51 \leq r_4(3) \leq 62$, where the upper bound was established by Fettes, Kramer and Radziszowski (2004) while the lower bound was obtained by Chung (1973). In the following proofs, we shall write r_n for $r_n(3)$.

Lemma 11.14 *Let $n \geq 2$ be an integer. Then*

$$r_n(3) - 1 \leq 1 + n(r_{n-1}(3) - 1),$$

namely, $r_n(3) \leq 2 + n(r_{n-1}(3) - 1)$.

Proof. Let $N = r_n - 1$. There is an edge coloring of K_N with colors $\{1, \dots, n\}$ such that there is no monochromatic triangles. Denote by $N_i(v)$ for the neighborhood of v in color i . Then any edge between the vertices in $N_i(v)$ is colored by a color in $\{1, \dots, n\} \setminus \{i\}$. Since there is no monochromatic triangle in the $N_i(v)$, whose edges are $(n-1)$ colored, so $d_i(v) = |N_i(v)| \leq r_{n-1} - 1$. Therefore,

$$N - 1 = r_n - 2 = \sum_{i=1}^n d_i(v) \leq n(r_{n-1} - 1).$$

The desired upper bound follows. $\square$

Corollary 11.2 *Let k be a positive integer, and let $c = c(k)$ as*

$$c = \frac{r_k(3) - 1}{k!} + \sum_{t>k} \frac{1}{t!}.$$

Then $r_n(3) < cn! + 1$ for $n \geq k$. In particular, $r_n(3) < en! + 1$ for $n \geq 1$.

Proof. Using Lemma 11.14 repeatedly,

$$\begin{aligned} r_n - 1 &\leq 1 + n(r_{n-1} - 1) \\ &\leq 1 + n[1 + (n-1)(r_{n-2} - 1)] \\ &\leq 1 + n + n(n-1) + \dots + n(n-1) \cdots (k+1)(r_k - 1) \\ &= n! \left(\frac{1}{n!} + \frac{1}{(n-1)!} + \dots + \frac{1}{(k+1)!} + \frac{r_k - 1}{k!} \right), \end{aligned}$$

and the desired upper bound follows. In particular, $c = e$ for $k = 1$. $\square$

By taking $r_4(3) \leq 62$, then $r_n(3) < cn! + 1$, where $c = e - 1/6 = 2.5516\dots$. Further improvement can be obtained by noting the following fact.

Lemma 11.15 *Let both n and p be even integers. If $r_{n-1}(3) \leq p$, then*

$$r_n(3) - 1 \leq n(p - 1).$$

In particular, when both n and $r_{n-1}(3)$ are even, then

$$r_n(3) - 1 \leq n(r_{n-1}(3) - 1).$$

Proof. If $r_{n-1} \leq p-1$, the assertion follows from Lemma 11.14. So we assume that $r_{n-1} = p$, which is even. Set $N = r_n - 1$, then $N \leq 1 + n(r_{n-1} - 1)$ by Lemma 11.14. Suppose to the contrary, the assertion is false, then $N = 1 + n(r_{n-1} - 1)$ is odd. There is an edge coloring of K_N with colors $\{1, \dots, n\}$ such that there is no monochromatic K_3 in any color. For a fixed color j , let E_j be the set of edges in color j . Since $\sum_x d_j(x) = 2|E_j|$ is even, where the sum is over N vertices and N is odd, so there exists a vertex v with $d_j(v)$ is even. Since $d_j(v) \leq r_{n-1} - 1$ so $d_j(v) \leq r_{n-1} - 2$. From the fact that $N - 1 = \sum_{j=1}^n d_j(v)$, thus

$$N - 1 \leq r_{n-1} - 2 + (n - 1)(r_{n-1} - 1) = n(r_{n-1} - 1) - 1,$$

yielding what as desired. $\square$

Theorem 11.11 *Let $k \geq 2$ be an even integer and $r_k(3) \leq p$ for some odd integer p . Then for any $n \geq k$,*

$$r_n(3) \leq n! \left(\frac{p-1}{k!} + \sum_{k < t \leq n, t \text{ odd}} \frac{1}{t!} \right) + 1.$$

Proof. The idea of the proof is a combination of that in Wan (1997), and Chung and Graham (1999). For fixed k and $n \geq k$, set

$$A_n = \frac{p-1}{k!} + \sum_{k < t \leq n, t \text{ odd}} \frac{1}{t!}.$$

Note that

$$A_{2m} = A_{2m-1} \quad \text{and} \quad A_{2m+1} = A_{2m} + \frac{1}{(2m+1)!}$$

We shall prove that $r_n \leq n!A_n + 1$ by induction on $n \geq k$. The assertion holds clearly for $n = k$, and $(k+1)!A_{k+1} = 1 + (k+1)(p-1)$, so the assertion for $n = k+1$ follows from Lemma 11.14. We now suppose that $n \geq k+2$.

Let $n = 2m \geq k+2$ be an even integer. Then $n-1 \geq k+1$, and $n-1$ is odd. It is easy to verify that

$$(n-1)!A_{n-1} = (n-1)! \left(\frac{p-1}{k!} + \frac{1}{(k+1)!} + \dots + \frac{1}{(n-1)!} \right)$$

is an odd integer. Assume $r_{n-1} \leq (n-1)!A_{n-1} + 1$, where the right-hand side is even. From Lemma 11.15, we have

$$r_n - 1 \leq n((2m-1)!A_{2m-1}) = n!A_{2m} = n!A_n.$$

So $r_n \leq n!A_n + 1$. For r_{2m+1} , from Lemma 11.14 and what just proved,

$$\begin{aligned} r_{2m+1} - 1 &\leq 1 + (2m+1)(r_{2m} - 1) \\ &\leq 1 + (2m+1)((2m)!A_{2m}) \\ &= (2m+1)!A_{2m+1}, \end{aligned}$$

completing the proof. $\square$

We are more concerned what is the limit $r_n(3)/n!$ as $n \rightarrow \infty$.

Proposition 11.4 *As $n \rightarrow \infty$, the limit of $r_n(3)/n!$ exists.*

Proof. Clearly, the limit of $r_n(3)/n!$ exists if and only if that of $(r_n(3) - 1)/n!$ does, and the limits are the same if they exist. Denote by ℓ for $\liminf_{n \rightarrow \infty} (r_n - 1)/n!$. For any $\epsilon > 0$, there are infinite many k such that $(r_k - 1)/k! < \ell + \epsilon/2$. Take such large k that $(r_n - 1)/n! > \ell - \epsilon$ for $n \geq k$ and that $\sum_{t > k} 1/t! < \epsilon/2$. From Corollary 11.2, we have

$$\ell - \epsilon < \frac{r_n - 1}{n!} < \frac{r_k - 1}{k!} + \sum_{t > k} \frac{1}{t!} < \ell + \epsilon.$$

Thus $\lim_{n \rightarrow \infty} (r_n - 1)/n! = \ell$. $\square$

We conclude this section with following problem.

Problem 11.1 *Prove or disprove that the limit of $r_n(3)/n!$ is zero as $n \rightarrow \infty$.*

11.5 Schur Numbers and $r_n(3)$

It is hard to find exact classical Ramsey numbers as stated. According to the communication theory, asymptotic bounds are more important. This is because the limit

$$\lim_{n \rightarrow \infty} (r_n(3) - 1)^{1/n} = \lim_{n \rightarrow \infty} \rho_n(3)^{1/n}$$

represents the maximal amount of Shannon capacity over all triangle-free (with clique number 2) graphs. We shall see that this limit exists.

Proposition 11.5 *The function $r_n(3) - 1$ is super-multiplicative. That is to say, for any positive integers m and n ,*

$$r_{m+n}(3) - 1 \geq (r_m(3) - 1)(r_n(3) - 1).$$

Therefore the limit of $(r_n(3) - 1)^{1/n}$ hence the limit of $r_n(3)^{1/n}$ exist, which both are equal to $\sup_n (r_n(3) - 1)^{1/n}$.

Proof. Set $N = r_n - 1$ and $M = r_m - 1$. Color the edges of K_N with n colors and color edges of K_M with other m colors so that there is no monochromatic triangles in any color. Then, “blow-up” one with another by replacing each vertex v of K_N with a colored K_M , denoted by H_v for this K_M . For any distinct vertices u and v of K_N , if $u' \in V(H_u)$, and $v' \in V(H_v)$, color edge between u' and v' with the color as the edge uv in K_N . We thus have a complete graph on NM vertices whose edges are colored with $n + m$ colors and there is no monochromatic triangles. Therefore $r_{m+n} - 1 \geq (r_m - 1)(r_n - 1)$. $\square$

It seems to be very difficult to determine the exact value of the limit of $r_n(3)^{1/n}$. From what we discussed in this chapter,

$$r_{2n}(3) - 1 = \rho_{2n}(3) = \max\{\alpha(\wedge^{2n}G) : \alpha(G) < 3\},$$

which is at least

$$\alpha(\wedge^{2n}C_5) \geq (\alpha(\wedge^2C_5))^n = 5^n,$$

So $\lim r_n(3)^{1/n} \geq \sqrt{5}$. However, by a lemma in Section 1 in this chapter,

$$\lim_{n \rightarrow \infty} (r_n(3) - 1)^{1/n} = \sup_{n \geq 1} (r_n(3) - 1)^{1/n},$$

so

$$\begin{aligned} \lim_{n \rightarrow \infty} r_n(3)^{1/n} &= \lim_{n \rightarrow \infty} (r_n(3) - 1)^{1/n} \\ &\geq (r_4(3) - 1)^{1/4} = 50^{1/4} = 2.6591 \dots \end{aligned}$$

Nevertheless, we shall do it better.

The lower bound for $r_n(3)$ is closely related to the Schur number s_n . A set of integers is said to be sum-free if $a, b \in S$ implies $a + b \notin S$ (a and b not necessarily distinct). A well known result of Schur (1916) states that if the integers $\{1, 2, \dots, \lfloor n!e \rfloor\}$ are partitioned in any manner into n classes, then at least one of the classes is not sum-free. Accordingly, we define *Schur number* s_n to be the largest positive integer such that $\{1, 2, \dots, s_n\}$ can be partitioned in some manner into n sum-free classes. No many exact values of Schur numbers have been known. It is not hard to verify that $s_1 = 1$, $s_2 = 4$ and $s_3 = 13$. With the aid of a computer, Baumert showed that $s_4 = 44$, reported in Abbott and Hanson (1972), and Exoo (1994) showed that $s_5 \geq 160$, who partitioned $[1, 160]$ into five sum-free sets. Since the partition is symmetric in the sense that i and $161 - i$ always belong to the same set, so only the integers from 1 to 80 are listed as follows.

Set 1	4 5 15 16 22 28 29 39 40 41 42 48 49 59
Set 2	2 3 8 14 19 20 24 25 36 46 47 51 62 73
Set 3	7 9 11 12 13 17 27 31 32 33 35 37 53 56 57 61 79
Set 4	1 6 10 18 21 23 26 30 34 38 43 45 50 54 65 74
Set 5	44 52 55 58 60 63 64 66 67 68 69 70 71 72 75 76 77 78 80

Table A partition of $[1, 160]$ into five sum-free sets.

Generally, Schur number s_n offers a good lower bound for Ramsey number $r_n(3)$ as follows.

Lemma 11.16 *Let n be a positive integer. Then*

$$r_n(3) \geq s_n + 2.$$

Proof. Set $N = s_n$. Then $\{1, 2, \dots, N\}$ can be partitioned into n sum-free sets $A_1, A_2, \dots, A_n$. We now color the edges of K_{N+1} with vertex set $\{1, 2, \dots, N+1\}$ in the following way: color the edge uv with color i if $|u - v| \in A_i$. Since $1 \leq |u - v| \leq N$ for any distinct u and v , all edges are colored. We then claim that there are no monochromatic triangles. In fact, if for some $1 \leq u < v < w \leq N+1$, the edges uv and vw are in color i , then $|u - v| = v - u \in A_i$ and $|v - w| = w - v \in A_i$, then $|u - w| = w - u = (w - v) + (v - u) \notin A_i$ since A_i is sum-free. So the edge uw are not in color i . Therefore u, v, w do not form a monochromatic triangle and the assertion holds. $\square$

Corollary 11.3 $r_5(3) \geq 162$.

Proof. The proof follows from the above lemma and the fact that $s_5 \geq 160$ immediately. $\square$

The following result is due to Abbott and Hanson (1972).

Theorem 11.12 *Let m and n be any positive integers. Then*

$$s_{m+n} \geq (2s_m + 1)s_n + s_m.$$

Proof. For $b = 0, 1, \dots, s_n$, set

$$A_b = \{b(2s_m + 1) + c \mid c = 1, 2, \dots, s_m\},$$

and for $c = 1, 2, \dots, s_m$, set

$$A^c = \{b(2s_m + 1) + c \mid b = 0, 1, \dots, s_n\}.$$

Let

$$A = \cup_{b=0}^{s_n} A_b = \cup_{c=1}^{s_m} A^c.$$

Partition $A_0 = \{1, 2, \dots, s_m\}$ into m sum-free classes $C_1, C_2, \dots, C_m$, and partition A into m classes $D_1, D_2, \dots, D_m$ by placing A^c in D_i if $c \in C_i$. We claim that each of these classes is sum-free. Suppose to the contrary, D_i is not sum-free, there are some elements $b_1M + c_1, b_2M + c_2, b_3M + c_3$ of D_i such that

$$(b_1M + c_1) + (b_2M + c_2) = b_3M + c_3,$$

where $M = 2s_m + 1$, and c_1, c_2, c_3 are all in C_i . We thus have

$$c_1 + c_2 = c_3 \pmod{M}.$$

However, $2 \leq c_1 + c_2 \leq 2s_m < M$, and $1 \leq c_3 \leq s_m < M$, so $c_1 + c_2 = c_3$, contradicting to the fact that C_i is sum-free.

For $b = 1, 2, \dots, s_n$, set

$$B_b = \{b(2s_m + 1) - c \mid c = 0, 1, \dots, s_m\}.$$

Let

$$B = \cup_{b=1}^{s_n} B_b.$$

Partition $\{1, 2, \dots, s_n\}$ into n sum-free classes $C_{m+1}, C_{m+2}, \dots, C_{m+n}$, and partition B into n classes $D_{m+1}, D_{m+2}, \dots, D_{m+n}$ by placing B_b in D_{m+i} if $b \in C_{m+i}$. We claim that each of these classes is sum-free. In fact, for any elements $b_1M - c_1, b_2M - c_2, b_3M - c_3$ of D_{m+i} , where $M = 2s_m + 1$, and b_1, b_2, b_3 are of C_{m+i} , satisfying $b_1 + b_2 \neq b_3$ since C_{m+i} is sum-free. If $b_1 + b_2 \geq b_3 + 1$, since $M - c_1 - c_2 \geq 1$ we have

$$(b_1M - c_1) + (b_2M - c_2) \geq b_3M + 1 > b_3M - c_3.$$

If $b_1 + b_2 \leq b_3 - 1$, then

$$(b_1M - c_1) + (b_2M - c_2) \leq (b_3 - 1)M < b_3M - c_3.$$

Therefore $(b_1M - c_1) + (b_2M - c_2) \neq b_3M - c_3$, which implies that D_{m+i} is sum-free.

It is easy to verify that $A_0, B_1, A_1, B_2, \dots, A_{s_n}$ is a partition of the set

$$A \cup B = \{1, 2, \dots, (2s_m + 1)s_n + s_m\}$$

in the natural order. The above set thus has been partitioned into $m + n$ sum-free classes, hence the proof is completed. $\square$

Corollary 11.4 *Let m be a fixed positive integer. Then we have*

$$s_n > c(2s_m + 1)^{n/m}$$

for any integer $n > m$, where $c = c(m) > 0$ is a constant.

Proof. Setting $n = km + r$ with $1 \leq r \leq m$, we have

$$\begin{aligned} s_n &\geq (2s_m + 1)s_{(k-1)m+r} + s_m \\ &\geq (2s_m + 1)^k s_r + (2s_m + 1)^{k-1} s_m + \cdots + (2s_m + 1)s_m + s_m \\ &> \frac{s_r}{(2s_m + 1)^{r/m}} (2s_m + 1)^{n/m}. \end{aligned}$$

Taking

$$c = \min\{s_r/(2s_m + 1)^{r/m} : 1 \leq r \leq m\},$$

the desired result follows. $\square$

From above two results, we know that s_n is super-multiplicative and

$$\lim_{n \rightarrow \infty} s_n^{1/n} \geq \sup_{m \geq 1} (2s_m + 1)^{1/m} \geq \sup_{m \geq 1} s_m^{1/m} = \lim_{n \rightarrow \infty} s_n^{1/n}.$$

Hence

$$\lim_{n \rightarrow \infty} s_n^{1/n} = \sup_{m \geq 1} (2s_m + 1)^{1/m} \geq (2s_5 + 1)^{1/5} \geq 321^{1/5},$$

where $s_5 \geq 160$ and $321^{1/5} = 3.17176 \cdots$. We thus have

$$s_n \geq c 321^{n/5},$$

and

$$321^{1/5} \leq \lim r_n(3)^{1/n} \leq \infty.$$

The following is a very old conjecture of Erdős .

Conjecture 11.1 (Erdős) *As $n \rightarrow \infty$, the limit $r_n(3)^{1/n}$ is infinity.*

11.6 A Related Bound

We now give an upper bound for $r_n(C_5)$. Recall a result in Chapter 3 that if a graph $G = (V, E)$ is C_{2m+1} -free, then

$$\alpha(G) \geq \frac{1}{(2m-1)2^{(m-1)/m}} \left(\sum_{v \in V} d(v)^{1/(m-1)} \right)^{(m-1)/m}.$$

In particular, if G is C_5 -free, then $\alpha(G) \geq \sqrt{Nd/18}$, where N is the order of G and d is its average degree.

Theorem 11.13 *For some constant $c > 0$,*

$$r_n(C_5) \leq c\sqrt{18^n n!}.$$

Proof. We write $r_n(C_5)$ as r_n in the proof. Set $N = r_n - 1$. There is a edge-coloring of K_N with n colors containing no monochromatic C_5 . The edge set of K_N is partitioned into n classes $E_1, E_2, \dots, E_n$, where E_i consists of all edges in color i . One of the classes, say E_1 , has the maximum cardinality, so $|E_1| \geq \frac{1}{n} \binom{N}{2}$. Denote by G for the graph with the same vertex set as the colored K_N and edge set E_1 . It is easy to verify that the average degree d of G satisfies

$$d \geq \frac{N-1}{n} = \frac{r_n-2}{n}.$$

Consider an independent set S of G with $|S| = \alpha(G)$, any edge of K_N between two vertices in S is colored in color $2, 3, \dots, n$. So the subgraph induced by S is an edge-colored complete graph in $n-1$ colors, which contains no monochromatic C_5 . Thus $|S| \leq r_{n-1} - 1$. By the result mentioned previously, we have

$$r_{n-1} - 1 \geq \sqrt{Nd/18} \geq \sqrt{(r_n-1)(r_n-2)/18n} > \frac{r_n-2}{\sqrt{18n}},$$

implying

$$r_n - 1 \leq 1 + \sqrt{18n}(r_{n-1} - 1).$$

Repeatedly using the above recursive upper bound, we have

$$\begin{aligned} r_n - 1 &\leq 1 + \sqrt{18n}(r_{n-1} - 1) \\ &\leq 1 + \sqrt{18n} + \sqrt{18^2 n(n-1)}(r_{n-2} - 1) \\ &\leq 1 + \sqrt{18n} + \dots + \sqrt{18^{n-1} n(n-1) \dots 2}(r_1 - 1) \\ &= \sqrt{18^n n!} \left(\frac{1}{\sqrt{18^n n!}} + \frac{1}{\sqrt{18^{n-1} (n-1)!}} + \dots + \frac{r_1 - 1}{\sqrt{18}} \right), \end{aligned}$$

where $r_1 = 5$. Clearly, the sum in the last parenthesis has a finite limit. Hence there exists some constant $c > 0$ such that the required upper bound holds. $\square$

We conclude this chapter with two problems.

Problem 11.2 Let m be a positive integer. Prove or disprove that $r_n(C_{2m+1}) = o((n!)^{1/m})$.

Problem 11.3 Let m be a positive integer. Prove or disprove that $r_n(C_{2m+1})^{1/n} \rightarrow \infty$ as $n \rightarrow \infty$.

A conjecture of Erdős and Graham that $r_n(C_{2m+3})/r_n(C_3) \rightarrow 0$ can be generalized as follows

Problem 11.4 Prove or disprove that $r_n(C_{2m+3})/r_n(C_{2m+1}) \rightarrow 0$ as $n \rightarrow \infty$.

Exercises

11.1 Prove $\alpha(\wedge^4 \overline{C_7}) \geq 17$. (T. Bohman and R. Holzman, A nontrivial lower bound on the Shannon capacities of the complements of odd cycles, *IEEE Trans. Inform. Theory*, **49** (2003), 721-722.

11.2 Prove $s_3 = 13$.

11.3 Give a constructive proof for $s_k \geq 3s_{k-1} + 1$ with $k \geq 2$ by partitioning $\{1, 2, \dots, 3s_{k-1} + 1\}$ into three classes, each of which consists of consecutive integers.

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Chapter 12

AP and Regularity Lemma

12.1 Van der Waerden's Numbers

Many results in early Ramsey theory are concerning integers. One of them is on Schur numbers, discussed in the last chapter. And another is van der Waerden's theorem (1927) on existence of an arithmetic progression (AP) in partition of integers as follows, in which the *length* of a sequence is the number of its terms.

Theorem 12.1 (Van der Waerden) *Let k and t be positive integers. Then if w is a sufficiently large integer, no matter how we partition $[w] = \{1, 2, \dots, w\}$ into k classes, one of the classes must contain an arithmetic progression of length t .*

As usual, partition of $[w]$ is called a coloring of $[w]$. Define $w_k(t)$ as the smallest integer w so that the mentioned property holds, and write $w(t) = w_2(t)$. It is trivial to see that $w_k(1) = 1$, $w_k(2) = k+1$ and $w_1(t) = t$. However, for general k and t , the functions $w_k(3)$ and $w_2(t)$ are not trivial at all. It is only known that $w(2) = 3$, $w(3) = 9$, $w(4) = 35$, $w(5) = 178$. In fact, very little is known about the growth of the functions $w(t)$ and $w_k(t)$. The current best upper bound for $w(t)$, which was obtained by Shelah in 1988, is a tower function as $2^{2^{\dots^2}}$, where the height of the tower is $w(t-1)$.

Rather than proving van der Waerden's theorem directly, we shall introduce a remarkable extension of the theorem by Hales and Jewett in 1963, which leads to the proof of Shelah.

The *cube* C_t^n is the set of sequences of length n formed from t symbols. The symbols are ordered, usually as $[t] = \{1, 2, \dots, t\}$ if no specified, or in example as $\{A, B, C, \dots, Z\}$ with $t = 26$. The symbol in the i th position of an element is called its i th coordinate. The elements are called *points* often.

Some points of C_{26}^5 : $P_1 : AAAAB$, $P_2 : AXYCN$, $P_3 : PPPPP$. The third coordinate of P_2 is Y .

The points of C_t^n can be viewed as the points of the n -dimensional discrete cube with t sides. For example, the points of C_4^2 with symbols 1, 2, 3, 4 can be arranged in a matrix.

11	12	13	14
21	22	23	24
31	32	33	34
41	42	43	44

A set of t points $L = \{P_1, P_2, \dots, P_t\}$ in C_t^n is called a *line* if there is a non-empty set I of indices such that for any $i \in I$, the i th coordinate of any point $P_j \in L$ is just j th symbol, and for $i \notin I$, the i th coordinates of all points of L are the same. The i th coordinates for $i \in I$ are called *moving* coordinates, and those for $i \notin I$ are called *constant* coordinates.

Let us look two lines L_1 and L_2 in C_{26}^5 as follows.

	L_1	L_2
$P_1 :$	$XAQAB$	$AAAAA$
$P_2 :$	$XBQBB$	$BABAA$
$P_3 :$	$XCQCB$	$CACAA$
$\vdots$	$\vdots$	$\vdots$
$P_{25} :$	$XYQYB$	$YAYAA$
$P_{26} :$	$XZQZB$	$ZAZAA$

In line L_1 , the first, third, and fifth coordinates are constant coordinates; the second and fourth are moving coordinates.

In C_4^2 , there are totally 9 lines, namely, each row and each column, and one diagonal line 11, 22, 33, 44. The last line does not have constant coordinates.

Clearly, every line has t points, and there are

$$\sum_{\emptyset \neq I \subseteq [n]} t^{n-|I|} = \sum_{J \subseteq [n], J \neq [n]} t^{|J|} = \sum_{j=0}^{n-1} \binom{n}{j} t^j = (t+1)^n - t^n$$

lines in C_t^n . For example, the cube C_t^2 contains $2t+1$ lines: t horizontal lines, t vertical lines, and one diagonal line.

The following is a result of Hales and Jewett in 1963.

Theorem 12.2 *For any positive integers k and t , there exists N such that if the points of C_t^N are colored in k colors in any fashion, there is a monochromatic line.*

The *Hales-Jewett function* $HJ(k, t)$ is defined the minimal value of N that will satisfy the above theorem. To see that the function makes sense, one needs to make sure that $N + 1$ will do the theorem. This is clear if we consider the subset of C_t^{N+1} consisting of those points whose last coordinates are $N + 1$, and none of other coordinates are $N + 1$. More importantly, we need to show that there is at least one such an N that is finite.

The theorem of van der Waerden follows Theorem 12.2 immediately.

Proof of Theorem 12.1. For given k and t , let M be an integer such that any coloring of C_t^M with k colors contains a monochromatic line, where the set of symbols of C_t^M is $\{0, 1, \dots, t-1\}$. Then each point can be viewed as a representation of a nonnegative integer in base t . The largest such integer is $t^M - 1$, denoted by N . This representation establishes a bijection from C_t^M to $\{0, 1, \dots, N\}$. A key observation is that a line in C_t^M is exactly an arithmetic progression of length t . Now for any coloring of $\{1, 2, \dots, N+1\}$ with k colors, yielding a natural coloring for $\{0, 1, \dots, N\}$ hence a coloring of C_t^M with k colors. From Theorem 12.2, we have a monochromatic line in C_t^M hence a monochromatic arithmetic progression of length t . $\square$

The above proof in fact gives an upper bound for $w_k(t)$.

Theorem 12.3 *For any positive integers k and t , $w_k(t) \leq HJ(k, t)$.*

The remaining part of this section is Shelah's proof of Theorem 12.2. However, in order to make the proof clearer, we shall give some examples and details to explain the concepts in the context.

A line L in C_t^n is called a *Shelah line* if for some i, j with $1 \leq i \leq j \leq n$, the following holds:

1. Coordinates $1, \dots, i-1$ are all equal to $t-1$ (constant);
2. Coordinates $i, \dots, j$ are moving;
3. Coordinates $j+1, \dots, n$ are all equal to t (constant).

Namely, a line is a Shelah line if the positions of moving coordinates are consecutive, and all coordinates are $t-1$ and t before and after moving

coordinates, respectively. In the above definition, if $i = 1$, the first condition disappears, and if $j = n$ the third condition disappears. A point is called *Shelah point* if it is on some Shelah line.

Let us look some Shelah lines in C_{26}^5 as follows.

L_1	L_2	L_3
YAAZZ	AAZZZ	AAAAA
YBBZZ	BBZZZ	BBBBB
YCCZZ	CCZZZ	CCCCC
$\vdots$	$\vdots$	$\vdots$
YYYZZ	YYZZZ	YYYYY
YZZZZ	ZZZZZ	ZZZZZ
$i = 2, j = 3$	$i = 1, j = 2$	$i = 1, j = 5$

The points $YYQZZ$, $YCCZ$, $HHHHZ$, $SSSSS$ are Shelah points, but $YFGZZ$, $AAYYY$ are not.

The following result is clear from the fact that a Shelah line is uniquely determined by i and j in the definition.

Lemma 12.1 *The cube C_t^n contains $\binom{n+1}{2}$ Shelah lines and at most $\binom{n+1}{2}t$ Shelah points.*

Assume $n = n_1 + n_2 + \cdots + n_s$, then

$$C_t^n = C_t^{n_1} \times C_t^{n_2} \times \cdots \times C_t^{n_s}.$$

Let L_j be a Shelah line of $C_t^{n_j}$ for $j = 1, 2, \dots, s$. We call

$$L_1 \times L_2 \times \cdots \times L_s$$

as a *Shelah s-space* of C_t^n . It contains t^s points of C_t^n .

For example, if $n_1 = 5, n_2 = 6$, then

$$\{Y\alpha\alpha ZZ \mid YY\beta\beta ZZ\},$$

where α and β are letters in alphabet and the vertical line is added for clarity of conjunction, forms a Shelah 2-space in C_{26}^{11} . It contains 26^2 points. If $n_1 = 3, n_2 = 4, n_3 = 4$, then

$$\{Y\alpha Z \mid Y\beta\beta Z \mid \gamma\gamma ZZ\}$$

forms a Shelah 3-space in C_{26}^{11} . It contains 26^3 points.

We now define a canonical isomorphism ϕ between a Shelah s-space of C_t^n and C_t^s as follows.

$$\phi : L_1 \times L_2 \times \cdots \times L_s \rightarrow C_t^s, \quad \phi(\eta) = \alpha_1\alpha_2 \cdots \alpha_s,$$

where α_j is the common value of the moving coordinate in L_j . Using the previous examples,

$$\begin{aligned}\phi(Y\alpha\alpha ZZ YY\beta\beta ZZ) &= \alpha\beta \in C_{11}^2 \\ \phi(Y\alpha Z Y\beta\beta Z \gamma\gamma ZZ) &= \alpha\beta\gamma \in C_{11}^3.\end{aligned}$$

It is easy to see that the map ϕ is really a bijection.

During the rest of the proof, χ is used as a coloring function, thus $\chi(P)$ is the color of the point $P \in C_t^n$. A coloring χ of the points of C_t^n is called *flip-top* coloring if $P = p_1 p_2 \cdots p_n$, $Q = q_1 q_2 \cdots q_n \in C_t^n$, for which there exists $I \subseteq [n]$ such that $p_i = t-1$, $q_i = t$ for $i \in I$, and $p_j = q_j$ for $j \notin I$, then $\chi(P) = \chi(Q)$, where $t-1$ and t are last two elements of the set of symbols. For example, a flip-top coloring χ of C_{26}^5 satisfies

$$\begin{aligned}\chi(BAZZO) &= \chi(BAYYO) \\ \chi(ZEZAK) &= \chi(HEYAK) \\ \chi(YYYYY) &= \chi(ZZZZZ).\end{aligned}$$

Let χ be a coloring of the points of $L_1 \times L_2 \times \cdots \times L_s$, and let ϕ be the canonical isomorphism from $L_1 \times L_2 \times \cdots \times L_s$ to C_t^s . Define a coloring χ' of the points of C_t^s as for $P \in C_t^s$,

$$\chi'(P) = \chi(\phi^{-1}(P)),$$

which is called the *derived coloring* of χ . The coloring χ is called *flip-top* on the Shelah s-space if χ' is flip-top on C_t^s .

For example, if χ is a flip-top coloring of the Shelah 2-space in the form of $\{Y\alpha\alpha ZZ | YY\beta\beta\beta Z\}$, then

$$\chi(Y\mathbf{Q}\mathbf{Q}ZZ | YY\mathbf{Y}\mathbf{Y}\mathbf{Y}Z) = \chi(Y\mathbf{Q}\mathbf{Q}ZZ | YY\mathbf{Z}\mathbf{Z}\mathbf{Z}Z)$$

since χ' is flip-top on C_{26}^2 thus $\chi'(QY) = \chi'(QZ)$.

If $s = 1$, the above definition gives flip-top coloring of a Shelah line. In this case ϕ maps a Shelah line L in C_t^n onto C_t^1 , and the derived coloring on C_t^1 must be flip-top. However, a flip-top coloring of C_t^1 simply says that the color of $t-1$ equals to the color of t . This gives the following remark.

Remark The points of a Shelah line $L = \{P_1, P_2, \dots, P_t\}$ in C_t^n are colored flip-top if and only if the “last two points” P_{t-1} and P_t have the same color.

Lemma 12.2 (Shelah Cube Lemma) *Let positive integers k, s, t be fixed. Then there exist $n_1, n_2, \dots, n_s$ with the following property: If the points of*

$C_t^n = C_t^{n_1+n_2+\dots+n_s}$ are colored in an arbitrary way with k colors, then this coloring is a flip-top coloring on a suitable Shelah s -space $L_1 \times L_2 \times \dots \times L_s$, where L_j is a Shelah line in $C_t^{n_j}$. In fact, the numbers n_j can be explicitly given by $n_1 = k^{t^{s-1}}$ and $n_{i+1} = k^{A_i}$ for $1 \leq i < s$, where

$$A_i = \left[\prod_{1 \leq j \leq i} \binom{n_j + 1}{2} \right] t^{s-1}.$$

Proof. We shall prove the lemma by induction on s . For $s = 1$, take $n_1 = n = k$. We have to show that in an arbitrary k -coloring of C_t^k there is a flip-top colored Shelah-line. Consider the following points in C_t^k :

$$\begin{array}{llllll} P_1 : & t-1 & t-1 & \cdots & t-1 & t-1 & t-1 \\ P_2 : & t-1 & t-1 & \cdots & t-1 & t-1 & t \\ P_3 : & t-1 & t-1 & \cdots & t-1 & t & t \\ \vdots & & & & & & \\ P_k : & t-1 & t & \cdots & t & t & t \\ P_{k+1} : & t & t & \cdots & t & t & t \end{array}$$

Each of these $k+1$ points consists of a block of $t-1$ followed by a block of t . Two of them, say P_i and P_j with $i < j$, are colored with the same color in our k -coloring. And P_i and P_j are the last two points of the following Shelah line determined by themselves:

$$\begin{array}{llll|lll|lll} Q_1 : & t-1 & \cdots & t-1 & 1 & \cdots & 1 & t & \cdots & t \\ Q_2 : & t-1 & \cdots & t-1 & 2 & \cdots & 2 & t & \cdots & t \\ \vdots & & & & & & & & & \\ Q_{t-2} : & t-1 & \cdots & t-1 & t-2 & \cdots & t-2 & t & \cdots & t \\ P_i : & t-1 & \cdots & t-1 & \mathbf{t-1} & \cdots & \mathbf{t-1} & t & \cdots & t \\ P_j : & t-1 & \cdots & t-1 & \mathbf{t} & \cdots & \mathbf{t} & t & \cdots & t \end{array}$$

By the previous remark, the above line is exactly what we want: a flip-top colored Shelah line.

For $s = 2$, let $n_1 = k^t$ and $n_2 = k^{A_1}$ with $A_1 = \binom{n_1+1}{2}t$. Let χ be a k -coloring of the points of

$$C_t^n = C_t^{n_1+n_2} = C_t^{n_1} \times C_t^{n_2}.$$

Write points of C_t^n as XY with $X \in C_t^{n_1}$ and $Y \in C_t^{n_2}$. We shall define new colorings χ_1 and χ_2 on $C_t^{n_1}$ and $C_t^{n_2}$, respectively.

Let us define χ_2 on $C_t^{n_2}$ first. Denote by m for number of Shelah points in $C_t^{n_1}$ and label these Shelah points as $X_1, X_2, \dots, X_m$. For any $Y \in C_t^{n_2}$, we assign a vector for its color as

$$\chi_2(Y) = (\chi(X_1Y), \chi(X_2Y), \dots, \chi(X_mY)).$$

Clearly two point $Y, Y' \in C_t^{n_2}$ have the same color if and only if $\chi(XY) = \chi(XY')$ for all Shelah points X of $C_t^{n_1}$.

Since $m \leq A_1 = \binom{n_1+1}{2}t$ as mentioned, and each of coordinate of $\chi_2(Y)$ is one of k colors, so coloring χ_2 uses at most $k^m \leq k^{A_1} = n_2$ colors. Referring to the case $s = 1$, coloring χ_2 is flip-top on a suitable Shelah line L_2 in $C_t^{n_2}$.

The Shelah line L_2 contains t points, say $L_2 = \{Y_1, Y_2, \dots, Y_t\}$. We then define coloring χ_1 on any point X of $C_t^{n_1}$ as a vector

$$\chi_1(X) = (\chi(XY_1), \chi(XY_2), \dots, \chi(XY_t)).$$

Thus two points $X, X' \in C_t^{n_1}$ have the same color in χ_1 if and only if $\chi(XY_\beta) = \chi(X'Y_\beta)$ for all $Y_\beta \in L_2$.

Clearly χ_1 uses at most $k^t = n_1$ colors. Again, referring the case $s = 1$, the coloring χ_1 is flip-top on a suitable Shelah line L_1 in $C_t^{n_1}$.

Next we verify that χ is flip-top on $L_1 \times L_2$: the derived coloring χ' of χ is a flip-top coloring on C_t^2 . That is to say, we need to verify

$$\begin{aligned} \chi'(\alpha(t-1)) &= \chi'(\alpha t), \\ \chi'((t-1)\beta) &= \chi'(t\beta), \\ \chi'((t-1)(t-1)) &= \chi'(tt) \end{aligned}$$

for any symbols α and β . We verify the first equality, and omit the similar proofs for the two others. By relabelling the Shelah points of $C_t^{n_1}$, we may assume that $L_1 = \{X_1, X_2, \dots, X_t\}$. Note that points of $L_1 \times L_2$ have form $X_\alpha Y_\beta$ with

$$X_\alpha = t-1 \dots t-1 \alpha \dots \alpha t \dots t, \quad Y_\beta = t-1 \dots t-1 \beta \dots \beta t \dots t.$$

Since χ_2 is flip-top on L_2 , we have $\chi_2(Y_{t-1}) = \chi_2(Y_t)$, implying $\chi(PY_{t-1}) = \chi(PY_t)$ for any Shelah point P in $C_t^{n_1}$ from the definition of χ_2 . In particular, we have $\chi(X_\alpha Y_{t-1}) = \chi(X_\alpha Y_t)$, which yields $\chi'(\alpha(t-1)) = \chi'(\alpha t)$.

The proof for general $s \geq 3$ is similar with the case $s = 2$, we thus omit it. $\square$

Lemma 12.3 (Induction Lemma) *Assume that $s = HJ(k, t-1)$ is defined, namely, in any k -coloring of C_{t-1}^s there is a monochromatic line. Then under any flip-top k -coloring of C_t^s , there is a monochromatic line.*

Proof. Consider a flip-top k -coloring χ on C_t^s . Since C_{t-1}^s is a subset of C_t^s and thus χ is a k -coloring on C_{t-1}^s , from the definition of s , there is a monochromatic line L in C_{t-1}^s , say $L = \{P_1, P_2, \dots, P_{t-1}\}$ in color 1, then

the following line

$$\begin{array}{lcl}
 P_1 : & \alpha & \cdots \alpha \\
 P_2 : & \alpha & \cdots \alpha \\
 \vdots & & \\
 P_{t-1} : & \alpha & \cdots \alpha \\
 P_t : & \alpha & \cdots \alpha
 \end{array}
 \left| \begin{array}{ccc}
 1 & \cdots & 1 \\
 2 & \cdots & 2 \\
 & & \\
 t-1 & \cdots & t-1 \\
 \mathbf{t} & \cdots & \mathbf{t}
 \end{array} \right|
 \begin{array}{lcl}
 \beta & \cdots & \text{color 1} \\
 \beta & \cdots & \text{color 1} \\
 & & \\
 \beta & \cdots & \text{color 1} \\
 \beta & \cdots & \text{new point,}
 \end{array}$$

is a monochromatic line, where P_t is a new point, and α and β are constant coordinates. The point P_t is colored in 1 since χ is flip-top on C_t^s . $\square$

Proof of Theorem 12.2. Now it is easy to see the existence of $HJ(k, t)$. We use the induction on t for fixed k .

Basis. It is a trivial fact that $HJ(k, 1) = 1$ for any k .

Inductive Step. Assume that $s = HJ(k, t-1)$ exists, that is $s = HJ(k, t-1) < \infty$. Define n which satisfies the cube lemma for the given k, t and s .

Claim. In any k -coloring of C_t^n , where n is defined above, there is a monochromatic line.

Proof of claim. Let χ be a k -coloring of C_t^n , the cube lemma shows that χ is flip-top on a suitable Shelah s-space $L_1 \times L_2 \times \cdots \times L_s$. By definition, the derived coloring χ' is flip-top on C_t^s . By the definition of $s = HJ(k, t-1)$, the induction lemma can be applied to χ' , there is a monochromatic line L in C_t^s . The $\phi^{-1}(L)$ is monochromatic line of C_t^n under coloring χ . $\square$

Remark. Define a sequence of functions $N \rightarrow N$ as $f_1(n) = 2n$ and

$$f_{m+1}(n) = \underbrace{f_m \circ f_m \circ \cdots \circ f_m}_n(1).$$

Some small values of $f_m(n)$ are listed in following table.

$f_m(n)$	$n = 1$	$n = 2$	$n = 3$	$n = 4$	n
$f_1(n)$	2	4	6	8	$2n$
$f_2(n)$	2	2^2	2^3	2^4	2^n
$f_3(n)$	2	2^2	2^{2^2}	$2^{2^{2^2}}$	$2^{2^{\cdots 2}} \} n$
$f_4(n)$	2	2^2	$2^{2^{2^2}}$	$2^{2^{\cdots 2}} \} f_4(3)$	$2^{2^{\cdots 2}} \} f_4(n-1)$
$f_5(n)$	2	2^2	$f_4(4)$	$f_4(f_4(4))$	

Table Some values of $f_m(n)$

The function $f_3(n)$ is called a “tower”, whose height is n . The function $f_4(n)$ grows much faster, which is called a *wowzer*, wow! Note that $f_4(n)$ is a tower of height $f_4(n-1)$. For example $f_4(4)$ is a tower of height $f_4(3) = 65536$. The value of $f_4(f_4(4) - 1)$ is the height of the tower of $f_5(4)$. The *Ackerman function* $A(n)$ is defined as diagonal value $f_n(n)$.

Shelah’s proof gives “wowzer” bound on $HJ(2, t)$ because it iterates the cube lemma, which gives a “tower” bound by iterating the exponential functions. In fact $HJ(k, t) \leq f_4(c(k+t))$ for some constant c . Even so, this proof for the van der Waerden theorem still reduces the original bound of van der Waerden greatly, who used double induction on k and t , even for $w(t) = w_2(t)$. The original upper bound was a Ackerman function.

An open problem of Graham (1995) was asking if the Cube Lemma can be improved to give an exponential bound instead of tower.

To generalized van der Waerden theorem in the 1930s, Erdős and Turán conjectured that that each set of positive integers with positive upper density contains arbitrarily long arithmetic progressions, that is to say, if a set A of positive integers satisfies $\lim_{n \rightarrow \infty} |A \cap [n]|/n > 0$, then A contains arbitrarily long arithmetic progression. The conjecture in case $\ell = 3$ was proved by Roth in 1953. The full conjecture was proved by Szemerédi in 1975 with a deep and complicated combinatorial argument. In this proof he used a result called the Regularity Lemma in bipartite version, and then he gave the general version in 1976. The lemma has now become a totally new tool in extremal graph theory. Sometimes the Regularity Lemma is called Uniformity Lemma, see e.g. *Modern Graph Theory* by Bollobás (1998). For its many applications, we refer the readers to the survey of Komlós and Simonovits (1996). We shall discuss this lemma in the next section.

12.2 Regularity Lemma

Let G be a graph and let X and Y be nonempty disjoint subsets of $V(G)$. Denote by $e(X, Y)$ for the number of edges between X and Y . The ratio

$$d(X, Y) = \frac{e(X, Y)}{|X||Y|}$$

is called the density of (X, Y) . It is the probability that any pair (x, y) selected randomly from $X \times Y$ is an edge. Clearly $0 \leq d(X, Y) \leq 1$. It is easy to see

$$d_G(X, Y) + d_{\bar{G}}(X, Y) = 1,$$

where $\bar{G}$ is the complement of G .

The density $d(X, Y)$ behaves in a fair continuous fashion.

Lemma 12.4 *Suppose X and Y are disjoint subsets of $V(G)$, and $X' \subseteq X$ and $Y' \subseteq Y$ with $|X'| > (1 - \delta)|X|$ and $|Y'| > (1 - \delta)|Y|$. Then*

$$|d(X', Y') - d(X, Y)| < 2\delta \quad \text{and} \quad |d^2(X', Y') - d^2(X, Y)| < 4\delta.$$

Proof. Note that

$$\begin{aligned} 0 &\leq e(X, Y) - e(X', Y') \\ &\leq e(X \setminus X', Y) + e(X, Y \setminus Y') < 2\delta|X||Y|, \end{aligned}$$

so $d(X, Y) - d(X', Y') < 2\delta$. Replace G by $\bar{G}$, we have

$$d(X', Y') - d(X, Y) = d_{\bar{G}}(X, Y) - d_{\bar{G}}(X', Y') < 2\delta.$$

Another inequality follows the above and the fact $d(X, Y) \leq 1$. $\square$

Let $\epsilon > 0$ be a real number. Then a pair (X, Y) is called ϵ -regular if for any $X' \subseteq X$ and $Y' \subseteq Y$ with $|X'| > \epsilon|X|$ and $|Y'| > \epsilon|Y|$,

$$|d(X, Y) - d(X', Y')| \leq \epsilon.$$

A partition $C_0, C_1, \dots, C_k$ of $V(G)$ is said to be equitable with exceptional set C_0 if $|C_1| = |C_2| = \dots = |C_k|$. An equitable partition with exceptional class C_0 is said to be ϵ -regular if $|C_0| \leq \epsilon n$, where $n = |V|$, and all but at most ϵk^2 pairs (C_i, C_j) with $1 \leq i < j \leq k$ are ϵ -regular.

Theorem 12.4 (Szemerédi's Regularity Lemma) *For any real $\epsilon > 0$ and any integer $m \geq 1$, there exists $M = M(\epsilon, m) > m$ such that every graph G on $n \geq m$ vertices has an ϵ -regular partition $C_0, C_1, \dots, C_k$ with exceptional set C_0 satisfying $m \leq k \leq M$.*

For any $M > m$, the theorem for $m \leq n \leq M$ is trivial since a partition with each class being a singleton is zero-regular. The crucial point for the regularity lemma is that the number of classes in partition is bounded for any graph. We will follow the original proof of Szemerédi (1976), and that reformulated in *Modern Graph Theory* by Bollobás (1998). A proof with slight change can be found in *Graph Theory (2nd ed.)* by Diestel (1997). The readers are encouraged to skip the proofs.

We will need the defect form of Cauchy-Schwarz inequality.

Lemma 12.5 *Let d_i be reals and $s > t \geq 1$ be integers. If*

$$\frac{1}{s} \sum_{i=1}^s d_i = \frac{1}{t} \sum_{i=1}^t d_i + \delta,$$

then

$$\frac{1}{s} \sum_{i=1}^s d_i^2 \geq \left(\frac{1}{s} \sum_{i=1}^s d_i \right)^2 + \frac{t\delta^2}{s-t} \geq \left(\frac{1}{s} \sum_{i=1}^s d_i \right)^2 + \frac{t\delta^2}{s}.$$

Proof. From the Cauchy-Schwarz inequality

$$\sum_{i=1}^s d_i^2 \sum_{i=1}^s b_i^2 \geq \left(\sum_{i=1}^s d_i b_i \right)^2,$$

by taking $b_i = 1$, we have $\sum_{i=1}^s d_i^2 \geq sD_s^2$, where $D_s = \frac{1}{s} \sum_{i=1}^s d_i$. Hence

$$\begin{aligned} \sum_{i=1}^s d_i^2 &= \sum_{i=1}^t d_i^2 + \sum_{i=t+1}^s d_i^2 \\ &\geq tD_t^2 + (s-t) \left(\frac{1}{s-t} \sum_{i=t+1}^s d_i \right)^2 \\ &= tD_t^2 + (s-t) \left(\frac{sD_s - tD_t}{s-t} \right)^2 \\ &= s \left(D_s^2 + \frac{t(D_s - D_t)^2}{s-t} \right), \end{aligned}$$

as claimed. $\square$

Given an equitable partition $\mathcal{P} = \{C_0, C_1, \dots, C_k\}$ with exceptional set C_0 , define

$$q(\mathcal{P}) = \frac{1}{k^2} \sum_{1 \leq i < j \leq k} d^2(C_i, C_j).$$

It is easy to see that $0 \leq q(\mathcal{P}) < 1/2$ since $d(C_i, C_j) \leq 1$.

The function $q(\mathcal{P})$ is a cornerstone in the proof of the regularity lemma. We will show that if $\mathcal{P}$ is not ϵ -regular, there is a partition $\mathcal{P}'$ with the new exceptional class a bit larger than the old one, but $q(\mathcal{P}') \geq q(\mathcal{P}) + \epsilon^5/2$. Do this again if $\mathcal{P}'$ is not ϵ -regular yet. The number of iterations is thus at most $1/\epsilon^5$ to guarantee occurrence of an ϵ -regular partition. Without loss of generality, we assume that $0 < \epsilon \leq 1/2$ since if $\epsilon > 1/2$, one can take $M(\epsilon, m) = M(1/2, m)$.

Lemma 12.6 *Let G be a graph of order n with an equitable partition $\mathcal{P} = \{C_0, C_1, \dots, C_k\}$ with exceptional class C_0 . Suppose*

$$|C_1| = \dots = |C_k| = c \geq 2^{3k+1}; \quad \text{and} \quad 2^k \geq 8/\epsilon^5,$$

and suppose $\mathcal{P}$ is not ϵ -regular, then there is an equitable partition $\mathcal{P}' = \{C'_0, C'_1, \dots, C'_\ell\}$ with exceptional class $C'_0 \supseteq C_0$ and $\ell = k(4^k - 2^{k-1})$, which satisfies

- (1) $|C'_0| \leq |C_0| + n/2^k$,
- (2) $q(\mathcal{P}') \geq q(\mathcal{P}) + \epsilon^5/2$.

Proof★. Separate all pairs (i, j) , $1 \leq i < j \leq k$, of indices into S and T , corresponding with that the pair (C_i, C_j) is ϵ -regular or not, respectively. For $(i, j) \in S$, set $C_{ij} = C_{ji} = \emptyset$, and for $(i, j) \in T$, set $C_{ij} \subseteq C_i$ and $C_{ji} \subseteq C_j$ with $|C_{ij}| > \epsilon c$, $|C_{ji}| > \epsilon c$, and

$$|d(C_{ij}, C_{ji}) - d(C_i, C_j)| > \epsilon.$$

Note that the pair (C_{ij}, C_{ji}) are not “very close” to (C_i, C_j) by Lemma 12.4.

For a fixed i , $1 \leq i \leq k$, consider the equivalence relation $\equiv$ on C_i as for $x, y \in C_i$, $x \equiv y$ means $x \in C_{ij}$ if and only if $y \in C_{ij}$ for any $j \neq i$. The equivalence classes are atoms of algebra induced by C_{ij} , $j \neq i$. Note that C_i has at most 2^{k-1} atoms.

Set $d = \lfloor c/4^k \rfloor$, then

$$d \geq 2^{k+1}, \quad \text{and} \quad 4^k d \leq c < 4^k(d+1).$$

Cut each atom in C_i into pairwise disjoint d -subsets. Denote by z for the maximal number of these d -subsets that one can take. Then

$$zd + 2^{k-1}(d-1) \geq c \geq 4^k d,$$

yielding $z \geq 4^k - 2^{k-1}$. Set $H = 4^k - 2^{k-1}$. We shall take *exactly* H such subsets and add the remainder to the “rubbish bin” to get a new exceptional set C'_0 . Label all these d -subsets as $D_{i1}, D_{i2}, \dots, D_{iH}$. Set $C'_0 = C_0 \cup [\cup_{i=1}^k (C_i \setminus \cup_{h=1}^H D_{ih})]$. Then $|C'_0| \leq |C_0| + k(c - Hd)$. Since

$$Hd \geq (4^k - 2^{k-1})\left(\frac{c}{4^k} - 1\right) \geq c - \frac{c}{2^{k+1}} - 4^k \geq c - \frac{c}{2^k}$$

as $c \geq 2^{3k+1}$, we have $|C'_0| \leq |C_0| + kc/2^k \leq |C_0| + n/2^k$. Rename D_{ih} as C'_j , $1 \leq j \leq \ell$, where $\ell = kH$. Set $\mathcal{P}' = \{C'_0, C'_1, \dots, C'_\ell\}$. All that remains is to show $q(\mathcal{P}') \geq q(\mathcal{P}) + \epsilon^5/2$.

For $i, j = 1, 2, \dots, k$, set

$$\overline{C_i} = \cup_{h=1}^H D_{ih} \quad \text{and} \quad \overline{C_{ij}} = \cup \{D_{ih} : D_{ih} \subseteq C_{ij}\}.$$

Set partition $\overline{\mathcal{P}} = \{C'_0, \overline{C_1}, \dots, \overline{C_k}\}$ with exceptional class C'_0 .

Claim 1 $q(\overline{\mathcal{P}}) \geq q(\mathcal{P}) - \epsilon^5/4$.

Proof of Claim 1 For any pair (C_i, C_j) ,

$$\frac{|C_i \setminus \overline{C_i}|}{|C_i|} = \frac{c - Hd}{c} \leq \frac{4^k(d+1) - Hd}{4^k(d+1)} < \frac{1}{2^k} \leq \frac{\epsilon^5}{8}. \quad (12.1)$$

By Lemma 12.4,

$$|d(\overline{C_i}, \overline{C_j}) - d(C_i, C_j)| \leq \frac{\epsilon^5}{4}. \quad (12.2)$$

Consequently

$$|d^2(\overline{C_i}, \overline{C_j}) - d^2(C_i, C_j)| \leq \frac{\epsilon^5}{2}.$$

Hence $d^2(\overline{C_i}, \overline{C_j}) \geq d^2(C_i, C_j) - \epsilon^5/2$, which implies that $q(\overline{\mathcal{P}}) \geq q(\mathcal{P}) - \epsilon^5/4$ as claimed.

To illustrate what can be obtained in the proof, we shall use $A \subset \approx B$ to represent for “ A is contained in B and close to B ”, and $A \subset \ll B$ for “ A is contained in B but not close to B ”. For a pair $(i, j) \in T$, we have

$$\begin{aligned} (\overline{C_{ij}}, \overline{C_{ji}}) \subset \approx (C_{ij}, C_{ji}) & \subset \ll (C_i, C_j) \\ (\overline{C_{ij}}, \overline{C_{ji}}) & \subset \ll (\overline{C_i}, \overline{C_j}) \subset \approx (C_i, C_j). \end{aligned}$$

So if $(i, j) \in T$, it is expected $|d(\overline{C_{ij}}, \overline{C_{ji}}) - d(\overline{C_i}, \overline{C_j})|$ is almost as large as $|d(C_{ij}, C_{ji}) - d(C_i, C_j)| > \epsilon$.

Claim 2 If $(i, j) \in T$, then $|d(\overline{C_{ij}}, \overline{C_{ji}}) - d(\overline{C_i}, \overline{C_j})| > \frac{15}{16}\epsilon$.

Proof of Claim 2 Since $C_i = C_{ij} \cup (C_i \setminus C_{ij})$, and subsets D_{ih} in $\overline{C_i}$ can be classified into two types: these are contains in C_{ij} , which form $\overline{C_i} \setminus \overline{C_{ij}}$, and these out of C_{ij} . so $C_{ij} \setminus \overline{C_{ij}} \subseteq C_i \setminus \overline{C_i}$. Then from (12.1),

$$\frac{|C_{ij} \setminus \overline{C_{ij}}|}{|C_{ij}|} \leq \frac{|C_i \setminus \overline{C_i}|}{|C_i|} \frac{|C_i|}{|C_{ij}|} \leq \frac{\epsilon^4}{8}, \quad (12.3)$$

which and Lemma 12.4 give

$$|d(\overline{C_{ij}}, \overline{C_{ji}}) - d(C_{ij}, C_{ji})| \leq \frac{\epsilon^4}{4}. \quad (12.4)$$

The definition of C_{ij} , the bounds (12.2) and (12.4) with the fact that $0 < \epsilon \leq 1/2$ yield

$$\begin{aligned} & |d(\overline{C_{ij}}, \overline{C_{ji}}) - d(\overline{C_i}, \overline{C_j})| \\ & \geq |d(C_{ij}, C_{ji}) - d(C_i, C_j)| - |d(\overline{C_{ij}}, \overline{C_{ji}}) - d(C_{ij}, C_{ji})| \\ & \quad - |d(\overline{C_i}, \overline{C_j}) - d(C_i, C_j)| \\ & \geq \epsilon - \frac{\epsilon^4}{4} - \frac{\epsilon^5}{4} \geq \frac{15}{16}\epsilon, \end{aligned}$$

as claimed. $\square$

Let us return to the partition $\mathcal{P}'$ in which each class is a d -subset D_{ih} except class C'_0 . It is easy to see that for any pair (C_i, C_j) ,

$$d(\overline{C_i}, \overline{C_j}) = \frac{1}{H^2} \sum_{1 \leq u, v \leq H} d(D_{iu}, D_{jv})$$

since $|\overline{C_i}| = |\overline{C_j}| = Hd$. Set

$$A(i, j) = \frac{1}{H^2} \sum_{1 \leq u, v \leq H} d^2(D_{iu}, D_{jv}).$$

Then from Cauchy-Schwarz inequality, for any pair (C_i, C_j) , we have

$$A(i, j) \geq d^2(\overline{C_i}, \overline{C_j}).$$

If $(i, j) \in T$, we have some gain. Let R be the set of indices (u, v) such that $D_{iu} \in \overline{C_{ij}}$ and $D_{jv} \in \overline{C_{ji}}$. Then

$$d(\overline{C_{ij}}, \overline{C_{ji}}) = \frac{1}{|R|} \sum_{(u, v) \in R} d(D_{iu}, D_{jv}).$$

So for $(i, j) \in T$, from Lemma 12.5 and Claim 2,

$$\begin{aligned} A(i, j) &\geq d^2(\overline{C_i}, \overline{C_j}) + \frac{|R|}{H^2} (d(\overline{C_i}, \overline{C_j}) - d(\overline{C_{ij}}, \overline{C_{ji}}))^2 \\ &\geq d^2(\overline{C_i}, \overline{C_j}) + \frac{|R|}{H^2} \left(\frac{15\epsilon}{16} \right)^2. \end{aligned}$$

Set H_{ij} is the number of $D_{iu} \in \overline{C_{ij}}$. Then by (12.3)

$$\begin{aligned} H_{ij}d &= |C_{ij}| - |C_{ij} \setminus \overline{C_{ij}}| \geq |C_{ij}| - |C_i \setminus \overline{C_i}| \\ &\geq (\epsilon - \epsilon^5/8)|C_i| \geq (1 - 2^{-7})\epsilon|C_i|. \end{aligned}$$

Since $|R| = H_{ij}H_{ji}$ and $|C_i||C_j| \geq (Hd)^2$, so if $(i, j) \in T$, $\frac{|R|}{H^2} \geq ((1 - 2^{-7})\epsilon)^2$, thus

$$\begin{aligned} A(i, j) &\geq d^2(\overline{C_i}, \overline{C_j}) + ((1 - 2^{-7})\epsilon)^2 \left(\frac{15\epsilon}{16} \right)^2 \\ &\geq d^2(\overline{C_i}, \overline{C_j}) + \frac{3}{4}\epsilon^4. \end{aligned}$$

Note $\ell = kH$, we have

$$q(\mathcal{P}') = \frac{1}{\ell^2} \sum_{1 \leq x < y \leq \ell} d^2(C'_x, C'_y) \geq \frac{1}{k^2} \sum_{1 \leq i < j \leq k} A(i, j),$$

where in the inequality the summands of form $d^2(D_{iu}, D_{iv})$ are ignored. Then from what just obtained,

$$\begin{aligned} q(\mathcal{P}') &\geq \frac{1}{k^2} \left(\sum_{(i,j) \in S} A(i, j) + \sum_{(i,j) \in T} A(i, j) \right) \\ &\geq \frac{1}{k^2} \left[\sum_{(i,j) \in S} d^2(\overline{C}_i, \overline{C}_j) + \sum_{(i,j) \in T} \left(d^2(\overline{C}_i, \overline{C}_j) + \frac{3}{4} \epsilon^4 \right) \right] \\ &= q(\overline{\mathcal{P}}) + \frac{3|T|}{4k^2} \epsilon^4. \end{aligned}$$

Since $\mathcal{P}$ is not ϵ -regular, so $|T| \geq \epsilon k^2$. From this fact and Claim 1, we have

$$q(\mathcal{P}') \geq q(\mathcal{P}) - \frac{\epsilon^5}{4} + \frac{3\epsilon k^2}{4k^2} \epsilon^4 = q(\mathcal{P}) + \frac{\epsilon^5}{2},$$

completing the proof. $\square$

Proof of Theorem 12.4 We shall use Lemma 12.6 repeatedly by showing that at most $t = \lfloor \epsilon^{-5} \rfloor$ iterations will yield a required partition. Let k_0 be an integer such that $k_0 \geq m$ and $2^{-k_0} \leq \epsilon^5/8$, and define $k_{i+1} = k_i(4^{k_i} - 2^{k_i-1})$. Set $M_i = k_i 2^{3k_i+2}$ and $M = M_t$.

Let G be a graph order $n \geq m$. We may assume that $n > M$ since otherwise the partition with each class being a singleton with empty exceptional set will do. Let $\mathcal{P}_0 = \{C_0^{(0)}, C_1^{(0)}, \dots, C_{k_0}^{(0)}\}$ be an equitable partition with $|C_1^{(0)}| = \dots = |C_{k_0}^{(0)}| = \lfloor n/k_0 \rfloor$. So $|C_0^{(0)}| < k_0 \leq M_0/(2^{3k_0+2}) < (\epsilon/2)n$. If $\mathcal{P}_0$ is ϵ -regular, we are at home. Otherwise, let $\mathcal{P}_1 = \{C_0^{(1)}, C_1^{(1)}, \dots, C_{k_1}^{(1)}\}$ guaranteed by Lemma 12.6. Then $|C_0^{(1)}| \leq |C_0^{(0)}| + n/2^{k_0} < \epsilon n$ and $q(\mathcal{P}_1) \geq q(\mathcal{P}) + \epsilon^5/2$. Continue the procedure, if $\mathcal{P}_1$ is not ϵ -regular yet, so we obtain $\mathcal{P}_2$. Suppose we obtain $\mathcal{P}_j$ with exceptional class $C_0^{(j)}$, then

$$|C_0^{(j)}| \leq |C_0^{(0)}| + n(2^{-k_0} + 2^{-k_1} + \dots) < \epsilon n,$$

and $q(\mathcal{P}_j) \geq q(\mathcal{P}_0) + j \frac{\epsilon^5}{2}$. Thus $j \leq t$ since $q(\mathcal{P}_j) < 1/2$, completing the proof. $\square$

The size of the exceptional class may be larger than that of normal ones. To see this, suppose m is much larger than $1/\epsilon$, then the size of normal classes is around n/k , which is much less than ϵn . One of reformulations of regularity lemma is as follows.

Theorem 12.5 *For any real $\epsilon > 0$ and any integer $m \geq 1$, there exists $M = M(\epsilon, m) > m$ such that every graph G on $n \geq m$ vertices has a partition $C_1, C_2, \dots, C_k$ with $|C_1| \leq |C_2| \leq \dots \leq |C_k| \leq |C_1| + 1$, and $m \leq k \leq M$, in which all but at most ϵk^2 pairs (C_i, C_j) , $1 \leq i < j \leq k$, are ϵ -regular.*

Proof For given $\epsilon > 0$ and $m \geq 1$, there is $M > m$ and an equitable and $\frac{\epsilon^2}{4}$ -regular partition $\mathcal{P} = \{C_0, C_1, \dots, C_k\}$ guaranteed by Theorem 12.4 with $1 \leq k \leq M$. Since $|C_0| < \frac{\epsilon^2}{4}n$, thus $\lfloor (1 - \epsilon^2/4)n/k \rfloor \leq |C_i| \leq n/k$. Partition C_0 into k classes $B_1, B_2, \dots, B_k$, such that $|B_i| = \lfloor |C_0|/k \rfloor$ or $|B_i| = \lceil |C_0|/k \rceil$. Set $C'_i = C_i \cup B_i$. Then the sizes of any C'_i and C'_j differ at most by one. We then prove that if a pair (C_i, C_j) is $\frac{\epsilon^2}{4}$ -regular, then (C'_i, C'_j) is ϵ -regular. So all but at most $\frac{\epsilon^2}{4}k^2 < \epsilon k^2$ pairs in the partition $\mathcal{P}' = \{C'_1, C'_2, \dots, C'_k\}$ are ϵ -regular.

In the remaining proof, we suppose that n is large enough, since otherwise we take sufficiently large $M_1 > M$, and let $n \geq M_1$. For $n \leq M_1$, we simply partition $V(G)$ into singletons.

First at all, we see $|C'_i| = \lfloor n/k \rfloor$ or $|C'_i| = \lceil n/k \rceil$. Let (C_i, C_j) , $1 \leq i < j \leq k$ be an $\frac{\epsilon^2}{4}$ -regular pair. Suppose $X'_i \subseteq C'_i$ with $|X'_i| > \epsilon |C'_i|$. Set $X_i = X'_i \setminus B_i \subseteq C_i$. Then

$$\frac{|X_i|}{|X'_i|} \geq 1 - \frac{|B_i|}{|X'_i|} \geq 1 - \frac{\lceil \epsilon^2 n / (4k) \rceil}{\epsilon \lfloor n/k \rfloor} > 1 - \frac{\epsilon}{3}.$$

Hence by Lemma 12.4,

$$|d(X'_i, X'_j) - d(X_i, X_j)| \leq \frac{2\epsilon}{3}. \quad (12.5)$$

The sizes of C_i and C'_i are close,

$$\frac{|C_i|}{|C'_i|} \geq \frac{\lfloor (1 - \epsilon^2/4)n/k \rfloor}{\lceil n/k \rceil} > 1 - \frac{\epsilon^2}{2},$$

thus

$$|d(C_i, C_j) - d(C'_i, C'_j)| \leq \epsilon^2. \quad (12.6)$$

Also, since that (C_i, C_j) is $\frac{\epsilon^2}{4}$ -regular and

$$\frac{|X_i|}{|C_i|} \geq \frac{|X_i|}{|X'_i|} \frac{|X'_i|}{|C'_i|} \geq \left(1 - \frac{\epsilon}{3}\right) \epsilon > \frac{\epsilon^2}{4},$$

we have

$$|d(X_i, X_j) - d(C_i, C_j)| \leq \frac{\epsilon^2}{4}. \quad (12.7)$$

Consequently, by inequalities (12.5), (12.6) and (12.7),

$$\begin{aligned} & |d(X'_i, X'_j) - d(C'_i, C'_j)| \\ & \leq |d(X'_i, X'_j) - d(X_i, X_j)| + |d(X_i, X_j) - d(C_i, C_j)| \\ & \quad + |d(C_i, C_j) - d(C'_i, C'_j)| \\ & \leq \frac{2\epsilon}{3} + \epsilon^2 + \frac{\epsilon^2}{4} < \epsilon. \end{aligned}$$

So the pair (C'_i, C'_j) is ϵ -regular. $\square$

Similar proof yields another formulation of the regularity lemma.

Theorem 12.6 *For any real $\epsilon > 0$ and any integer $m \geq 1$, there exists $M = M(\epsilon, m) > m$ such that every graph G on $n \geq m$ vertices has a partition $C_0, C_1, \dots, C_k$ with $|C_0| \leq k - 1$, $|C_1| = |C_2| = \dots = |C_k|$, and $m \leq k \leq M$, in which all but at most ϵk^2 pairs (C_i, C_j) , $1 \leq i < j \leq k$, are ϵ -regular.*

The bound on $M(\epsilon, m)$ given in Regularity Lemmas is enormous, unless m is much larger than $1/\epsilon$, it is about $2^{2^{\dots^2}}$, where the height of the tower may be up to $1/\epsilon^5$ since the iterations in proof. This seems to be very bad. However, a celebrated result of Gowers (1997) proved that $M(\epsilon, 2)$ grows at least as a such tower of height about $1/\epsilon^{1/16}$. His argument is powerful.

Exercises

12.1 Try to show $w(3) = 9$.

12.2 Try to sketch the proof of Theorem 12.6. In particular, define the function $q(\mathcal{P})$ for a partition $\mathcal{P}$.

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Chapter 13

Ramsey Linearity

13.1 Ramsey Linear Families

We now discuss Ramsey linearity. A family $\mathcal{G}$ of graphs is said to be *Ramsey linear* if there exists a constant $c = c(\mathcal{G}) > 0$ such that $r(G) \leq c|V(G)|$ for any G in $\mathcal{G}$, where $r(G) = r(G, G)$.

For dense graph G on n vertices, $r(G)$ tends to grow exponentially in n . For example, the extreme case of $G = K_n$ has $r(K_n)$ lying roughly between $2^{n/2}$ and 4^n as discussed in previous Chapters. However, for relatively sparse graphs, $r(G)$ grows much more modestly. A particular class which has been investigated from this perspective is the class of graphs G having n vertices and maximum degree at most Δ , where Δ is fixed. This section contains a well known application of the regularity lemma of Chvátal, Rödl, Szemerédi, and Trotter that the family of graphs with bounded maximum degrees is Ramsey linear. Their result does not cover the subdivided graphs, which will be discussed in the next section.

In last section of the chapter, we return to Ramsey goodness, which is a topic in Chapter 1. All results on Ramsey goodness can be viewed as the generalizations of Chvátal result that $r(T_m, K_n) = (n-1)(m-1) + 1$, proved in Chapter 1.

Usually, the regularity lemma is applied for large, dense graphs, which is exactly the case in Ramsey graphs with fixed number of colors. We shall denote the maximum degree of a graph G by $\Delta(G)$.

A simple lemma that is needed in the proof for the major result is as follows.

Lemma 13.1 *Let (A, B) be an ϵ -pair of density $d \in (0, 1]$, and $Y \subseteq B$ with $|Y| \geq \epsilon|B|$. Then there exists a subset $A' \subseteq A$ with $|A'| \geq (1 - \epsilon)|A|$,*

each vertex in A' is adjacent to at least $(d - \epsilon)|Y|$ vertices in Y .

Proof. Let X be the set of vertices with fewer than $(d - \epsilon)|Y|$ neighbors in Y . Then $e(X, Y) < (d - \epsilon)|X||Y|$, so $d(X, Y) < d - \epsilon$. Since (A, B) is ϵ -regular, this implies that $|X| < \epsilon|A|$. $\square$

Theorem 13.1 *For any integer $\Delta \geq 1$, there exists a constant $c = c(\Delta) > 0$ such that if G is a graph on n vertices with $\Delta(G) \leq \Delta$, then $r(G) \leq cn$.*

Proof. Let $\Delta \geq 1$ be a fixed integer. Let m be an integer and $\epsilon = 1/(6m)$ such that $m \geq r(\Delta + 1, \Delta + 1)$ and

$$(1 - \Delta\epsilon)(1/2 - \epsilon)^\Delta m \geq 1 \quad \text{hence} \quad (1 - \Delta\epsilon)(1/2 - \epsilon)^\Delta \geq \epsilon.$$

Let $M = M(\epsilon, 2m) > 2m$ be the integer determined by ϵ and $2m$ in the regularity lemma. Finally, let $c = mM$ that is constant determined completely by Δ . We shall show that if G is a graph of order n with $\Delta(G) \leq \Delta$, then $r(G) \leq cn$.

Consider an edge coloring in two colors, red and blue, of complete graph K_{cn} on vertex set V . Let H be the graph spanned by red edges on V . Then the graph spanned by blue edges is $\overline{H}$, the complement of H .

Since H has $cn > 2m$ vertices, from Regularity Lemma, there is a partition $\{C_1, C_2, \dots, C_k\}$ of V with $||C_i| - |C_j|| \leq 1$, and $2m \leq k \leq M$, in which all but at most ϵk^2 pairs (C_i, C_j) , $1 \leq i < j \leq k$, are ϵ -regular. Let H_1 be the graph on vertex set $\{1, 2, \dots, k\}$, in which (i, j) is an edge if and only if (C_i, C_j) are ϵ -regular. Then the number of edges of H_1 is at least

$$\binom{k}{2} - \epsilon k^2 \geq \frac{m-1}{m} \frac{k^2}{2} \geq t_m(k).$$

By Turán's theorem in Chapter 8, $ex(k; K_m) = t_{m-1}(k) < t_m(k)$, so H_1 contains a complete graph H_2 on m vertices. Color an edge (i, j) of H_2 green if $d(C_i, C_j) \geq 1/2$, or white if $d(C_i, C_j) < 1/2$. Since $m \geq r(\Delta + 1, \Delta + 1)$, so H_2 contains a monochromatic complete graph H_3 on $\Delta + 1$ vertices, without loss of generality, suppose that H_3 is green for otherwise we consider the graph $\overline{H}$. By relabelling the sets in the partition if necessary, we assume that the vertex set of H_3 is $\{1, 2, \dots, \Delta + 1\}$. Thus we assume that for all pairs (C_i, C_j) with $1 \leq i < j \leq \Delta + 1$, are ϵ -regular, and $d_H(C_i, C_j) \geq 1/2$.

Note that if $Y_i \subseteq C_i$ with $|Y_i| \geq (1 - \Delta\epsilon)(1/2 - \epsilon)^\Delta |C_i|$, then $|Y_i| \geq \epsilon |C_i|$, which is the preparation for using Lemma 13.1, and

$$|Y_i| \geq (1 - \Delta\epsilon)(1/2 - \epsilon)^\Delta \frac{cn}{M} \geq n,$$

which will give us enough space to turn around. Assume that $V(G) = \{u_1, u_2, \dots, u_n\}$. We shall show that the red graph H contains G as a subgraph constructively. We will choose $v_1, v_2, \dots, v_n$ from the sets $C_1, \dots, C_{\Delta+1}$. Since the coloring number of G is at most $\Delta+1$, so $V(G)$ can be partitioned into $\Delta+1$ coloring classes, this defines a map $\sigma: \{1, \dots, n\} \rightarrow \{1, \dots, \Delta+1\}$, where $\sigma(i)$ is the index of the color of vertex u_i . Our aim is to define an embedding $u_i \rightarrow v_i \in C_{\sigma(i)}$, such that $v_i v_j$ will be an edge of H whenever $u_i u_j$ is an edge of G .

Our plan is to choose the vertices $v_1, \dots, v_n$ inductively. Throughout the induction, we shall have a target set $Y_i \subseteq C_{\sigma(i)}$ assigned to each i . Initially, Y_i is the entire $C_{\sigma(i)}$. As the embedding proceeds, Y_i will get smaller and smaller. Some vertices will be deleted in procedure. But any $C_{\sigma(i)}$ will really have some vertices deleted at most Δ times. To make this approach work, we have to ensure Y_i do not get too small.

Let us begin the initial step. Set

$$Y_1^0 = C_{\sigma(1)}, Y_2^0 = C_{\sigma(2)}, \dots, Y_n^0 = C_{\sigma(n)}.$$

Note that Y_i^0 and Y_j^0 are not necessarily distinct sets.

We then begin the first step by considering u_1 , which will be selected from Y_1^0 , and its neighbors, $u_\alpha, \dots, u_\beta$, say. Suppose that the degree of u_1 is d . By using Lemma 13.1 repeatedly, we know that there exists a subset $Y_1^1 \subseteq Y_1^0$ with $|Y_1^1| \geq (1 - d\epsilon)|Y_1^0| \geq n$, such that each vertex in Y_1^1 has to least $(1/2 - \epsilon)|Y_j^0|$ neighbors in Y_j^0 , where $j = \alpha, \dots, \beta$. Choose an arbitrary vertex v_1 from Y_1^1 . For $j = \alpha, \dots, \beta$, define Y_j^1 be the neighborhood of v_1 in Y_j^0 . For $j \geq 2, j \neq \alpha, \dots, \beta$, define $Y_j^1 = Y_j^0$, that is, no vertices are deleted from such Y_j^0 . In this step, v_1 has been chosen and it completely adjacent to Y_j^1 in H whenever u_1 and u_j are adjacent in G .

In a general step, we consider u_i and its neighbors. We will choose u_i from Y_i^{i-1} . Suppose that u_i has d_1 neighbors in $\{u_1, \dots, u_{i-1}\}$, and d_2 neighbors, $u_\alpha, \dots, u_\beta$, say, in $\{u_{i+1}, \dots, u_n\}$. Then $d_1 + d_2 \leq \Delta$, and $|Y_i^{i-1}| \geq (1/2 - \epsilon)^{d_1} |Y_i^0|$. That is to say, the current set Y_i^{i-1} are obtained from Y_i^0 by deleting some vertices d_1 times before this step. By using Lemma 13.1 repeatedly again, we have a subset $Y_i^i \subseteq Y_i^{i-1}$ with $|Y_i^i| \geq (1 - d_2\epsilon)|Y_i^{i-1}|$ so that each vertex in Y_i^i has at least $(1/2 - \epsilon)|Y_j^{i-1}|$ neighborhood in Y_j^{i-1} , where $j = \alpha, \dots, \beta$. Since

$$\begin{aligned} |Y_i^i| \geq (1 - \epsilon)^{d_1} |Y_i^{i-1}| &\geq (1 - \epsilon)^{d_1} (1/2 - d_2\epsilon) |Y_i^0| \\ &\geq (1 - \Delta\epsilon)(1/2 - \epsilon)^\Delta |C_i| \\ &\geq n, \end{aligned}$$

we can choose a vertex v_i from Y_i^i , which is distinct from $v_1, \dots, v_{i-1}$ those have been chosen before this step, and some may be from Y_i^i . For $j =$

$\alpha, \dots, \beta$, define Y_j^i to be the neighborhood of v_i in Y_j^{i-1} . For $j \geq i+1, j \neq \alpha, \dots, \beta$, define $Y_j^i = Y_j^{i-1}$, that is, no vertices are deleted from such Y_j^{i-1} . Note that v_i is adjacent to any v_j , where $j < i$ and u_j is adjacent to u_i , and v_i is completely connected with each set Y_j^i , in which a neighbor of v_i will be selected after this step.

It is easy to check that the condition for using Lemma 13.1 can be satisfied since $(1 - \Delta\epsilon)(1/2 - \epsilon)^\Delta \geq \epsilon$. We thus finished the general step hence the proof. $\square$

The result just proved can be generalized to multi-color Ramsey numbers as follows.

Theorem 13.2 *For any fixed positive integers s and Δ , there exists a constant $c = c(s, \Delta) > 0$ such that $r_s(G) \leq cn$ for any graph G of order n with $\Delta(G) \leq \Delta$.*

The proof uses the following form of Regularity Lemma. In fact, a special case, in which the edges of a complete graph is colored in several colors, is needed for above theorem.

Theorem 13.3 (Many-Color Regularity Lemma) *For any real $\epsilon > 0$ and any positive integers m and s , there exists $M = M(\epsilon, m, s) > m$ such that if the edges of a graph G on $n \geq m$ vertices are colored in s colors, all monochromatic graphs have a same partition $C_0, C_1, \dots, C_k$ that is ϵ -regular with exceptional set C_0 satisfying $m \leq k \leq M$.*

Proof. Using the original proof, but modify the definition of $q(\mathcal{P})$ by summing the indices over each color: for a partition of $V(G)$ into $\mathcal{P} = \{C_0, C_1, \dots, C_k\}$ with exceptional set C_0 , let

$$q(\mathcal{P}) = \frac{1}{k^2} \sum_{\ell=1}^s \sum_{1 \leq i < j \leq k} d_\ell^2(C_i, C_j),$$

where $d_\ell(C_i, C_j)$ is the edge density in the ℓ -th color. $\square$

A problem needed to consider is that the huge constant obtained in Theorem 13.1 by using regularity lemma. Some improvements on the constant have been down by Eaton (1998); and further by Graham, Rödl, and Ruciński (2000), in which they showed that the constant can be bounded above by $2^{c\Delta(\log \Delta)^2}$; Kostochka and Rödl (2001) gave some smaller constant for some special bipartite graphs, and they (2004) gave a very reasonable constant.

Let d be a positive integer. A graph G is called d -degenerate if any subgraph of G has minimum degree at most d . The following conjecture is made by Burr and Erdős (1973).

Conjecture 13.1 *The family of d -degenerate graphs is Ramsey linear.*

There are two prominent families included in the above conjecture: the family of planar graphs and that of subdivisions, whose maximum degrees are not bounded. A graph G is called a *subdivision* of H if it is obtained by replacing each edge of H by a path; and G is called an *essential subdivision* of H if the length of each subdividing path is two. An essential subdivision of K_k has $k + \binom{k}{2}$ vertices and $2\binom{k}{2}$ edges, and it is 2-degenerate with the maximum degree k .

Chen and Schelp (1993) made a generalization of Theorem 13.1. In particular, they showed that the family of planar graphs is Ramsey linear. In the next section, we will see that how Alon proved that the family of essential subdivided graphs is Ramsey linear with an appropriate constant.

13.2 Subdivided Graphs

The following result is due to N. Alon (1994), whose original constant was 12. The improved constant 6 is due to Li, Rousseau, and Šoltés (1997). Note that if G is an essential subdivided graph, then all vertices on G of degree at least three are independent.

Theorem 13.4 *If G is a graph of order $n \geq 3$ in which all vertices of degree at least three are independent, then $r(G) \leq 6n - 12$.*

Corollary 13.1 *The family of essential graphs is Ramsey linear.*

We need two lemmas for the proof.

Lemma 13.2 *Let G be a graph without isolated vertices. Then*

$$r(K_3, G) \leq 3q,$$

where q is the number of edges of G .

Proof. For $q = 1$, the assertion is trivial. It is easy to see that if $r(K_3, G_1) \leq r(K_3, G_2)$, then

$$r(K_3, G_1 \cup G_2) \leq r(K_3, G_1) + r(K_3, G_2),$$

so we may assume that G is connected and $q \geq 2$.

Suppose to the contrary, there is a graph G on n vertices and q edges, where q is minimal such that $r(K_3, G) > 3q$. Let us write $N = 3q$. From the definition, there is a red-blue edge coloring of K_N on vertex set V such that there is neither red K_3 nor blue G . Denote by R and B for the subgraphs on V with edge sets consisting of all red edges or all blue edges, respectively. Note that for any vertex $u \in V$, $\deg_R(u) \leq n - 1$ since R is triangle-free thus the subgraph induced by $N_R(u)$ is completely blue.

Let $\delta = \delta(G) \geq 1$ be the minimum degree of G , and let G' be the subgraph of G by deleting a vertex v of degree δ . By the minimality of q , we have $r(K_3, G') \leq 3(q - \delta) < 3q$ thus B contains G' as a subgraph. Let $X \subseteq V$ be the subset of vertices not belonging to $V(G')$ in B . Since B contains no G and $\deg_G(v) = \delta \geq 1$, so each vertex $u \in X$ is adjacent to at least one vertex in $N_G(v)$ in R . However, each vertex in $N_G(v)$ can be adjacent to at most $n - 1$ vertices of X in R , so $|X| = N - (n - 1) \leq \delta(n - 1)$, implying $N \leq (\delta + 1)(n - 1)$, which with the fact that $\delta n \leq 2q$ and $n - 1 \leq q$ yields

$$N \leq 3q - \delta \leq 3q - 1,$$

which is a contradiction. $\square$

The above lemma has been improved as $r(K_3, G) \leq 2q + 1$ by Sidorenko (1993), and Goddard and Kleitman (1994), which was conjectured by Harary.

Another lemma for the proof of main result is as follows.

Lemma 13.3 *Let $m \geq 1$ and $n \geq 2$ be integers. Then*

$$r(K_n, mK_2) = n + 2m - 2.$$

Proof. The graph $K_{n-2} + \overline{K}_{2m-1}$ does not contain K_n and its complement, $\overline{K}_{n-2} \cup K_{2m-1}$ does not contain m independent edges. Hence $r(K_n, mK_2) \geq n + 2m - 2$.

On the other hand, let G be a graph of order $n + 2m - 2$ that does not contain K_n . We shall prove that $\overline{G}$ contain m independent edges by induction on m . This is clear if $m = 1$. For general $m \geq 2$, by deleting a pair of non-adjacent vertices u and v , which form an edge in $\overline{G}$, we have a graph H of order $n + 2(m - 1) - 2$, which does not contain no K_n . By

induction hypothesis, the complement of H contains $m - 1$ independent edges, which with the edge uv give m independent edges of $\overline{G}$. $\square$

Recall that $r(K_{2,n}) \leq 4n - 2$ and the equality holds for infinitely many n , proved in Chapter 8.

Proof of Theorem 13.4. With the fact $r(K_3) = 6$, we may assume that $n \geq 4$. We also assume that G is not a subgraph of $K_{2,n-2}$ by the upper bound just mentioned. Let $V = \{v_1, v_2, \dots, v_n\}$ be the vertex set of G , without loss of generality, let

$$I = \{v_1, v_2, \dots, v_m\} \subseteq V$$

be a maximal independent set containing all vertices of degree at least three, where $m < n$. Since I is maximal, each vertex in $V \setminus I$ has at least one neighbor in I . Thus each component C of $G - I$ is a single vertex or an edge. If C is a single vertex, it has either one or two neighbors in I ; and if C is an edge, then each of its end-vertices has exactly one neighbor in I .

We now define a new graph H with vertex set $V(H) = \{1, 2, \dots, m\}$, and $\{i, j\}$ is an edge of H if v_i and v_j are commonly in the neighborhood of some component of $G - I$. Since there are less n components of $G - I$, thus $|E(H)| < n$. Consider a red-blue edge coloring of complete graph of order $6n - 12$ on vertex set $U = \{u_1, u_2, \dots, u_{6n-12}\}$. Denote by R and B for the subgraph with vertex set U and edge set consisting of all red edges or all blue edges, respectively. By symmetry, we may assume that at least half of the vertices in U have red degrees at least $3n - 6$. Therefore there is a subset $U_1 \subseteq U$, say $U_1 = \{u_1, u_2, \dots, u_{3n-6}\}$, such that $|N_R(u_i)| \geq 3n - 6$ for $1 \leq i \leq 3n - 6$. We then define a yellow graph Y on vertex set U_1 with u_i and u_j are connected by a yellow edge if and only if $|N_R(u_i) \cap N_R(u_j)| \geq n - 3$.

We claim that the independent number of the graph Y is at most two. In fact, if u_1, u_2 and u_3 are independent in Y , then

$$\begin{aligned} & \left| \bigcup_{i=1}^3 N_R(u_i) \right| \\ &= \sum_{i=1}^3 |N_R(u_i)| - \sum_{1 \leq i < j \leq 3} |N_R(u_i) \cap N_R(u_j)| + \left| \bigcap_{i=1}^3 N_R(u_i) \right| \\ &\geq 3(3n - 6) - 3(n - 4) > 6n - 12, \end{aligned}$$

a contradiction. Then by the fact from Lemma 13.2, we know that Y contains H as a subgraph including all isolated vertices since Y has enough vertices.

Therefore, by the definitions of H and Y , we obtain a subset

$$I' = \{v'_1, v'_2, \dots, v'_m\} \subseteq U_1$$

with $|N_R(v'_i)| \geq 3n - 6$ for $1 \leq i \leq m$, and $|N_R(v'_i) \cap N_R(v'_j)| \geq n - 3$ if both v_i and v_j are commonly in the neighborhood of some component of $G - I$.

We shall seek to embed G into red graph R such that $v_i \rightarrow v'_i$ for $1 \leq i \leq m$. Suppose that we have constructed the appropriate embedding of some components of $G - I$. We now wish to extend this embedding to one more component C .

Case 1. $C = \{v\}$ is a single vertex.

Subcase 1.1. v has two neighbors v_i and v_j in I with $1 \leq i < j \leq m$. Since we have chosen at most $n - 3$ vertices previously that are not all in $N_R(v'_i) \cap N_R(v'_j)$ as $G \not\subseteq K_{2,n-2}$, there is at least one vertex in $N_R(v'_i) \cap N_R(v'_j)$ left that can be chosen for v .

Subcase 1.2. v has only one neighbor v'_i in I . Simply take any vertex that has not been chosen in $N_R(v'_i)$.

Case 2. C is an edge e .

Subcase 2.1. Two end-vertices of e have a common neighbor v_i in I with $1 \leq i \leq m$. We can choose any red edge in $N_R(v'_i)$ for e . If no such red edge available, then $N_R(v'_i)$ induces a complete blue graph of order at least $3n - 6 \geq n$. We thus have a blue graph G .

Subcase 2.2. Each end-vertex of e has a (different) neighbor v_i and v_j in I with $1 \leq i < j \leq m$. We seek a red path of length three with end-vertices v'_i and v'_j , and one internal vertex in $N_R(v'_i)$ and one in $N_R(v'_j)$, which have not been chosen previously. Suppose that no such path exists, then any edge between $N_R(v'_i)$ and $N_R(v'_j)$ is blue unless one of its end vertices has been chosen previously. We shall construct a blue G in another way. Let W be the set of vertices chosen so far in the embedding of G into red graph R , then $|W| \leq n - 4$. And let

$$\begin{aligned} X &= (N_R(v'_i) \cap N_R(v'_j)) \setminus W, \\ Y &= N_R(v'_i) \setminus (W \cup N_R(v'_j)), \\ Z &= N_R(v'_j) \setminus W. \end{aligned}$$

We may assume that $|X| \leq n - 1$ since otherwise we have a complete blue graph induced by X of order at least n hence a blue G . Since $Y = N_R(v'_i) \setminus (W \cup X)$ and $|W \cup X| \leq 2n - 5$, so $|Y| \geq 3n - 6 - (2n - 5) = n - 1$. Also $|Z| \geq 3n - 6 - (n - 4) = 2n - 2$. By Lemma 13.3, $r(K_n, (n - 1)K_2) = 2n - 4$, we assume that there is a blue matching M of size $n - 1$ in Z . Take a subset $J \subseteq Y$ with $|J| = |I| = m < n$. Then all edges between J and M are blue.

We can use J and M to construct a blue G with J corresponding to I . This completes the proof. $\square$

The similar proof yields the following result.

Theorem 13.5 *Let d be a positive integer. Let $\mathcal{F}$ be a family of graphs G that contain an independent set I such that each component of $G - I$ contains at most d vertices and has at most two neighbors in I . Then $\mathcal{F}$ is a Ramsey family.*

13.3 Ramsey Goodness

Recall Chvátal's result in Chapter 2

$$r(T_m, K_n) = (m-1)(n-1) + 1,$$

where T_m is a tree on m vertices.

Burr (1981) generalized Chvátal's result in the following way. Denote by $s(G)$ for the minimum number of vertices in a color class among all proper vertex colorings of G with $\chi(G)$ colors and call $s(G)$ the *chromatic surplus*. For example, $s(K_n) = 1$, $s(C_{2m}) = m$ and $s(C_{2m+1}) = 1$.

Theorem 13.6 *Let G be graph with $\chi(G) = k$ and let H be a connected graph of order $n \geq s(G)$, then*

$$r(G, H) \geq (k-1)(n-1) + s(G),$$

where $s(G)$ is the chromatic surplus of G .

Proof. Let $s = s(G)$ and $N = (k-1)(n-1) + s - 1$. Color the edges of K_N red and blue such that the blue graph is isomorphic to $(k-1)K_{n-1} \cup K_{s-1}$, so it contains no H . The chromatic number of the red graph is k , and the smallest vertex color class has size $s-1$, so it contains no G . Thus $r(G, H) \geq N + 1$ as desired. $\square$

In view of the general lower bound in Theorem 13.6, we describe the connected graph H of order $n \geq s(G)$ as G -good if

$$r(G, H) = (\chi(G) - 1)(n - 1) + s(G).$$

A K_k -good graph is also called a k -good graph. It was conjectured by Burr and Erdős that for a fixed k , any large sparse graph is k -good. So

any tree is k -good, but the edge density of a tree is less than 1. We shall prove F_n is 3-good, where $F_n = K_1 + nK_2$ is an n -fan, or a friendship graph, see Chapter 9. Let us have a lemma first.

Lemma 13.4 *For any integer $n \geq 1$,*

$$r(K_3, nK_2) = 2n + 1.$$

Proof. The graph $K_{1,2n-1}$ gives $r(K_3, nK_2) \geq 2n + 1$. For the inverse inequality, we use induction on n . Since $r(K_3, K_2) = 3$, so we suppose that $r(K_3, nK_2) \leq 2n + 1$. Let $N = 2n + 3$. For any coloring of edges of K_N in red and blue. Assume that there is no red K_3 , then there is at least one blue edge uv , say. Deleting vertices u and v , we have a two colored K_{N-2} , in which there is no red K_3 . Since $N - 2 = 2n + 1$, by induction, we get a blue nK_2 , hence a blue $(n + 1)K_2$ together with the blue edge uv . $\square$

The following result is due to Li and Rousseau (1996).

Theorem 13.7 *Let $n \geq 2$ be an integer, then F_n is 3-good, that is*

$$r(K_3, F_n) = 4n + 1.$$

Proof. Theorem 13.6 again yields $r(K_3, F_n) \geq 4n + 1$. We then verify the inverse inequality. For an arbitrary two-coloring of edges of K_{4n+1} , let R and B be the subgraph induced by all red edges and all blue edges, respectively. Suppose that R contains no K_3 and B contains no F_n . For any vertex u , the absence of K_3 implies that $N_R(u)$ induces a complete graph in B , and thus $|N_R(u)| \leq 2n$. Also, by Lemma 13.4, the absence of a blue F_n implies $|N_B(u)| \leq 2n$. It follows that both graphs R and B are regular of degree $2n$.

Suppose that u and v are adjacent in R . Then $N_R(u)$ and $N_R(v)$ are disjoint sets, each inducing a complete graph in B of order $2n$, and $N_B(u) \cap N_B(v)$ has a single vertex, which we denote by w . If w were adjacent to one or more vertices of $N_R(u)$ or to one or more vertices of $N_R(v)$, there would yield a blue F_n . It follows that the degree of w in B , already determined to be $2n$, is 2, hence $n = 1$, a contradiction. This proves that $r(K_3, F_n) \leq 4n + 1$. $\square$

We have seen that any tree T is k -good and $F_n = K_1 + nK_2$ is 3-good if $n \geq 2$. We shall discuss Ramsey goodness more in this section. Bondy and Erdős (1973) proved that a long cycle (hence a long path) is C_m good and $K_r(t)$ -good. Particularly, they obtained

$$r(G, C_n) = \begin{cases} 2n - 1 & G = C_{2m+1}, n > (m + 1)(2m + 1); \\ n + m - 1 & G = C_{2m}, n > 4m^2 - m + 2; \\ (m - 1)(n - 1) + 1 & G = K_m, n \geq m^2 - 2. \end{cases}$$

Burr (1981) generalized the above results from C_n to a graph H that contains a long suspended path. A path of H is called *suspended* if the degree of each internal vertex is two.

Lemma 13.5 *Let G be a graph of order m . Let H be a connected graph of order n that contains a suspended path of length ℓ . Let G_1 be a graph from G by deleting an independent set of t vertices, and let H_1 be a graph from H by shortening the suspended path by 1. If $\ell \geq (m-2)(m-t) + t + 1$, then*

$$r(G, H) \leq \max\{r(G, H_1), r(G_1, H) + n - 1\}.$$

Proof. For $m = 1$ or $m = 2$, the assertion holds trivially. So we assume that $m \geq 3$. Write the right-hand side of the above by N . Consider a red-blue edge coloring of K_N . We shall prove that there is either a red G or a blue H . Since $N \geq r(G, H_1)$, we are done unless there is a blue H_1 . Delete $n - 1$ vertices of this blue H_1 , there are at least $r(G_1, H)$ vertices left, so we may assume that there are a red G_1 . Thus we obtain a blue H_1 and a red G_1 . Let X and Y be their disjoint vertex sets with $|X| = n - 1$ and $|Y| = m - t$, respectively.

The blue H_1 has a suspended path with ℓ vertices, say

$$X' = \{x_1, x_2, \dots, x_\ell\} \subseteq X$$

in order. Write $X'' = \{x_1, x_2, \dots, x_{\ell-1}\} \subseteq X'$. For any $y \in Y$, consider all $\ell - 1$ edges between y and X'' . We assume that no two consecutive edges yx_i and yx_{i+1} are both blue, since otherwise we have a blue H . Furthermore, suppose that $m - 1$ of these edges are blue, say $yx_{i_1}, yx_{i_2}, \dots, yx_{i_{m-1}}$ are blue. Consider any edge $x_{i_j+1}x_{i_k+1}$, if this edge is blue, we have a blue H with the new suspended path

$$x_0 \dots x_{i_j} y x_{i_k} x_{i_{k-1}} \dots x_{i_{j+1}} x_{i_k+1} \dots x_\ell.$$

If all edges $x_{i_j+1}x_{i_k+1}$ are red, then $x_{i_1+1}, x_{i_2+1}, \dots, x_{i_{m-1}+1}$ and y will form a red K_m hence a red G . Consequently, we may assume that any $y \in Y$ is connected with X'' in at most $m - 2$ blue edges. Therefore there are at most $(m - 2)(m - t)$ blue edges between Y and X'' . Hence at least

$$\ell - 1 - (m - 2)(m - t) \geq t$$

vertices in X'' are connected with each vertex of Y in red completely. These vertices and Y yields a red G , completing the proof. $\square$

Theorem 13.8 *Let G be any graph and let H be a connected graph of order p . Choose an edge of H and form a sequence of graphs H_n by putting $n - p$ extra vertices to that edge. If n is sufficiently large, then H_n is G -good.*

Proof. Set $k = \chi(G)$, $s = s(G)$ and $m = |V(G)|$. We shall use the induction on k to show that $r(G, H) = (k - 1)(n - 1) + s$ for sufficiently large n . The assertion is trivial for $k = 1$ since $r(\overline{K}_s, H_n) = s$ if $n \geq 2$, so we assume that $k \geq 2$ and the assertion holds for $k - 1$. Note that H_n has a suspended path of length $n - p + 1$, which would yield graph H_{n-1} by shortening the suspended path by 1. Let H_n and H_{n-1} play the roles of H and H_1 in Lemma 13.5. Consider a vertex coloring of G with k colors such that there is a color class with s vertices. Let $t \geq s$ be the number of vertices in some other color class and let G_1 be the graph from G by deleting those t vertices. Applying Lemma 13.5, we have

$$r(G, H_n) \leq \max\{r(G, H_{n-1}), r(G_1, H_n) + n - 1\}.$$

By the induction hypothesis, $r(G_1, H_n) = (k - 2)(n - 1) + s$ when $n \geq n_0$ for some $n_0 \geq p$. Consequently, for $n \geq n_0$,

$$r(G, H_n) \leq \max\{r(G, H_{n-1}), (k - 1)(n - 1) + s\}.$$

Using Lemma 13.5 repeatedly, we have

$$r(G, H_{n-1}) \leq \max\{r(G, H_{n-2}), (k - 1)(n - 1) + s\},$$

and

$$\begin{aligned} r(G, H_n) &\leq \max\{r(G, H_{n-2}), (k - 1)(n - 1) + s\} \\ &\leq \max\{r(G, H_{n_0}), (k - 1)(n - 1) + s\} \end{aligned}$$

for all $n \geq n_0$. Hence if $(k - 1)(n - 1) + s \geq r(G, H_{n_0})$, then

$$r(G, H_n) = (k - 1)(n - 1) + s,$$

completing the proof. $\square$

Recall that a graph H is called a subdivision of F if it is obtained by replacing each edge of F by a path. We shall say that H is *homeomorphic* to F if they have isomorphic subdivisions.

Theorem 13.9 *Let G be a graph and let H be a connected graph. Let H_n be a graph of order n which is homeomorphic to H . Then if n is large enough, then H_n is G -good.*

Proof. Let $k = \chi(G)$, $s = s(G)$, $m = |V(G)|$ and $p = |V(H)|$. We shall proceed the proof by induction on k as that in the proof of Theorem 13.8. The assertion is trivial for $k = 1$, so we assume that $k \geq 2$ and the assertion holds for $k - 1$. If H has q edges, then H_n has a suspended path of length at least $(n - p)/q + 1$. Let H_{n-1} be formed from H_n by shortening the suspended path by 1 and let G_1 be a graph as defined in the proof of Theorem 13.8. Therefore, if $(n - p)/q + 1 \geq (m - 2)(m - t) + t + 1$, then Lemma 13.5 gives that

$$r(G, H_n) \leq \max\{r(G, H_{n-1}), r(G_1, H_n) + n - 1\}.$$

Applying the induction hypothesis on $r(G_1, H_n)$, we know that there is some $n_0 \geq p$, such that for all $n \geq n_0$,

$$r(G, H_n) \leq \max\{r(G, H_{n-1}), (k - 1)(n - 1) + s\}.$$

The proof concludes as before. $\square$

Let G be a fixed graph. We may wonder whether or not a enough sparse large connected graph is G -good. The answer for general graph G is negative such as $G = C_4$.

Recall graph G_q constructed by Erdős, Rényi, Sós and Brown in Chapter 9. The order of G_q is $q^2 + q + 1$. G_q contains no C_4 as a subgraph. From the fact that each vertex has a degree q or $q + 1$, we know that the maximum degree of $\bar{G}_q$ is q^2 . Hence for $n = q^2 + 1$,

$$r(C_4, K_{1,n}) \geq n + \sqrt{n - 1} + 1.$$

Parsons in fact proved that the equality holds as we have seen in Chapter 9. The above lower bound means that $r(C_4, K_{1,n}) \neq n + 2$ for $n = q^2 + 1$.

Proposition 13.1 *The star $K_{1,n}$ is not C_4 -good for $n = q^2 + 1$.*

Theorem 13.10 *Let G be a graph without isolated vertex and let H be a connected graph. If H is not G -good, then it is not $(K_1 + G)$ -good.*

Proof. Set $k = \chi(G)$, $s = s(G)$ and $n = |V(H)|$. Since H is not G -good, so $r(G, H) \geq (k - 1)(n - 1) + s + 1$, there is a graph F of order $(k - 1)(n - 1) + s$ such that F contains no G and its complement contains no H . It is easy to see that $\bar{K}_{n-1} + F$ contains no $K_1 + G$ and its complement contains no H , thus $r(K_1 + G, H) \geq k(n - 1) + s + 1$ so H is not $(K_1 + G)$ -good. $\square$

Denote by $W_m = K_1 + C_m$, which is called a *wheel*. We have seen that a large star is not C_4 -good, so we know that for a fixed $k \geq 1$, any large star is not $(K_k + C_4)$ -good. In particular, a large star is not W_4 -good.

However, we can find some star-like graphs are G -good for any G . Before giving a result, we define the *upper chromatic surplus* $\bar{s}(G)$ as maximum number of vertices in a color class among all proper vertex coloring of G using $\chi(G)$ colors. Clearly, $\bar{s}(G) \geq s(G)$. Also, a pendant edge of a graph H is an edge that has an end vertex of degree one.

Theorem 13.11 *Let G be any graph without isolates and let H be a connected graph of order $p \geq \bar{s}(G)$. For any $U \subseteq V(H)$ with $|U| = \bar{s}(G)$, let $\mathcal{H}_n$ denote the class of graphs obtained from H by adding $n - p$ pendent edges joining with vertices in U . If n is sufficiently large, then H_n is G -good for some $H_n \in \mathcal{H}_n$.*

Proof. Let $k = \chi(G)$ and $s = s(G)$. We shall prove that for some $H_n \in \mathcal{H}_n$,

$$r(G, H_n) \leq (k - 1)(n - 1) + s$$

if n is sufficiently large. For $k = 1$, $G = K_1$ and the assertion is trivial. Here we admit that the only vertex in K_1 is not an isolate. Suppose that $k \geq 2$ and the assertion holds for $k - 1$. Consider a vertex coloring of G in k colors with color classes $C_1, C_2, \dots, C_k$ satisfying

$$s = |C_1| \leq |C_2| \leq \dots \leq |C_k| \leq \bar{s}(G).$$

Denote by G' for the graph from G by deleting C_k . Then $\chi(G') = k - 1$ and $s(G') = s$. By the induction hypothesis, there exists $N > 0$ such that if $n \geq N$ we can find some specific $H' \in \mathcal{H}_n$ that is G' -good, namely

$$r(G', H') = (k - 2)(n - 1) + s.$$

Take $N_1 \geq N$ such that $(k - 1)(N_1 - 1) + s \geq r(G, H)$. For $n \geq N_1$ set $q = (k - 1)(n - 1) + s$ and let (R, B) be an edge-coloring of K_q in red and blue. We want to show that either $G \subset \langle R \rangle$ or $H'' \subset \langle B \rangle$ for some $H'' \in \mathcal{H}_n$.

Suppose to the contrary, that $\langle R \rangle$ contains no G and $\langle B \rangle$ contains no any member of $\mathcal{H}_n$. Since $q \geq r(G, H)$, we have $H \subset \langle B \rangle$. We thus assume that there is some m with $p \leq m < n$ such that $\langle B \rangle$ contains some member of $\mathcal{H}_m$ but no member of $\mathcal{H}_{m+1}$. Then there is a partition $X \cup Y$ of vertex set K_q with $|X| = m$ and

$$|Y| = q - m \geq (k - 2)(n - 1) + s = r(G', H')$$

such that $\langle X \rangle_B$ (the blue graph induced by X) contains some member of $\mathcal{H}_m$ and all edges xy with $x \in U \subseteq V(H) \subseteq X$ and $y \in Y$ are red. Since $\langle Y \rangle_B$ contains no H' , so $\langle Y \rangle_R$ contains G' . Finally from the fact

$|U| = \bar{s}(G) \geq |C_k|$, we obtain a red G . This contradiction completes the proof. $\square$

We have seen that any tree is k -good for any k , and a long path is G -good for any fixed G . Negatively, all large star is not C_4 -good. So we do not expect that a large sparse graph is G -good for a non-complete graph G . But we may believe that $r(G, K_{1,n})$ achieves the maximum value among all $r(G, T_n)$, where T_n is a tree of n edges.

However, for fixed k and d , if H is d -degenerate, then $r(K_k, H)$ grows linearly on the order of H .

Theorem 13.12 *Let k and d be fixed positive integers. There exists a constant $c = c(k, d) > 0$ such that for any d -degenerate graph H of order n ,*

$$r(K_k, H) \leq cn.$$

proof. For $k = 1$, the assertion is trivial. Noting the fact that $r(K_2, H) = n$, we recursively define a sequence of constants $\{c_i\}$ by setting $c_2 = 1$ and $c_k = dc_{k-1} + 1$ for $k \geq 3$. We shall prove that $r(K_k, H) \leq c_k n$. Suppose to the contrary that there is a red-blue coloring of edges of K_N , where $N = c_k n$, that contains neither a red K_k nor a blue G . Moreover, suppose $k \geq 3$ is the smallest integer with this property. If the colored K_N has a vertex v incident with at least $c_{k-1}n$ red edges, since $r(K_{k-1}, H) \leq c_{k-1}n$ with the choice of k , then we obtain a blue H or a red K_{k-1} extendable to a red K_k by the addition of v . Both cases violate our assumption. Thus each vertex is incident with at most $c_{k-1}n - 1$ red edges.

Now we shall show that any d vertices have at least n common blue neighbors. To see this, fix d vertices and remove them and their red neighbors. Since each vertex is incident with at most $c_{k-1}n - 1$ red edges, we delete at most $dc_{k-1}n$ vertices. The remaining set, which is the intersection of blue neighborhoods of fixed vertices, contains at least $c_k n - dc_{k-1}n = n$ vertices.

Since H is d -degenerate, we may set $V(H) = \{v_1, v_2, \dots, v_n\}$ with the property that any vertex v_i has at most d neighbors in $\{v_1, \dots, v_{i-1}\}$. Let H_m be the subgraph of H induced by $\{v_1, \dots, v_m\}$. Since the colored K_N contains neither red K_k nor blue H , we may assume that m is the largest integer such that a blue H_m exists with $1 \leq m \leq n - 1$. However, v_{m+1} has at most d neighbors in $\{v_1, \dots, v_m\}$, we can easily obtain a blue H_{m+1} since any d vertices have at least n common blue neighbors, and fewer than n of them have been used. This is a contradiction. $\square$

Let us conclude this section with a conjecture of Burr and Erdős (1983).

Conjecture 13.2 *If k and d are fixed, then all sufficiently large connected graphs with average degree of any subgraph no more than d are k -good.*

Exercises

13.1 Let a, b, k be positive integers and let $G_1, \dots, G_k$ be graphs of order n . If in each G_i the number of vertices with degrees greater than a is at most b , then there exists a constant $c = c(a, b, k)$ such that $r(G_1, \dots, G_k) \leq cn$.

13.2 Complete the proof of Theorem 13.5.

13.3 Prove that B_n is 3-good for $n \geq 2$.

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Chapter 14

Size Ramsey Numbers and Other Variants

14.1 Size Ramsey Numbers of K_n and P_n

Ramsey's theorem has inspired many striking and difficult problems. There are a lot of variations. We discuss some of them in this chapter. We shall begin with reviewing the definition of Ramsey numbers in a variant way.

For graphs G , G_1 and G_2 , let

$$G \rightarrow (G_1, G_2)$$

signify that in any coloring of $E(G)$ in red and blue, there is either monochromatic red G_1 or a monochromatic blue G_2 . With this notation, we know that the Ramsey number $r(G_1, G_2)$ is the smallest N such that $K_N \rightarrow (G_1, G_2)$. Clearly,

$$\begin{aligned} r(G_1, G_2) &= \min\{N : K_N \rightarrow (G_1, G_2)\} \\ &= \min\{|V(G)| : G \rightarrow (G_1, G_2)\}, \end{aligned}$$

in which the second equality holds since $G \rightarrow (G_1, G_2)$ implies $K_N \rightarrow (G_1, G_2)$ with $N = |V(G)|$. As the number of edges in a graph G , denoted by $e(G)$, is often called the size of the graph, Erdős, Faudree, Rousseau and Schelp (1978) introduced an idea of measuring minimality with respect to size rather than order of the graphs with $G \rightarrow (G_1, G_2)$. The *size Ramsey number* $\hat{r}(G_1, G_2)$ is defined as

$$\hat{r}(G_1, G_2) = \min\{e(G) : G \rightarrow (G_1, G_2)\}.$$

Directly from the definition, we know that the upper bound $\hat{r}(G_1, G_2) \leq q$ is equivalent to the existence of a graph G of size q such that $G \rightarrow (G_1, G_2)$; and the lower bound $\hat{r}(G_1, G_2) > q$ is equivalent to that for any graph G of size q , there is a coloring of $E(G)$ in red and blue, such that there is neither red G_1 nor blue G_2 . Here the existence arguments appear in the proofs for upper bound of $\hat{r}(G_1, G_2)$, which is different from that for Ramsey number $r(G_1, G_2)$. Needless to say, an edge coloring in two colors is equivalent to a partition of edge set into two subsets.

Lemma 14.1 *For any graphs G_1 and G_2 ,*

$$\hat{r}(G_1, G_2) \leq \binom{r(G_1, G_2)}{2}.$$

Proof. Set $r = r(G_1, G_2)$. Then $K_r \rightarrow (G_1, G_2)$, so

$$\hat{r}(G_1, G_2) \leq e(K_r) = \binom{r}{2}.$$

□

The following result is due to Chvátal, reported in Erdős, Faudree, Rousseau and Schelp (1978), which indicates that the problem is not new if both G_1 and G_2 are complete graphs.

Theorem 14.1 *For positive integers m and n , let $r = r(K_m, K_n)$, then*

$$\hat{r}(K_m, K_n) = \binom{r}{2}.$$

Furthermore, if G is a connected graph with $e(G) \leq \binom{r}{2}$ such that $G \rightarrow (K_m, K_n)$, then $G = K_r$.

Proof. Let us begin with an observation. Let the edge set of a graph G be colored in red and blue and let u and v be two non-adjacent vertices of G . Consider the induced edge colorings of graphs $G - u$ and $G - v$. If there is neither a monochromatic K_m in $G - u$ nor a monochromatic K_n in $G - v$, then the same is true in G . The reason is simple. Any assumed monochromatic complete graph in G cannot contain both u and v , since these two vertices are not adjacent. Hence, any such monochromatic complete graph would appear in the induced coloring of either $G - u$ or $G - v$.

Set $R = \binom{r}{2}$. Let $G = (V, E)$ be a connected graph of size q with $q \leq R$. We shall prove the following claim first.

Claim. If $G \neq K_r$, then $G \not\rightarrow (K_m, K_n)$.

We use induction on order p of G . The claim is certainly true for $p < r = r(K_m, K_n)$, since $K_p \not\rightarrow (K_m, K_n)$ induces a coloring of G for $G \not\rightarrow (K_m, K_n)$. So we assume that $p \geq r$. Since $q \leq R$ and $G \neq K_r$, we know that G is not a complete graph. Let u and v be two non-adjacent vertices of G and set

$$W = V \setminus \{u, v\}, \quad H = G - \{u, v\}.$$

If there exist red-blue edge colorings of $G - u$ and $G - v$, respectively, which agree on H , such that there is neither red K_m nor blue K_n , then, by the observation made before, $G \not\rightarrow (K_m, K_n)$. The proof is then to establish such an edge coloring.

Let $X = N_G(u) \cup N_G(v)$. Denote by H' for the graph obtained from H by adding a new vertex x and joining x and each vertex in X . That is to say, the vertices u and v are contracted to a new vertex x in G . Note that the order of H' is $p - 1$ and its size is at most q .

For the reasons which follow, we may assume $H' \neq K_r$, where $r = r(K_m, K_n)$. In fact, if $H' = K_r$, then $H = K_{r-1}$ and $X = W$. It is clear $N(u) \cap N(v) = \emptyset$ since $q \leq R$. By taking $w \in W$ with $\text{dist}(u, w) = 2$ so $w \in N(v)$, we may simply consider $G - u$ and $G - w$ at the beginning since u and w are not adjacent and $N(u) \cap N(w) \neq \emptyset$. Hence, we can apply the induction hypothesis on H' . The existence of the desired edge coloring which agree on H is manifest. Clearly, the deletion of edges so that H' returns to $G - u$ and $G - v$ spoils nothing so the desired edge coloring of G has been constructed. This proves the claim.

Now let $q < R$. For any graph G of size q , if G is complete, then its order is less than r , so $G \not\rightarrow (K_m, K_n)$. If G is not complete, the claim shows that $G \not\rightarrow (K_m, K_n)$. Thus $\hat{r}(K_m, K_n) > q$, in particular $\hat{r}(K_m, K_n) > R - 1$, which and Lemma 14.1 yield the desired assertion. $\square$

The following theorem says that $\hat{r}(K_{1,m}, K_{1,n})$ is very close to $r(K_{1,m}, K_{1,n})$.

Theorem 14.2 For any positive integers m and n ,

$$\hat{r}(K_{1,m}, K_{1,n}) = m + n - 1.$$

Proof. Since in any edge coloring of $K_{1,m+n-1}$ in red and blue, there is either a red $K_{1,m}$ or a blue $K_{1,n}$, so $\hat{r}(K_{1,m}, K_{1,n}) \leq m + n - 1$. In what follows, we suppose $m \leq n$. Let G be a graph of size $q \leq m + n - 2$. It is clear that G has at most one vertex v with $\deg(v) \geq n$. If this is the case, then every other vertex u satisfies $\deg(u) \leq m - 1$. We may color all edges of $G - v$ red. Then at most $m - 1$ edges incident with v have been colored.

We shall color other edges incident with v in such a way that there is neither red $K_{1,m}$ nor blue $K_{1,n}$. If every vertex has degree at most $n-1$, we may color all edges of G in blue. We thus reach a conclusion that $G \not\rightarrow (K_{1,m}, K_{1,n})$. Since G is arbitrary so $\hat{r}(K_{1,m}, K_{1,n}) \geq m+n-1$. $\square$

As usual, we write $\hat{r}(G)$ for $\hat{r}(G, G)$. The above theorem gives that $\hat{r}(K_{1,n}) = 2n-1$. However, it is difficult to find the exact values of $\hat{r}(P_n)$. With an impressive proof, Beck (1983) showed that $\hat{r}(P_n) \leq cn$ for some constant $c > 0$. In the proof he used transforms defined by Pósa (1976), who needed them for finding an upper threshold function of form $p = c \log n/n$ for random graph $G \in \mathcal{G}(n, p)$ being Hamiltonian.

Let G be a graph and let $P_0 = x_0x_1 \cdots x_n$ be a maximal length path in G beginning at x_0 . Denote by

$$X = V(P_0) = \{x_0, x_1, \dots, x_n\}$$

for the vertex set of P_0 . Let P be a path on vertex set X beginning at x_0 and let $P(k)$ is the vertex which is at distance k from x_0 on the path P . If for some $k \leq n-2$, $P(k)$ is adjacent to $P(n)$ in G , then the path P' obtained from P by adding the edge $P(k)P(n)$ and deleting $P(k)P(k+1)$ will be called a *simple transform* of P . Clearly if P' is a transform of P then P is a transform of P' , and P_0 has exactly $d(x_n)-1$ simple transforms. The result of doing a finite sequence of simple transforms is called a *transform*. Denote by $\mathcal{F}$ for the family of all transforms of P_0 . Note that $V(P) = x_0$ and the length of P is n for any $P \in \mathcal{F}$.

We now partition $V(G)$ into Red, Write and Blue sets as follows.

$$\begin{aligned} R &= \{P(n) \in X : P \in \mathcal{F}\}. \\ W &= \{x_k \in X \setminus R : x_{k-1} \in R, \text{ or } x_{k+1} \in R\}. \\ B &= V(G) \setminus (R \cup W). \end{aligned}$$

Note that $R \neq \emptyset$ as $x_n \in R$. If $x_0x_n \in E(G)$, then $x_1 \in R$ and hence $x_0 \in W$ since $x_0 \notin R$, and in this case if $x_2 \notin R$ then $x_2 \in W$. If $x_1 \notin R$, then $x_0 \in B$. Certainly $V(G) \setminus X \subseteq B$. Let us call an edge between R and B as an $R-B$ edge.

Lemma 14.2 (Pósa) *There are no $R-B$ edges in G .*

Proof. Clearly, there is no edge between $V(G) \setminus X$ and R by the maximality of length of P_0 . Suppose that $x_j \in B \cap X$ is adjacent to $x_k \in R$. Since $x_k \in R$, there is a transform P in which $P(n) = x_k$. If $x_j = P(i)$, then $P(i+1) \in R$. [Just consider the simple transform of P obtained

by adding $P(i)P(n) = x_jx_k$ and deleting $P(i)P(i+1)$.] It follows that $P(i+1) \notin \{x_{j-1}, x_{j+1}\}$. Otherwise, $x_j = P(i) \in W$. In other words, either $x_{j-1}x_j$ or x_jx_{j+1} is deleted in the sequence of simple transforms which lead from P_0 to P . However, by considering the first point at which an edge of P_0 is deleted in a simple transform, we see that one of its vertices must be in R and the other one must be in $R \cup W$. This contradicts the fact that $x_j \in B$ and completes the proof. $\square$

In the following corollary, $N(S) = \cup_{x \in S} N(x)$ is the union of neighborhoods of vertices in S .

We write P_n for a path on n vertices.

Lemma 14.3 *Suppose that each $S \subseteq V(G)$ with $|S| \leq n$ satisfies*

$$|S \cup N(S)| \geq 3|S|.$$

Then G contains P_{3n} .

Proof. In this proof, X , R , W and B have the same meaning as in the preceding lemma. Note that $N(R) \subseteq X$ by the maximality of length of P_0 , and thus $R \cup W \subseteq X$. Furthermore, $N(R) \setminus R = W$ and $|W| \leq 2|R| - 1$ since each vertex in R can possibly account for two vertices in W except for x_n which can account at most one (namely, x_{n-1}). Hence

$$|R \cup N(R)| = |R| + |W| \leq 3|R| - 1.$$

In view of the above hypothesis, we must conclude that $|R| > n$. Choose $S \subseteq R$ with $|S| = n$. Since there are no edges between R and B ,

$$S \cup N(S) \subseteq X,$$

which yields

$$|X| \geq |S \cup N(S)| \geq 3|S| = 3n.$$

The desired result follows immediately. $\square$

In the proof of above lemma, we did not use Lemma 14.2 strictly. What we needed was just the fact that there are no edges between $S \subseteq R$ and $V(G) \setminus X$, which can be seen easily from the maximality of length of P_0 . Of course the fact is a spacial case of Lemma 14.2.

We shall introduce some notation before proving Beck's result. The number of G and the subgraph induced by X in G will be denoted by $e(G)$

and $e_G(X)$, respectively. The average degree of G will be denoted by $d(G)$. Let $\bar{e}_G(X)$ be the number of edges of G which are incident with vertices in X . Note that the edges with both end-vertices in X can be counted exactly once, so,

$$\bar{e}_G(X) = \sum_{x \in X} \deg(x) - e_G(X).$$

Write

$$\bar{d}_G(X) = \frac{\bar{e}_G(X)}{|X|}.$$

Thus $\bar{d}_G(X)$ is the average degree of vertices in X if X is an independent set. Note that for $X = V(G)$ it is $\frac{1}{2}d(G)$ and for $X = \{x\}$ this quantity is $\deg(x)$.

Lemma 14.4 *Let G be an arbitrary graph. Then G contains a subgraph H such that for every non-empty $X \subseteq V(H)$,*

$$\bar{d}_H(X) \geq \frac{1}{2}d(G).$$

Proof. Let H be a minimal vertex-induced subgraph of G on vertex set Y which satisfies

$$d(H) \geq d(G).$$

This implies $e(H) \geq d(G)|Y|/2$. We shall show that H satisfies the assertion.

Suppose to the contrary, there is some non-empty set $X \subseteq Y$ such that $\bar{d}_H(X) < d(G)/2$. Then $\bar{e}_H(X) < d(G)|X|/2$ and $Y \setminus X \neq \emptyset$. Thus the subgraph H' induced by $Y \setminus X$ satisfies

$$\begin{aligned} e(H') &= e(H) - \bar{e}_H(X) \\ &> \frac{1}{2}d(G)|Y| - \frac{1}{2}d(G)|X| = \frac{1}{2}d(G)|Y \setminus X|. \end{aligned}$$

It follows that $d(H') > d(G)$. Since this violates the minimality condition, we have proved that H has the desired property. $\square$

We will say that G has *Property Q_n* if for every $X \subseteq Y \subseteq V(G)$ with $|X| \leq n$ and $|Y| \leq 3|X| - 1$ satisfies

$$\bar{d}_{\langle Y \rangle}(X) < \frac{1}{4}d(G),$$

where $\langle Y \rangle$ is the subgraph induced by Y . Roughly speaking, graph G has property Q_n means that a bigger proportion of neighbors of X are outside of Y . We have seen this sort of property in random graphs as described in a theorem in Chapter 6.

Lemma 14.5 *If G has property Q_n , then every subgraph of G with at least $\frac{1}{2}e(G)$ edges contains P_{3n} .*

Proof. Let G' be a subgraph of G with $e(G') \geq \frac{1}{2}e(G)$. Then $d(G') \geq \frac{1}{2}d(G)$ and, according to Lemma 14.4, there exists a subgraph H of G' such that for every non-empty $X \subseteq V(H)$,

$$\bar{d}_H(X) \geq \frac{1}{2}d(G') \geq \frac{1}{4}d(G).$$

We claim that H satisfies the condition of Lemma 14.3, so H (hence G') contains P_{3n} . If not, then for some $X \subseteq V(H)$ with $|X| \leq n$,

$$|X \cup N_H(X)| \leq 3|X| - 1.$$

Setting $Y = X \cup N_H(X)$, we have $|Y| \leq 3|X| - 1$ and

$$d_{\langle Y \rangle}(X) = d_H(X) \geq \frac{1}{2}d(G') \geq \frac{1}{4}d(G),$$

which contradicts the fact that G has property Q_n . $\square$

Theorem 14.3 (Beck) *For all large n ,*

$$\hat{r}(P_n) \leq 708n.$$

Specifically, for all large n there exists a graph G with $e(G) \leq 708n$ such that every subgraph G' of G with $e(G') \geq \frac{1}{2}e(G)$ contains P_n .

Proof. We shall prove that $\hat{r}(P_{3n}) \leq 2123n$ for large n so

$$\hat{r}(P_n) \leq \hat{r}(P_{3m}) \leq 2123m < 708n$$

where $m = \lceil n/3 \rceil$. In view of Lemma 14.5, it suffices for us to prove that whenever n is large, there exists a graph with at most $2123n$ edges which has property Q_n . Set $N = \lfloor 58.8n \rfloor$ and consider a random graph G_p in $\mathcal{G}(N, p)$ with $p = 72.1/N$. The number of edges in $G = G_p$ has binomial distribution $B(\binom{N}{2}, p)$. It follows from Chernoff's inequalities in Chapter 4 that almost every graph $G = G_p$ satisfies $d(G) \geq 72$ and $e(G) \leq 36.1N < 2123n$. Let F (failure) be the following event:

$$F : d(G) \geq 72 \text{ and } G \text{ fails to satisfy } Q_n.$$

We shall show $\Pr(F) \rightarrow 0$ as $n \rightarrow \infty$.

If $G \in F$ then there exists $X \subseteq V(G)$ with $|X| \leq n$ and $Y \supseteq X$ with $|Y| \leq 3|X| - 1$ such that

$$\bar{d}_{\langle Y \rangle}(X) \geq \frac{1}{4}d(G) \geq 18,$$

yielding

$$\bar{e}_{\langle Y \rangle}(X) \geq 18|X|.$$

If $|X| = k$, then $\bar{e}_{\langle Y \rangle}(X)$ is a random variable with binomial distribution $B(M, p)$, where $M \leq \binom{k}{2} + k(2k - 1) < 5k^2/2$. Note that the expected value of $\bar{e}_{\langle Y \rangle}(X)$ is Mp that satisfies

$$Mp \leq \left(\frac{5k^2}{2}\right) \left(\frac{72.1}{N}\right) \leq 180.25 \frac{k}{N} |X| \leq 3.07|X|,$$

where $k/N \leq n/N$. It follows that the event $\bar{e}_{\langle Y \rangle}(X) \geq 18|X|$ is very unlikely. Recall a result in Chapter 4, if T is a random variable with binomial distribution $B(M, p)$ and $k \geq Mp$, then

$$\Pr(T \geq k) \leq \left(\frac{Mpe}{k}\right)^k.$$

Now we have

$$\left(\frac{Mpe}{18k}\right)^{18k} < \left(\frac{5(72.1)e}{36} \frac{k}{N}\right)^{18k} < \left(\frac{27.25k}{N}\right)^{18k}.$$

Thus we bound the probability of F by

$$\begin{aligned} \Pr(F) &\leq \sum_{k=1}^n \binom{N}{k} \binom{N-k}{2k} \left(\frac{27.25k}{N}\right)^{18k} \\ &\leq \sum_{k=1}^n \left(\frac{Ne}{k}\right)^k \left(\frac{Ne}{2k}\right)^{2k} \left(\frac{27.25k}{N}\right)^{18k} \\ &= \sum_{k=1}^n \left[c \left(\frac{k}{N}\right)^{15} \right]^k, \end{aligned}$$

where $c = e^3(27.25)^{18}/4$. It is clear that the terms in the sum with $k \leq \sqrt{N}$ contribute $o(1)$ since for these terms the expression in the square braces is $O(1/N^{15/2})$. In the range $\sqrt{N} < k \leq n$ it suffices to check that the expression in the square braces remains less than $1 - \epsilon$ for some $\epsilon > 0$. It follows that $\Pr(F) \rightarrow 0$ as $n \rightarrow \infty$. Thus almost every random graph G_p is a graph with at most $2123n$ edges which has property Q_n . $\square$

14.2 Size Ramsey Number of $K_{m,n}$

Friedman and Pippenger (1987) generalized Beck's linear bound of $\hat{r}(P_n)$ for proving that there is some constant $c > 0$ such that $\hat{r}(T_n) \leq cn$ for any tree T_n on n vertices. Beck (1990) even asked if there is some constant $c = c(\Delta) > 0$ such that $\hat{r}(G) \leq cn$ for any graph with n vertices and maximum degree at most Δ . Haxell, Kohayahawa and Łuczak (1995) proved that $\hat{r}(C_n) \leq cn$. Rödl and Szemerédi (2000) answered the question of Beck negatively for even $\Delta = 3$. However, $\hat{r}(K_{m,n})$ has a linear upper bound if m is fixed.

Theorem 14.4 *For any fixed positive integer m , if n is sufficiently large, then*

$$\frac{1}{2e}m2^mn \leq \hat{r}(K_{m,n}) \leq 4m^22^mn.$$

Let us have a lemma first.

Lemma 14.6 *Suppose that G is a subgraph of $K_{M,N}$ with $e(G) \geq Np$ and*

$$N \binom{p}{m} > (n-1) \binom{M}{m},$$

then G contains $K_{m,n}$.

The proof of the above lemma is similar to the proof of Theorem 8.4. The key for the proof is so called “double counting argument”. Without loss generality, we assume that G is a spanning subgraph of G that contains no $K_{m,n}$. Let $d_1, d_2, \dots, d_N$ be the degree sequence of vertices in one part X of N vertices of G . For any vertex $v \in X$, an m -set in neighborhood of v is covered by at most $n-1$ vertices in X . So

$$\sum_{k=1}^N \binom{d_k}{m} \leq (n-1) \binom{M}{m}.$$

The left hand side is at least $N \binom{p}{m}$ by the convexity of the function $\binom{x}{m}$ since $(\sum d_k)/N = e(G)/N \geq p$, which leads to a contradiction.

Proof of Theorem 14.4. The assertion is obvious for $m = 1$ by Theorem 14.2, so we assume $m \geq 2$. For the upper bound, let us consider a complete bipartite graph $K_{M,N}$ on bipartition (A, B) and an edge partition (E_1, E_2) . We may assume that $|E_1| \geq MN/2$. Hence, by setting $p = M/2$ in Lemma 14.6, the subgraph induced by E_1 contains $K_{m,n}$ if

$$N \binom{M/2}{m} > (n-1) \binom{M}{m}.$$

This will certainly be the case if we set $N = \left\lfloor \binom{M}{m} n / \binom{M/2}{m} \right\rfloor$. It follows that for all $M \geq 2m$,

$$\hat{r}(K_{m,n}) \leq MN \leq Cn,$$

where

$$C = \frac{M \binom{M}{m}}{\binom{M/2}{m}} = 2^{m-1} M \frac{(M-1)(M-2) \cdots [M-(m-1)]}{(M-2)(M-4) \cdots [M-2(m-1)]}.$$

By taking $M = \lfloor m^2/2 \rfloor$, we have $C \leq 4m^2 2^m$ for $m \leq 8$. For $m \geq 9$,

$$\begin{aligned} C &\leq m^2 2^{m-2} \left[1 + \frac{2(m-1)}{(m-2)^2} \right]^{m-1} \\ &\leq m^2 2^{m-2} \exp \left(\frac{2(m-1)^2}{(m-2)^2} \right) < 4m^2 2^m, \end{aligned}$$

where we use the facts that $1+x \leq e^x$ and the minimum value of $\frac{2(m-1)^2}{(m-2)^2}$ attains at $m=9$, so the desired upper bound follows.

The proof of the lower bound employs the probabilistic method. Suppose that $G=(V,E)$ is a graph in which every edge coloring in two colors produces a monochromatic $K_{m,n}$. Let

$$V_k = \{v \in V : \deg(v) \geq k\}.$$

Then $|V_k| \leq 2|E|/k$. If G contains $K_{m,n}$ on bipartition (A,B) as a subgraph, then, clearly, $A \subseteq V_n$ and $B \subseteq V_m$. Hence, setting $M = |V_n|$ and $N = |V_m|$, it must be true that every two-coloring of edges of $K_{M,N}$ produces a monochromatic $K_{m,n}$. However, in a random edge coloring of $K_{M,N}$ in red and blue in which

$$\Pr[e \text{ is red}] = \Pr[e \text{ is blue}] = \frac{1}{2},$$

the probability that there is a monochromatic $K_{m,n}$ is at most

$$\begin{aligned} \frac{2 \binom{M}{m} \binom{N}{n}}{2^{mn}} &\leq 2 \left(\frac{eM}{m} \right)^m \left(\frac{eN}{n} \right)^n \frac{1}{2^{mn}} \\ &= 2 \left(\frac{eM}{m} \right)^m \left(\frac{eN}{2^m n} \right)^n. \end{aligned}$$

Now suppose that $e(G) < m2^{m-1}n/e$. Then $M = |V_n| < m2^m/e$ and $N = |V_m| < n2^m/e$ so that $eN/(2^m n) \leq 1 - \epsilon$ for some $\epsilon > 0$. It follows that if n is sufficiently large, then the probability there is a monochromatic

$K_{m,n}$ is less than one. Hence, for any graph G if $e(G) < m2^{m-1}n/e$, then $G \not\rightarrow (K_{m,n})$, which follows by the desired lower bound follows. $\square$

It is natural to believe that $\hat{r}(K_{n,n}) = (2 + o(1))^n$, which is exactly the case. The following result is due to Erdős and Rousseau (1993).

Theorem 14.5 *For all large integer n ,*

$$\frac{1}{30}n^22^n < \hat{r}(K_{n,n}) < \frac{3}{2}n^32^n.$$

Proof. The proof for the upper bound is similar with that in Theorem 14.4 as if M and N satisfy

$$N \binom{M/2}{n} > (n-1) \binom{M}{n},$$

then $K_{M,N} \rightarrow K_{n,n}$. In particular, if we set $N = \lfloor n^2/2 \rfloor$ and $M = 3n2^n$, then the above holds for all $n \geq 6$ and we have the upper bound as desired.

The proof of the lower bound depends on the following counting result.

Lemma 14.7 *A graph with q edges contains at most*

$$\left(\frac{2eq}{n}\right) \left(\frac{2e^2q}{n^2}\right)^n$$

copies of $K_{n,n}$.

Proof. Let $G = (V, E)$ be a graph with q edges and let

$$m = \left\lfloor \frac{n}{2} \log \left(\frac{2q}{n^2} \right) \right\rfloor.$$

Without loss generality, we assume that G contains no isolated vertices. To distinguish the magnitude of the degrees of vertices, we set

$$d_{-1} = 1, d_k = ne^{k/n} \ (k = 0, 1, \dots, m), d_{m+1} = \infty,$$

and set

$$X_k = \{x \in V : d_k \leq \deg(x) < d_{k+1}\}$$

for $k = -1, 0, \dots, m$. Then $X_{-1}, X_0, \dots, X_m$ form a partition of V . Let

$$W_k = \cup_{j=k}^m X_j = \{x \in V : \deg(x) \geq d_k\}.$$

Let us say that a subgraph $K_{n,n}$ in G on vertex set $U \subseteq V$ is of *type k* if k is the smallest index j for which $X_j \cap U \neq \emptyset$. Equivalently $K_{n,n}$

is of type k if and only if each $v \in U$ satisfies $d_k \leq \deg_G(v) < d_{k+1}$, where $\deg_G(v)$ is the degree of v in G not that in $K_{n,n}$. Denote by M_k for the number of type k copies of $K_{n,n}$ in G . Note there is no $K_{n,n}$ of type -1 and $M = \sum_{k=0}^m M_k$ is the total number of copies of $K_{n,n}$ in G . In a type k copy of $K_{n,n}$ every vertex belongs to W_k and at least one vertex belongs to X_k . Thus one side of the $K_{n,n}$ is an n -element subset of the neighborhood of a vertex in X_k and the other side is an n -element subset of W_k . It follows that

$$M_k \leq |X_k| \binom{d_{k+1}}{n} \binom{|W_k|}{n}.$$

Using the facts that $\binom{N}{n} \leq (eN/n)^n$ and $|W_k| \leq 2q/d_k$, we find that for $k = 0, 1, \dots, m-1$,

$$M_k \leq |X_k| \left(\frac{ed_{k+1}}{n} \right)^n \left(\frac{2eq}{d_k n} \right)^n = e|X_k| \left(\frac{2e^2 q}{n^2} \right)^n.$$

Since $d_m \geq \sqrt{2q}$ so $|X_m| \leq \sqrt{2q}$, and each vertex of a type m subgraph must belong to X_m , thus we have

$$M_m \leq \binom{|X_m|}{n}^2 \leq \left(\frac{2e^2 q}{n^2} \right)^n.$$

If $|X_m| = 0$ then $M_m = 0$. Hence we can write

$$M_m \leq e|X_m| \left(\frac{2e^2 q}{n^2} \right)^n,$$

which coincides with the upper bound of M_k with $0 \leq k \leq m-1$. Therefore we obtain

$$\begin{aligned} M &= \sum_{k=0}^m M_k \leq \sum_{k=0}^m e|X_k| \left(\frac{2e^2 q}{n^2} \right)^n \\ &= e|W_0| \left(\frac{2e^2 q}{n^2} \right)^n \leq \left(\frac{2eq}{n} \right) \left(\frac{2e^2 q}{n^2} \right)^n, \end{aligned}$$

completing the proof. $\square$

Proof for the lower bound in Theorem 14.5. Now let G be an arbitrary graph with q edges with $q \leq n^2 2^n / 30$. Consider a random red-blue edge coloring of G in which each edge is red with probability $1/2$ and colorings of distinct edges are independent. In view of the lemma, we find

that the probability P that such a random coloring yields a monochromatic $K_{n,n}$ satisfies

$$\begin{aligned} P &< 2 \left(\frac{2eq}{n} \right) \left(\frac{2e^2q}{n^2} \right)^n 2^{-n^2} \\ &\leq \frac{2en}{15} \left(\frac{4e^2q}{n^2} \right)^n 2^{-n^2} \leq n \left(\frac{2e^2}{15} \right)^n \rightarrow 0 \end{aligned}$$

as $n \rightarrow \infty$. Thus $G \not\rightarrow (K_{n,n})$, and the desired lower bound follows. $\square$

14.3 Induced Ramsey Theorem ★

In the definition of Ramsey number $r(K_n)$, we ask what the smallest N such that in any red-blue edge coloring of K_N , there is either a red K_n or a blue K_n . Here the clique K_n is an induced subgraph. Let us change the problem for a general graph H and ask what for a graph G such that in any red-blue edge coloring of G , there is a monochromatic *induced* H . This slightly modification changes the problem dramatically. A big question is whether or not such graph G exists for an arbitrary given graph H . We call such a graph G a *Ramsey graph* for H . This should be different from a definition in Chapter 1, where a graph G was called as a Ramsey graph for a Ramsey number $r(H)$ if the order of G is $r(H) - 1$ and neither G nor $\bar{G}$ contains H as a subgraph. The subgraph is not necessarily induced.

The following existence theorem was proved by Deuber. Independent proofs were due to Erdős, Hajnal and Pósa, and to Rödl. We adopt one of them from the book by Diestel.

Theorem 14.6 *Every graph has a Ramsey graph.*

The remainder of the section is the proof for this theorem, which begins with its version on bipartite graphs.

We shall write a bipartite graph B as triples (V_1, V_2, E) , where V_1 and V_2 are bipartition of B and E is the its edge set. Given another bipartite graph $\bar{B} = (\bar{V}_1, \bar{V}_2, \bar{E})$, if B is isomorphic to an *induced* subgraph of $\bar{B}$, in which V_i corresponds $\bar{V}_i$ for $i = 1, 2$, then there is an injective map $\phi : V_1 \cup V_2 \rightarrow \bar{V}_1 \cup \bar{V}_2$ so that $\phi(V_i) \subseteq \bar{V}_i$ for $i = 1, 2$, and $\phi(v_1)\phi(v_2) \in \bar{E}$ if and only $v_1v_2 \in E$. We will call such a map ϕ as an *embedding* of B in $\bar{B}$, denoted by $\phi : B \rightarrow \bar{B}$.

As before, let $X^{(k)}$ denote the family of all k -subsets of X . Define a bipartite graph $(X, X^{(k)}, E_k)$, in which the edge set E_k contains all edges of the form xY with $Y \in X^{(k)}$ and $x \in Y$.

Lemma 14.8 *Every bipartite graph $B = (V_1, V_2, E)$ can be embedded in a bipartite graph of the form $\bar{B} = (X, X^{(k)}, E_k)$.*

Proof. Let B be a bipartite graph with vertex classes $V_1 = \{a_1, a_2, \dots, a_n\}$ and $V_2 = \{b_1, b_2, \dots, b_m\}$. Denote by X for a set of $2n + m$ vertices as

$$X = \{x_1, \dots, x_n, y_1, \dots, y_n, z_1, \dots, z_m\}.$$

Let $\bar{B} = (X, X^{(n+1)}, E_{n+1})$. We shall show that there is an embedding ϕ of B in $\bar{B}$.

Set $\phi(a_i) = x_i$ for $i = 1, \dots, n$ first. Then for any $j = 1, \dots, m$, a member $Y \in X^{(n+1)}$ will be chosen as $\phi(b_j)$, which is denoted by Y_j . We in fact construct Y_j as follows. Since $\phi(b_j) = Y_j$, then the neighbors of b_j in B should be mapped into Y_j , that is to say,

$$\phi(N(b_j)) \subseteq Y_j.$$

In order to distinguish different Y_j , we put z_j into Y_j as a label. So far we have chosen

$$|\phi(N(b_j))| + 1 = d(b_j) + 1 \leq n + 1$$

elements for Y_j . If the inequality is strict, we then simply fill Y_j with elements from $\{y_1, \dots, y_n\}$ until Y_j has $n + 1$ elements.

Note that $\phi(V_1) = \{x_1, \dots, x_n\}$ and $\phi(V_2) = \{Y_1, \dots, Y_m\}$, and x_i is a neighbor of Y_j in $\bar{B}$ if and only if a_i is a neighbor of b_j in B . The other neighbors of Y_j are out of $\{x_1, \dots, x_n\}$. Thus B is isomorphic to the subgraph induced by $\{x_1, \dots, x_n\}$ and $\{Y_1, \dots, Y_m\}$ in $\bar{B}$, so the map ϕ is an embedding of B in $\bar{B}$ as desired. $\square$

The next lemma is the bipartite case of Theorem 14.6, which says that every bipartite graph has a bipartite Ramsey graph.

Lemma 14.9 *Let B be bipartite graph. Then there exists a bipartite graph $\bar{B}$ such that for any two-coloring of edges of $\bar{B}$ there is an embedding $\phi : B \rightarrow \bar{B}$ in which all edges of $\phi(B)$ are monochromatic.*

Proof. It is easy to see that if ϕ_1 is an embedding of B_1 in B_2 and ϕ_2 is an embedding of B_2 in B_3 , then $\phi_2\phi_1$ is an embedding of B_1 in B_3 . Thus by Lemma 14.8 we may assume that $B = (X, X^{(k)}, E_k)$. Let $n = |X|$, $\ell = 2k - 1$ and $s = 2\binom{\ell}{k}$. Let $\bar{X}$ be a set whose cardinality is an s -color Ramsey number for ℓ -uniform hypergraph as

$$|\bar{X}| = r_s^{(\ell)}(kn + k - 1).$$

We shall show the assertion in the lemma with $\bar{B} = (\bar{X}, \bar{X}^{(\ell)}, E_\ell)$.

Let us fix a coloring on E_ℓ with colors α and β . Among $\ell = 2k - 1$ edges incident to a vertex $\bar{Y} \in \bar{X}^{(\ell)}$ in $\bar{B}$, at least k of them are monochromatic. Define $\bar{Z} \subseteq \bar{Y}$ with $|\bar{Z}| = k$ so that all edges $\bar{x}\bar{Y}$ for $\bar{x} \in \bar{Z}$ are in the same color. The color and the set $\bar{Z}$ are called as the color and the set associated with $\bar{Y}$, respectively.

Assign a linear order to $\bar{X}$ as

$$\bar{X} = \{\bar{x}_1, \bar{x}_2, \bar{x}_3, \dots\}.$$

For every $\bar{Y} \in \bar{X}^{(\ell)}$ with $\bar{Y} = \{\bar{x}_{i_1}, \dots, \bar{x}_{i_\ell}\}$, denote by $\sigma_{\bar{Y}}$ for the order-preserving map with $\sigma_{\bar{Y}}(\bar{x}_{i_j}) = j$, then $\sigma_{\bar{Y}}(\bar{Z}) \in [\ell]^{(k)}$.

We now color $\bar{X}^{(\ell)}$ with $s = 2\binom{\ell}{k}$ elements of the set $[\ell]^{(k)} \times \{\alpha, \beta\}$ as colors. For a given $\bar{Y} \in \bar{X}^{(\ell)}$, color $\bar{Y}$ with the pair $(\sigma_{\bar{Y}}(\bar{Z}), \gamma)$, where $\bar{Z}$ and $\gamma \in \{\alpha, \beta\}$ are the set and the color associated with $\bar{Y}$, respectively. By the definition of Ramsey number for uniform hypergraph, we know that there is $W \subseteq \bar{X}$ with $|W| = kn + k - 1$ such that all elements of $W^{(\ell)}$ are monochromatic. Thus there exists $S \in [\ell]^{(k)}$ and a color γ , say α , such that all $\bar{Y} \in W^{(\ell)}$ are colored (S, α) . That is to say, all $\bar{Y} \in W^{(\ell)}$ satisfy that $\sigma_{\bar{Y}}(\bar{Z}) = S$ and they are all associated with the same color α .

We now construct the desired embedding ϕ of B in $\bar{B}$.

The elements of W have an order preserved from that in $\bar{X}$. Without loss of generality, we assume that W contains the first $kn + k - 1$ elements of $\bar{X}$,

$$W = \{\bar{x}_i : i = 1, 2, \dots, kn + k - 1\}.$$

Set

$$X = \{x_1, x_2, \dots, x_n\},$$

and define $\phi(x_i) = \bar{x}_{ik}$. Denote by w_i for $\bar{x}_{ik}$, so $\phi(X) = \{w_1, \dots, w_n\}$ and there are exactly $k - 1$ elements of W between w_i and w_{i+1} in the order.

We then define ϕ on $X^{(k)}$. Given $Y \in X^{(k)}$, we shall choose $\phi(Y) = \bar{Y} \in \bar{X}^\ell$ so that the neighbors of $\bar{Y}$ among the vertices in $\phi(X)$ are precisely the images of the neighbors of Y in B , i.e. the vertices $\phi(x)$ with $x \in Y$, and so that all these edges incident to $\bar{Y}$ in $\bar{B}$ are colored α . To find such $\bar{Y}$, we first construct its subset $\bar{Z}$ as $\{\phi(x) : x \in Y\}$, which are k vertices of type w_i . Then extend $\bar{Z}$ by $\ell - k$ further vertices $u \in W \setminus \phi(X)$ to a set $\bar{Y} \in W^{(\ell)}$, in such a way that $\sigma_{\bar{Y}}(\bar{Z}) = S$. This is possible since there are $k - 1 = \ell - k$ other vertices of W between w_i and w_{i+1} . Then

$$\bar{Y} \cap \phi(X) = \bar{Z} = \{\phi(x) : x \in Y\},$$

so $\bar{Y}$ has the right neighbors in $\phi(X)$, and all the edges between $\bar{Y}$ and these neighbors are colored α . The images $\bar{Y}$ of different vertices Y are

distinct since their intersections with $\phi(X)$ differ, so ϕ is an injective map on $X^{(k)}$. Hence, the map ϕ is indeed an embedding of B in $\bar{B}$ satisfied the desired condition. $\square$

Proof of Theorem 14.6. The idea of the proof is to reduce the general case of the theorem to the bipartite case, where Lemma 14.9 can be employed. Let H be a given graph of order s and let $n = r(s, s)$. Then in any edge coloring of K_n in two colors, there is a monochromatic K_s hence a monochromatic copy of H . Denote by K for this K_n . Note that the monochromatic subgraph H in K may be not an induced subgraph in that color.

We will construct a graph G_0 . Let $\ell = \binom{n}{s}$. Arrange the vertices of K in a column, and replace every vertex by a row of ℓ vertices. In each of the ℓ columns, choose an s -set so that any pair of such sets contain vertices coming from different rows. Let us furnish each column in the chosen s -set with the edges of a copy of H . The graph G_0 consists of ℓ disjoint copies of H and $(n - s)\ell$ isolated vertices.

We define G_0 formally as follows. Assume that $V(K) = \{1, \dots, n\}$ and choose copies $H_1, \dots, H_\ell$ of H in K with pairwise distinct (not necessarily disjoint) vertex sets. We then define

$$\begin{aligned} V(G_0) &= \{(i, j) : i = 1, \dots, n; j = 1, \dots, \ell\} \\ E(G_0) &= \cup_{j=1}^{\ell} \{(i, j)(i', j) : ii' \in E(H_j)\}, \end{aligned}$$

where the end vertices of each edge $(i, j)(i', j)$ are in the same column.

Applying Lemma 14.9 iteratively to all the pairs of rows of G_0 , we construct a large graph G such that for every edge coloring of G there is an induced copy of G_0 in G that is monochromatic on all the bipartite subgraph induced by its pairs of rows. By contracting its rows the projection of this $G_0 \subseteq G$ to $\{1, \dots, n\}$, we define an edge coloring of K . Thus one of the $H_j \subseteq K$ will be monochromatic. But this H_j occurs with the same coloring in the j th column of G_0 , and it is an induced subgraph of G_0 hence an induced subgraph of G . We omit the formal definition of desired embedding map described the above procedure. $\square$

14.4 Bipartite Ramsey Numbers

It is likely that there are some similar behaviors between Ramsey numbers of complete graphs and that of complete bipartite graphs, such as

$r(K_m, K_n)$ and $r(K_{m-1, m-1}, K_{n, n})$, and $r(K_n)$ and $r(K_{n, n})$. The similarity even appears in other variant called *bipartite Ramsey number*, which is defined to be the smallest integer N such that in any red-blue edge-coloring of $K_{N, N}$, there is a monochromatic red $K_{m, n}$ or a monochromatic blue $K_{s, t}$. Let us denote the defined N as $br(K_{m, n}, K_{s, t})$, and write

$$b(m, n) = br(K_{m, n}, K_{m, n}),$$

and

$$\bar{b}(m, n) = br(K_{m, m}, K_{n, n}).$$

Note that $b(n, n) = \bar{b}(n, n)$.

The following result is due to A. Thomason (1982), in which the special cases with $m = 2$ and $m = 3$ are due to Beineke and Schwenk (1976), and Irving (1978), respectively.

Theorem 14.7 *For any integers $n \geq m \geq 1$,*

$$b(m, n) \leq 2^m(n - 1) + 1.$$

The result will follow the following lemma immediately. Let $K_{N, N}$ be a complete bipartite graph, whose first vertex class is X and the second is Y . In the following lemma, $K_{m, n}$ is a complete bipartite graph on vertex classes $P \cup Q$ with $P \subseteq X$, $|P| = m$, and $Q \subseteq Y$, $|Q| = n$. It is important to note here that a $K_{m, n}$ is distinguish from a $K_{n, m}$. In other word, we consider the orders of the vertex classes of the bipartite graphs.

Lemma 14.10 *Let $N = 2^m(n - 1)$ with $1 \leq m \leq n$. Suppose that the edges of $K_{N, N}$ are colored with red and blue such that there is neither a red $K_{m, n}$ nor a blue $K_{n, m}$. Then each vertex of $K_{N, N}$ is incident with exactly $N/2$ red edges and $N/2$ blue edges (unless $m = n = 2$). Furthermore, any two-coloring of edges of $K_{N, N+1}$ yields a red $K_{m, n}$ or a blue $K_{n, m}$.*

Proof. Observe that the second part of the lemma follows from the first. Suppose that the edges of $K_{N, N+1}$ are colored without a red $K_{m, n}$ or a blue $K_{n, m}$. Choose $x \in X$ and $y_1, y_2 \in Y$ such that xy_1 is red and xy_2 is blue. Then $K_{N, N+1} - y_i$, $i = 1, 2$, has an edge coloring without red $K_{m, n}$ or blue $K_{n, m}$ so x is incident with exactly $N/2$ red edges in each. This is impossible since x is incident with more red edges in $K_{N, N+1} - y_2$ than that in $K_{N, N+1} - y_1$.

The proof for the first part is by induction on m . Let $K_{N, N}$ be a complete bipartite graph on $X \cup Y$ with $|X| = |Y| = N = 2^m(n - 1)$ that contains neither a red $K_{m, n}$ nor a blue $K_{n, m}$. If $m = 1$, then $N = 2(n - 1)$

and $K_{N,N}$ has at most $N(n-1)$ red edges and at most $N(n-1)$ blue edges. So

$$N^2 = \sum_{x \in X} d_r(x) + \sum_{y \in Y} d_b(y) \leq 2N(n-1) = N^2,$$

hence the equalities hold throughout, implying that $d_r(x) = d_b(y) = n-1$ for any $x \in X$ and $y \in Y$.

Let $m \geq 2$ be fixed. Suppose that the theorem holds for smaller values of m . Let $t = N/2 = 2^{m-1}(n-1)$. Suppose that there is vertex $x_0 \in X$ with $d_r(x_0) > t$.

Case 1 There is a vertex $y_0 \in Y$ with $d_b(y_0) > t$. We assume that the edge $x_0 y_0$ is red, say. Consider the subgraph induced by $N_r(x_0)$ and $N_b(y_0) \setminus \{x_0\}$. This is a complete bipartite graph. Its vertex class in X has at least t vertices and that in Y has at least $t+1 = 2^{m-1}(n-1) + 1$ vertices. We thus have a complete bipartite graph $K_{t+1,t}$. From the induction hypothesis, we have either a red $K_{m-1,n}$ or a blue $K_{n,m-1}$. Together with the vertex x_0 or y_0 , we get a red $K_{m,n}$ or a blue $K_{n,m}$. This is impossible. The situation is same if we assume that the edge $x_0 y_0$ is blue.

Case 2 Each vertex $y \in Y$ satisfies that $d_b(y) \leq t$. We shall show that for $1 \leq s \leq m-1$, a red $K_{s, 2^{m-s}(n-1)+1}$ yields a red $K_{s+1, 2^{m-s-1}(n-1)+1}$. The existence of such a red $K_{s, 2^{m-s}(n-1)+1}$ for $s=1$ is just the condition $d_r(x_0) > t$ as given. Suppose that there is a red $K_{s, 2^{m-s}(n-1)+1}$ on vertex classes P and Q with $P \subseteq X$, $|P| = s$, $Q \subseteq Y$, $|Q| = 2^{m-s}(n-1) + 1 = 2\lambda + 1$, where $\lambda = 2^{m-s-1}(n-1)$. Then the number of red edges between $X \setminus P$ and Q is at least $|Q|(N/2 - s)$ since each $y \in Y$ satisfies $d_r(y) \geq N/2$.

Subcase 2.1 Each $x \in X \setminus P$ satisfies $|N_r(x) \cap Q| \leq \lambda = 2^{m-s-1}(n-1)$. We have

$$|X \setminus P| \lambda \geq |Q|(N/2 - s),$$

which gives

$$(N-s)\lambda \geq (2\lambda+1)(N/2-s).$$

We thus have

$$s(\lambda+1) \geq N/2 = 2^{s+1}\lambda,$$

which is impossible unless $s=1$ and $m=n=2$.

Subcase 2.2 Some $x \in X \setminus P$ satisfies $|N_r(x) \cap Q| \geq 2^{m-s-1}(n-1) + 1$. Thus $N_r(x) \cap Q$ and $P \cup \{x\}$ induce a red $K_{s+1, 2^{m-s-1}(n-1)+1}$ as desired. We hence obtain a red $K_{m,n}$ with $s=m-1$.

Therefore in any edge-coloring of $K_{N,N}$ that contains neither a red $K_{m,n}$ nor a blue $K_{n,m}$, there is no vertex x with $d_r(x) > N/2$. So each vertex $x \in X$ satisfies $d_r(x) \leq N/2$. Similarly each $y \in Y$ satisfies $d_r(y) \leq N/2$. The same argument will yield that each vertex v satisfies $d_b(v) \leq N/2$, so the desired statement follows immediately. $\square$

For fixed m and large n , the asymptotic formula of $b(m, n)$ is $2^m n$, which is also that of $r(K_{m,n})$.

Theorem 14.8 *For fixed m , as $n \rightarrow \infty$,*

$$b(m, n) \sim 2^m n.$$

Proof. The upper bound follows from Theorem 14.7. The proof for the desired lower bound $b(m, n) \geq (1 - o(1))2^m n$ is similar to that for $r(K_{m,n})$ in Chapter 5, so we omit it. $\square$

However, it is difficult to obtain an asymptotic formula of $\bar{b}(m, n)$. The situation is similar to that for $r(m, n)$ that we have encountered. The following result is due to Y. Caro and C. Rousseau (2001).

Theorem 14.9 *Let integer $m \geq 2$ be fixed. Then there are positive constant A and B such that*

$$A \left(\frac{n}{\log n} \right)^{(m+1)/2} \leq \bar{b}(m, n) \leq B \left(\frac{n}{\log n} \right)^m.$$

Proof. The upper bound is based on well-known results for the Zarankiewicz numbers $z(N, M; s, t)$ that is defined in Chapter 4. Denote by $z(N; s)$ for $z(N, N; s, s)$, then we have

$$z(N; s) \leq \left(\frac{s-1}{N} \right)^{1/s} N(N-s+1) + (s-1)N.$$

To prove $\bar{b}(m, n) \leq N$ it suffices to show that $z(N; m) + z(N; n) < N^2$. Take $\epsilon > 0$ and $N = c(n/\log n)^m$, where $c = (1 + \epsilon)/(m-1)^{m-1}$. Then

$$\begin{aligned} \frac{z(N; m)}{N^2} &< \left(\frac{m-1}{N} \right)^{1/m} \left(1 - \frac{m-1}{N} \right) + \frac{m-1}{N} \\ &= \left(\frac{m-1}{c} \right)^{1/m} \frac{\log n}{n} + O \left(\left(\frac{\log n}{n} \right)^m \right). \end{aligned}$$

To bound $z(N; n)/N^2$, we first have

$$\left(\frac{n-1}{N} \right)^{1/n} = \left(\frac{(n-1)(\log n)^m}{cn^m} \right)^{1/n} = 1 - \frac{(m-1)\log n}{n} + O \left(\frac{\log \log n}{n} \right).$$

Hence

$$\begin{aligned} \frac{z(N; n)}{N^2} &\leq \left(\frac{n-1}{N} \right)^{1/n} \left(1 - \frac{n-1}{N} \right) + \frac{n-1}{N} \\ &= 1 - \frac{(m-1)\log n}{n} + O \left(\frac{\log \log n}{n} \right). \end{aligned}$$

Adding the above bounds, we obtain

$$\begin{aligned} & \frac{z(N; m) + z(N; n)}{N^2} \\ = & 1 - \left(1 - \frac{1}{(1 + \epsilon)^{1/m}}\right) \frac{(m-1) \log n}{n} + O\left(\frac{\log \log n}{n}\right), \end{aligned}$$

so $(z(N; m) + z(N; n))/N^2 < 1$ for all large n as required.

To prove the lower bound, we use the Lovász local lemma in the manner as we used for the lower bound of $r(m, n)$ in Chapter 6. Consider a random edge-coloring of $K_{N,N}$ in red and blue, in which each edge is colored red with probability p , independently. For each set S of $2m$ vertices, m from each vertex class of the $K_{N,N}$. Let R_S denote the event that S induces a monochromatic red $K_{m,m}$. Similarly, for each set T consisting of n vertices from each vertex class, let B_T denote the event that T induces a monochromatic blue $K_{n,n}$. For each of the $\binom{N}{m}^2$ of S , write $\Pr(R)$ for the common value $\Pr(R_S) = p^{m^2}$. In the same way, $\Pr(B) = \Pr(B_T) = (1-p)^{n^2}$ for each of $\binom{N}{n}^2$ possible $B = B_T$ events. Let S be fixed and let D_{RR} denote the number of events $R_{S'}$ such that R_S and $R_{S'}$ are dependent, that is the red bipartite graphs induced by S and S' share at least one edge. Similarly, let D_{RB} denote the number of events B_T such that R_S and B_T are dependent. In the same way, for fixed T , we define the dependence numbers D_{BR} and D_{BB} . By the Local Lemma, the probability that a random coloring has neither a red $K_{m,m}$ nor a blue $K_{n,n}$ is positive if there exist positive numbers x and y such that $x \Pr(R) < 1$ and $y \Pr(B) < 1$ and

$$\begin{aligned} \log x &> x D_{RR} \Pr(R) + y D_{RB} \Pr(B) \\ \log y &> x D_{BR} \Pr(R) + y D_{BB} \Pr(B). \end{aligned}$$

With positive constants c_1, c_2, c_3, c_4 to be chosen, set

$$\begin{aligned} p &= c_1 N^{-2/(m+1)}, & n &= c_2 N^{2/(m+1)}, \\ x &= c_3, & y &= \exp(c_4 N^{2/(m+1)} (\log N)^n). \end{aligned}$$

To prove that there are choices of the constants $c_1, \dots, c_4$, we note the following bounds

$$\begin{aligned} D_{RR} &\leq m^2 \binom{N}{m-1}^2 < N^{2(m-1)} \\ D_{BR} &\leq n^2 \binom{N}{m-1}^2 < n^2 N^{2(m-1)} \\ D_{RB}, D_{BB} &\leq \binom{N}{n}^2 < \left(\frac{eN}{n}\right)^{2n}. \end{aligned}$$

Similar computation as in Chapter 6 justifies the lower bound statement with

$$A = \frac{1 - \epsilon}{m^{1/(m-1)}} \left(\frac{m-1}{m^2} \right)^{(m+1)/2},$$

where $\epsilon > 0$ is arbitrary. $\square$

Theorem 14.10 *For all large n ,*

$$(1 - o(1)) \frac{\sqrt{2}n}{e} 2^{n/2} \leq b(n, n) = \bar{b}(n, n) \leq 2^n(n-1) + 1.$$

Proof. The upper bound follows from Theorem 14.7. The lower bound can be proved by the symmetric form of Local Lemma in the same manner as that for the lower bound of $r(n, n)$ in Chapter 6, we thus omit it. (See Hattingh and Henning (1998)). $\square$

Problem 14.1 *Determine the limit $b(n, n)^{1/n}$ as $n \rightarrow \infty$ if it exists, and determine the order of $\bar{b}(m, n)$ for fixed m .*

We believe the above problem is easier than the same problem for $r(n, n)$ and $r(m, n)$.

14.5 Ramsey Numbers on Parameters

We have seen that it is usually difficult to estimate the value of $r(G, H)$, particularly that of $r(m, n)$. Many researchers made some generalization on Ramsey numbers. For example, Benedict, Chartand, and Lick (1977) defined $\mu(f; m, H)$ to be the least integer N such that any graph G of N , either $f(G) \geq m$ or $\bar{G}$ contains H as a subgraph.

Let f_1 and f_2 be graph parameters and let m and n be integers. Define the Ramsey number $r(f_1 \geq m; f_2 \geq n)$ as the least integer N such that for any graph G of order N either $f_1(G) \geq m$ or $f_2(\bar{G}) \geq n$. We call $r(f_1 \geq m, f_2 \geq n)$ as the *Ramsey number on f_1 and f_2* , or the *mixed Ramsey number* if f_1 and f_2 are different. Thus $r(m, n) = r(\omega \geq m, \omega \geq n) = r(\alpha \geq m, \alpha \geq n)$, where $\omega(G)$ is the clique number of G , and $\alpha(G)$ is the independence number of G . For simplicity, if $f_H(G)$ is the number of subgraphs H in G , we write $r(H; \dots)$ for $r(f_H \geq 1; \dots)$. Thus $r(f_H \geq 1; f_F \geq 1)$ can be written as $r(H; F)$.

The most natural question is that for which pair f_1 and f_2 , the defined Ramsey number $r(f_1 \geq m; f_2 \geq n)$ exists. The answer is not always

positive. For example, $r(\chi \geq 2, \alpha \geq 2)$ does not exist since the empty graph $N_n = \overline{K}_n$ satisfies that $\chi(N_n) = 1$ and $\alpha(\overline{N}_n) = 1$. However, for many pairs of parameters, the existence of the Ramsey number on these parameters can be easily verified.

Theorem 14.11 *Let f_1 and f_2 be graph parameters. Then for any positive integers m and n , $r(f_1 \geq m; f_2 \geq n)$ exists if and only if*

$$\lim_{k \rightarrow \infty} \left(\min_{|V(G)|=k} (f_1(G) + f_2(G)) \right) = \infty. \quad (14.1)$$

Proof. Suppose that for any positive integers m and n , $r(f_1 \geq m; f_2 \geq n)$ exists. Then for any integer $M > 0$, $K(M) = r(f_1 \geq M; f_2 \geq M)$ exists. So for any graph G of order $k \geq K(M)$, either $f_1(G) \geq M$ or $f_2(\overline{G}) \geq M$. Therefore

$$\min_{|V(G)|=k} (f_1(G) + f_2(G)) \geq M + 1$$

for $k \geq K(M)$, so (14.1) holds.

Conversely, suppose (14.1) holds. Then for any positive integers m and n , there exists $K = K(m, n)$ such that if $k \geq K$,

$$\min_{|V(G)|=k} (f_1(G) + f_2(G)) \geq m + n.$$

So for any graph G of order $k \geq K$, $f_1(G) + f_2(\overline{G}) \geq m + n$, hence $f_1(G) \geq m$ or $f_2(\overline{G}) \geq n$. Minimizing such K shows the existence of $r(f_1 \geq m; f_2 \geq n)$. $\square$

As an example, we shall verify an easy formula.

Theorem 14.12 *Let m and n be positive integers. Then*

$$r(\chi \geq m, \chi \geq n) = (m - 1)(n - 1) + 1,$$

where χ signifies the chromatic number of a graph.

Proof. The assertion is obviously true if one of m and n is one. Assume that $m, n \geq 2$. Let G be a graph consisting of disjoint $n - 1$ copies of K_{m-1} . Then $\chi(G) = m - 1$ and $\chi(\overline{G}) = n - 1$, yielding that

$$r(\chi \geq m, \chi \geq n) \geq (m - 1)(n - 1) + 1.$$

On the other hand, if G is a graph of order $N = (m - 1)(n - 1) + 1$, then by the fact that $\chi(G)\alpha(G) \geq N$ we have

$$\alpha(G) \geq \left\lfloor \frac{N}{\chi(G)} \right\rfloor = \left\lfloor \frac{(m - 1)(n - 1) + 1}{m - 1} \right\rfloor \geq n.$$

Therefore $\chi(\overline{G}) \geq \omega(\overline{G}) = \alpha(G) \geq n$, proving

$$r(\chi \geq m, \chi \geq n) \leq (m-1)(n-1) + 1.$$

□

Let G' be a graph obtained by deleting a vertex from G . Similarly as that for proving

$$r(F, H) \leq r(F', H) + r(F, H'), \quad (14.2)$$

we have the following bound.

Theorem 14.13 *For a fixed graph F , define $f_F(G)$ be the number of subgraphs isomorphic to F with different vertex sets in graph G . Then for any graphs F and H of orders at least two and any positive integers m and n ,*

$$r(f_F \geq m, f_H \geq n) \leq r(f_{F'} \geq m, f_H \geq n) + r(f_F \geq m, f_{H'} \geq n).$$

□

By setting $m = n = 1$, the above theorem becomes (14.2).

Let $\mathcal{F}_i$ is a family of graphs for $i = 1, 2$. Define the *class Ramsey number* $r(\mathcal{F}_1, \mathcal{F}_2)$ as the minimum integer N such that any edge coloring of K_N in red and blue contains some red $F_1 \in \mathcal{F}_1$ or some blue $F_2 \in \mathcal{F}_2$. The class Ramsey number is associated to Ramsey number on parameters if so is $\mathcal{F}_i$. Let us call a graph as (p, q) graph if it has p vertices and q edges. Denote by $\mathcal{F}(n, s)$ for the family of connected graphs with n vertices and s edges for $n-1 \leq s \leq \binom{n}{2}$. We then get a definition of $r(n, s; m, t)$ as $r(\mathcal{F}(n, s), \mathcal{F}(m, t))$. General results on class Ramsey numbers are very difficult since it covers the classic Ramsey numbers $r(m, n)$.

14.6 Different Coloring Types

In traditional definitions of graph Ramsey numbers, we asked the minimum number N such that in some edge coloring of K_N contains some *monochromatic* graphs. If we change the coloring types either in K_N or in subgraphs that are contained, we may have some other definitions of Ramsey numbers. We introduce some of them without details of discussion.

Local k -coloring

Let G be a graph and let $k \geq 0$ be an integer. A subset $S \subseteq V(G)$ is said to be a k -independent set if the subgraph induced by S has a maximum

degree at most k . Correspondingly, an edge-coloring of G in any number of colors is called a local k -coloring if each vertex of G is adjacent to at most k distinct colors.

There are two ways to define Ramsey numbers with k -independence. One way is to ask what is the minimum integer N such that any local edge k -coloring of K_N contains a monochromatic G . See, for example, Gyárfás, Lehel, Nešetřil, Rödl, Schelp, and Tuza (1987), and Caro and Tuza (1993). Other way is to ask what is the minimum integer N such that any edge coloring of K_N in m colors contains a local k -colored G .

Zero-sum coloring

Most recent combinatorial research on zero-sum problems is related to a result of Erdős, Ginzburg and Ziv (1961) as follows.

Theorem 14.14 *Let $m \geq k \geq 2$ be integers such that $k|m$. Then for any sequence of integers $\{a_1, a_2, \dots, a_{m+k-1}\}$, there exists a subset I of indices of $\{1, 2, \dots, m+k-1\}$ such that $|I| = m$ and $\sum_{i \in I} a_i = 0 \pmod{k}$.*

Let Z_k be the additive group modulo k . A Z_k -coloring of edges of graph G is a function $f : E(G) \rightarrow Z_k$. We say that G is zero-sum, relative to f , if $\sum_{e \in E(G)} f(e) = 0 \pmod{k}$.

The zero-sum Ramsey number is to ask the minimum integer N such that any Z_k -coloring of edges of K_N contains a zero sum G . For this topic, see a survey by Caro (1996).

Anti-Ramsey coloring

We call an edge coloring of G to be *anti-Ramsey* if each pair of edges have distinct colors. Weakening this condition, an edge coloring of G is called a (p, q) -coloring if every subset of p vertices spans at least q colors. Erdős (1981) asked what is the minimum number $f(n, p, q)$ of colors such that there exist a (p, q) -coloring of K_n . To avoid the trivial cases, we assume that $2 \leq q \leq \binom{p}{2}$. Clearly $f(n, p, \binom{p}{2}) = \binom{n}{2}$. Observe that $(p, 2)$ -coloring are equivalent to coloring without monochromatic K_p . As an example, the result $r_3(3) = 17$ can be expressed in terms of f as $f(16, 3, 2) = 3$, and $f(17, 3, 2) = 4$. For more information, see Erdős and Gyárfás (1997).

We have to be careful to ask the anti-Ramsey subgraphs. Some researchers ask what is the minimum N such that any edge coloring of K_N contains a monochromatic G_1 or an anti-Ramsey G_2 . For most pairs of (G_1, G_2) , these numbers do not exist.

Regular coloring

A graph G is called regular by k if $\Delta(G) - \delta(G) \leq k$. Recall many Ramsey graph we met are regular or regular by 1. We believe that the

minimum integer N such that any regular coloring by 1 contains K_m or independent set of order n is very close to $r(m, n)$.

Exercises

14.1 Prove the lower bounds in Theorem 14.8 and in Theorem 14.10, respectively.

14.2 Find upper bound and lower bound for $r(\chi \geq n, \chi \geq n)$. (Refer Bollobás' result on chromatic numbers of random graphs in Chapter 6 for the lower bound.)

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