Fundamental Graph Theory Test (I)

10,17

Choose exactly 6 problems to work on, 20 points each.

1. Find all non-isomorphic graphs of order 5 and size 7. (Explain your answer.)
2. The eigenvalues of a graph G are the eigenvalues of its adjacency matrix. Find the eigenvalues of C_6.
3. Prove that in a set of n points in the plane with no pair more that distance 1 apart, then the maximum number of pairs separated by distance more than $\frac{1}{\sqrt{2}}$ is $\lfloor \frac{n^2}{3} \rfloor$.
4. Let $< d_1, d_2, \ldots, d_p >$ be a sequence of p positive integers such that $\sum_{i=1}^{p} d_i = 2(p-1)$. Prove that there exists a tree G of order p such that $< d_1, d_2, \ldots, d_p >$ is a degree sequence of G.
5. Prove that $rad(G) \leq diam(G) \leq 2rad(G)$, moreover, for any two positive integers m and n satisfying $m \leq n \leq 2m$, there exists a graph G with $rad(G) = m$ and $diam(G) = n$.
6. If G is a simple graph, then $diam(G) \geq 3$ implies that $diam(\bar{G}) \leq 3$.
7. Let G be a graph with $\delta(G) = m - 1$ and T be a tree of order m. Prove that G contains a subgraph H which is isomorphic to T.

Fundamental Graph Theory, Test (II)

11,28

Choose exactly 6 problems to work on, 20 points each.

1. Prove that a connected multigraph has an eulerian circuit if and only if each vertex of the graph is of even degree.

2. Prove that for each planar graph there exists a drawing with each edge a straight line segment and there are no crossings.

3. Let G^2 be the graph obtain from G by adding the edge uv into G whenever $d_G(u, v) = 2$. Prove that if G is 2-connected, i.e., G has no cut-vertices, then G^2 is a hamiltonian graph.

4. Prove that there are exactly five regular polyhedra.

5. Let a, b, c be three positive integers such that $a < b < c$. Find a graph with $\delta(G) = c$, $\kappa_1(G) = b$ and $\kappa(G) = a$. ($\kappa_1(G)$ is the edge-connectivity of G.)

6. Prove that if $diam(G) = 2$, then $\delta(G) = \kappa_1(G)$.

7. Let $\beta(G)$ denote the maximum number of vertices which are not adjacent mutually, i.e., the independence number. Prove that if $\kappa(G) \geq \beta(G)$, then G is hamiltonian.

Fundamental Graph Theory, Test (III)

1,9

Choose exactly 6 problems to work on, 20 points each.

1. Let G be embedded on a surface with genus $\gamma(G)$ and $|F(G)|$ regions. Prove that $|V(G)| - |E(G)| + |F(G)|$ $= 2 - 2\gamma(G)$.
2. Prove that K_6 has a 2-cell embedding on S_3. (S_3 is a surface with 3 handles.)
3. Prove that a graph G is of chromatic number 2 if and only if G contains no odd cycles.
4. Find a graph G with contains no K_3's but $\chi(G) = 5$.
5. Let $\omega(G)$ be the clique number of G. Prove that if G is an interval graph, then $\chi(G) = \omega(G)$.
6. Let P be the Petersen graph. Prove that $\chi'(P) = 4$.
7. Let G be a k-edge-colorable graph. Prove that G has an equalized k-edge-coloring.
8. Prove that $\chi''(K_{2m}) = \chi''(K_{2m+1}) = 2m + 1$.