Fundamental Graph Theory Homework (III)

1. Prove that $\gamma(K_6) = 1$.
2. Find a 2-cell embedding of K_6 on surface such that the embedding has exactly one region.
3. Prove that for $p \geq 3$, $\gamma(K_p) \geq \lceil \frac{(p-3)(p-4)}{12} \rceil$ and for $m, n \geq 2$, $\gamma(K_{m,n}) \geq \lceil \frac{(m-2)(n-2)}{4} \rceil$.
4. Let G be embedded on a surface with genus $\gamma(G)$ and r regions. Prove that $p - q + r = 2 - 2\gamma(G)$. ($p = |V(G)|$ and $q = |E(G)|$.)
5. Find a 2-cell embedding of $K_{4,5}$ on S_2.
6. Explain why a disconnected graph can not have a 2-cell embedding.
7. Define $\theta_1(G) = min\{n|$ G can be decomposed into n planar graphs$\}$. Prove that $\theta_1(K_{10}) = 3$. (Find a lower bound for $\theta_1(G)$ first.)
8. Determine $\theta_1(K_{6,6})$.
9. Prove that a graph G is of chromatic number 2 if and only if G contains no odd cycles.
10. Find a graph G which contains no K_3's but $\chi(G) = 5$.
11. Prove that $\chi(G) \leq \Delta(G) + 1$.
12. Let G be a graph which is not an odd cycle or a complete graph. Then, prove that $\chi(G) \leq \Delta(G)$.
13. Let G be a planar graph. Prove that $\chi(G) \leq 5$.

14. Let $\chi(G;k)$ denote the number of different vertex colorings of G with k colors. Prove that

    (a) if T is a tree with p vertices, then $\chi(G;k) = k(k-1)^{p-1}$; and

    (b) $\chi(G;k) = \chi(G-e;k) - \chi(G\cdot e;k)$ where $G\cdot e$ is the graph obtained by contracting the edge e into one vertex.

15. Prove that $\chi(G;k)$ is a (degree $|V(G)|$) polynomial with indeterminate k, moreover, the second highest degree term has coefficient $-|E(G)|$.

16. Let $\omega(G)$ be the clique number of G. Prove that if G is an interval graph, then $\chi(G) = \omega(G)$.

17. Prove that $\chi'(K_{2m}) = 2m-1$ and $\chi'(K_{2m+1}) = 2m+1$.

18. Prove that $\chi'(G) \le \frac{3}{2}\Delta(G)$.

19. A graph G is overfull if $|E(G)| > \Delta(G) \cdot \lfloor \frac{|V(G)|}{2} \rfloor$. Prove that if G is overfull, then $\chi'(G) > \Delta(G)$.

20. Prove that $K_{m(n)}$ is of class 1 if and only if $K_{m(n)}$ is not overfull.

21. Prove that if G is a graph with $\binom{n+1}{2}$ edges and $\Delta(G) \le \frac{n}{2}$, then G can be decomposed into n graphs $G_1, G_2, \cdots, G_n$ where $G_i, i = 1, 2, \cdots, n$, is a matching with i edges.

22. Let P be the Petersen graph. Prove that $\chi'(P) = 4$.

23. If G is a regular graph with an odd number of vertices, then prove that $\chi'(G) > \Delta(G)$.

24. Let $\chi''(G)$ denote the total chromatic number of G. Prove that $\chi''(K_{2m}) = \chi''(K_{2m+1}) = 2m+2$ and $\chi''(K_{m,m}) = m+2$.

25. Prove that $\chi''(K_{m,n}) = \Delta(K_{m,n}) + 1$ if and only if $m \ne n$.

26. Let T be a tree. Find $\chi''(T)$.

27. Find a graph G such that $\chi''(G) = \chi'(G)+1 = \chi(G)+2 = 5$.

28. A labeling φ of a graph G is called a prime labeling if $(\varphi(u), \varphi(v)) = 1$ whenever $uv \in E(G)$. Prove that every caterpillar has a prime labeling.

29. Prove or disprove that Q_n has a prime labeling.

30. Prove of disprove $P_m \times P_n$ has a prime labeling.