Fundamental Graph Theory Homework (II)

1. Use three different ways to prove that a connected multigraph has an eulerian circuit if and only if each vertex of the graph is of even degree.

2. Prove that if G is an eulerian graph (G has an eulerian circuit), then its line graph $L(G)$ also has an eulerian circuit, but the converse statement may not be true.

3. Construct a binary de Bruijn sequence (2,5)-sequence, i.e., consecutive 5 digits are all distinct.

4. Prove that the complete graph K_p has p^{p-2} distinct spanning trees.

5. Find the number of distinct spanning trees in $K_p - e$ where e is an arbitrary edge in K_p.

6. Prove that if G is a planar graph with k components, then $p - q + r = 1 + k$ where p, q and r are the number of vertices, edges and regions in G respectively.

7. Prove that for each planar graph there exists a drawing with each edge a straight line segment and there are no crossings.

8. Let G be a maximal planar graph of order p and p_i denote the number of vertices of degree i. Prove that $3p_3 + 2p_4 + p_5 = p_7 + 2p_8 + \cdots + (\triangle(G) - 6)p_{\triangle(G)} + 12$.

9. Prove that the Petersen graph is not a planar graph.

10. Prove that K_6 is not a planar graph and in any drawing of K_6 there are at least three crossings.

11. Find the minimum number of crossings in drawing $K_{4,5}$.

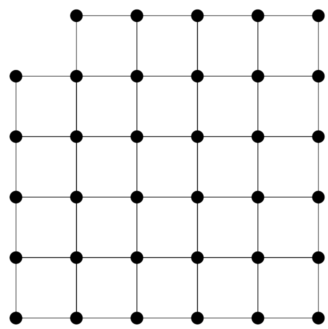
12. Decompose K_{2m+1} into m Hamiltonian cycles.

13. Let m and n be odd positive integers. Decompose $K_{m(n)}$ into $m(n-1)/2$ Hamiltonian cycles.

14. If G and H are hamiltonian graphs, i.e., G and H have a Hamiltonian cycle respectively, then prove that $G \times H$ is also a hamiltonian graph.

15. Let p be an even positive integer. Prove that K_p can be decomposed into $\frac{p}{2}$ Hamiltonian paths.

16. Determine whether the following graph is a hamiltonian graph.



17. Let G^2 be the graph obtain from G by adding the edge uv into G whenever $d_G(u, v) = 2$. Prove that if G is 2-connected, i.e., G has no cut-vertices, then G^2 is a hamiltonian graph.

18. Prove that Q_n (n-cube) is a hamiltonian graph.

19. Prove that in a graph (non-empty) there are at least two vertices which are not cut-vertices.

20. Prove that there are exactly five regular polyhedra.

21. Show that all regular polyhedra are hamiltonian graphs.

22. The connectivity of a graph G is $\kappa(G) = min\{|S| \mid S \subseteq V(G)$ and $G - S$ is not connected or a vertex $\}$. Give an example that G is 3-regular, $diam(G) = 4$ and $\kappa(G) = 1$.

23. Let a, b, c be three positive integers such that $a < b < c$. Find a graph with $\delta(G) = c$, $\kappa_1(G) = b$ and $\kappa(G) = a$. ($\kappa_1(G)$ is the edge-connectivity of G.)

24. Prove that if $diam(G) = 2$, then $\delta(G) = \kappa_1(G)$.

25. Prove that K_{2m} can be decomposed into $2m - 1$ perfect matchings. (A perfect matching of a graph G is a spanning 1-regular subgraph of G.)

26. Let T be the graph as in Figure 1. Prove that K_{21} can be decomposed into 21 isomorphic subgraphs and each graph is isomorphic to T.

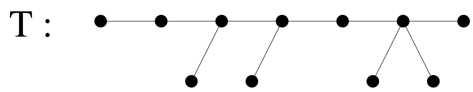


Figure 1: T

27. Prove that K_{101} can be decomposed into isomorphic copies of T.

28. Prove that every 2-connected 3-regular graph has a perfect matching.

29. Let $\beta(G)$ denote the maximum number of vertices which are not adjacent mutually, i.e., the independence number. Prove that if $\kappa(G) \geq \beta(G)$, then G is hamiltonian.

30. Find a "formula" to estimate $\beta(G)$.